PHYSICAL REVIEW D VOLUME 2, NUMBER 6 15 SEPTEMBER 1970

Rest-Frame Sum Rules from Quark Model and Field Algebra

G. J. Komen*
International Centre for Theoretical Physics, Trieste, Italy

AND

S. Pallua
Institute "Rudjer Bošković," Zagreb, Yugoslavia

(Received 28 January 1970; revised manuscript received 27 February 1970)

The extrapolation of low-energy theorems for certain weak quantities from the soft point to the ρ mass is studied. A parabola in the plane of variables, essentially connected with the mass and energy of the particles involved, is taken for the path of extrapolation, as proposed by Fubini and Furlan. As a consequence, sum rules involving the $\pi\rho$ and ρN elastic scattering matrices at threshold are obtained, in which the most important contributions are explicitly exhibited. An important term arises from the right-hand side of the equal-time commutation relation of space components of vector currents, and thus the resulting sum rules are particularly suitable for discussing models which predict such commutators. The same results are obtained in the saturation approach, starting from commutators of electric dipoles and space-space commutators of vector currents. Combining the $\rho\pi$ scattering matrix sum rule with the Weinberg formula, an expression for $\frac{1}{6}\langle r_\pi^2\rangle$ is found. In this particular case and under the simplest assumptions, both field algebra and quark model are compatible with the data within present experimental errors. The investigation is extended to the algebra of tensor currents and together with a hypothesis of partial conservation of tensor current and local $SU(2)$ algebra, this leads to a value for the difference of the mean-square electromagnetic pion and nucleon radius as an exact consequence.

I. INTRODUCTION

Current algebra, as is well known, provides us with low-energy theorems for certain weak quantities. In order to transform these predictions into statements for strong quantities in physical points, some extrapolation procedure has to be chosen. This choice is, of course, independent of current algebra itself. Recently, Fubini and Furlan[1] proposed a general dispersion formulation for this extrapolation. For particles at rest, their choice is equivalent to the rest-frame saturation of equal-time commutators. The main applications have so far been done for the extrapolation from vanishing to physical pion mass. It would be interesting, however, to go beyond the pion case; in particular, to apply the method to vector mesons.

Another problem, and, as it will turn out, a related one, is that of the saturation of equal-time commutators as suggested by higher symmetry[2] and field-algebra models.[3] Whereas the chiral $SU(3)\times SU(3)$ algebra is experimentally rather well confirmed, the validity of the former models is still open. Since these algebras involve "bad" operators,[2,4] saturation in the infinite-momentum frame was difficult. In the rest frame, the validity of a simple saturation assumption was doubted because of the possible presence of semidisconnected contributions.[4] Since the method of Ref. 1 takes these contributions into account, it provides us with a suitable framework for the investigation of these higher commutators. It turns out that in order to get a statement for an on-mass-shell strong quantity, one needs in general two different equal-time commutation relations. For this purpose, sum rules derived from the equal-time commutator of the space components of axial-vector currents and from the commutator of the time derivatives of these currents have been combined.[5] It would be desirable to obtain sum rules in which the space-space commutator is combined with a better-known second commutator. In looking for such a combination one has some choice. In fact, it is only necessary that both commutators lead to the same strong quantity, which appears as a mass singularity, at the mass of the particle bearing the quantum numbers of the currents. It is a general feature of finite-frame saturation of current algebra that, by virtue of Haag's theorem,[6] different algebras having the same quantum numbers lead indeed to the same strong quantity. This will enable us to use in different combinations the following equal-time commutators:

$$[V_i^\alpha, V_j^\beta], \quad \int d^3x \int d^3x' [x_i V_0^\alpha(\mathbf{x},0), x_j' V_0^\beta(\mathbf{x}',0)],$$

and

$$[\mathcal{T}_{0i}^\alpha, \mathcal{T}_{0j}^\beta],$$

where the V's are vector currents and the $\mathcal{T}$'s are tensor charges. These commutators are given, respectively, by field algebra or the $SU(6)$ quark model, by local $SU(2)$, and by the $SU(12)$ quark model.

* On leave from Instituut voor theoretische fysica, Universiteit van Amsterdam, The Netherlands.

[1] S. Fubini and G. Furlan, Ann. Phys. (N. Y.) **48**, 322 (1968); V. de Alfaro and C. Rossetti, Nuovo Cimento Suppl. **6**, 575 (1968).

[2] S. Fubini, G. Segrè, and Y. D. Walecka, Ann. Phys. (N. Y.) **39**, 381 (1966).

[3] T. D. Lee and B. Zumino, Phys. Rev. **163**, 1667 (1967).

[4] R. F. Dashen and M. Gell-Mann, in *Proceedings of the Third Coral Gables Conference on Symmetry Principles at High Energy* (Freeman, San Francisco, 1966).

[5] G. J. Komen and S. Pallua, Nucl. Phys. B20, 493 (1970).

[6] R. Haag, Phys. Rev. **112**, 669 (1958).

In the first part of this paper, we derive an expression for the elastic $\rho\pi$ scattering matrix at threshold, extrapolating a low-energy value for a weak quantity. In the derivation of the low-energy theorem the local $SU(2)$ algebra is used, and in the extrapolation procedure the $[V_i,V_j]$ commutator enters. We show that the same final result can be obtained by rest-frame saturation of electric dipoles and the commutator $[V_i,V_j]$. We give analogous results for ρN scattering.

In Sec. III, we investigate the relation of the tensor algebra to the $[V_i,V_j]$ and dipole commutators. In order to interpret the matrix elements of the tensor currents, we assume a form of partial conservation of tensor current (PCTC). We rederive an exact relation between the constant Γ entering in PCTC and the nucleon isovector radius. We give an analogous relation between Γ and the pion radius. This leads then to

$$\tfrac{1}{6}\langle r_\pi^2\rangle=\tfrac{1}{6}\langle r_N^2\rangle_V+0.69/m_\rho^2.$$

In Sec. IV we confine ourselves to the $\pi\rho$ system. With the help of the Weinberg formula we eliminate the $\pi\rho$ scattering matrix from the sum rule derived in Sec. II, and we get an expression for the mean square pion radius, in terms of the physical contributions to our sum rules. In the rest of that section we try to make some approximations in order to give an estimate for the numerical value of $\langle r_\pi^2\rangle$. Under the simplest assumptions we find, e.g., from field algebra $\langle r_\pi^2\rangle^{1/2}=0.77$ fm, which is compatible with experiment.

In Appendix B we work out some consequences for the three-point functions from the rest-frame saturation of current algebra.

II. SUM RULES

In this section we derive sum rules for the vector-meson–pion and vector-meson–nucleon scattering matrices at threshold. They follow from a local $SU(2)$ low-energy theorem for a certain weak quantity and from a space-space equal-time commutator of vector currents. This commutator will appear through use of the Bjorken[7] limit for the asymptotic behavior of the weak quantity in question. We give a dispersion formulation of the sum rule; the path along which we shall integrate is a parabola in a plane of two variables essentially connected to the energy and mass of the vector mesons. Let us first confine our attention to the pion–vector-meson sum rule.

In order to do this, we want to derive briefly the low-energy theorem we need. The current-algebra content of this low-energy theorem is equivalent to that of the Cabibbo-Radicati sum rule (for pion targets).[8] We shall pay attention to the limiting procedures we shall encounter.

[7] J. D. Bjorken, Phys. Rev. **148**, 1467 (1966).

[8] N. Cabibbo and L. Radicati, Phys. Letters **19**, 697 (1966); G. Furlan, in *Elementary Particle Physics and Scattering Theory*, edited by M. Chrétien and S. S. Schweber (Gordon and Breach, New York, 1970).

Kinematics

Let us consider the quantity[9]

$$T_{\mu\nu}{}^{\alpha\beta}=i\int dx\, e^{iq_1x}\langle p_1,\sigma|\,T(V_\mu{}^\alpha(x)V_\nu{}^\beta(0))\,|p_2,\tau\rangle\,,\quad(2.1)$$

where $|p,\sigma\rangle$ are the meson states at rest with momentum p and Cartesian isospin component σ. $V_\mu{}^\alpha$ is the vector current with isospin component α. The absorptive part is

$$t_{\mu\nu}{}^{\alpha\beta}=\tfrac{1}{2}\int dx\, e^{iq_1x}\langle p_1,\sigma|\,[V_\mu{}^\alpha(x),V_\nu{}^\beta(0)]\,|p_2,\tau\rangle\,.\quad(2.2)$$

Let us introduce the standard notation

$$\begin{gathered}P=\tfrac{1}{2}(p_1+p_2)\,,\quad Q=\tfrac{1}{2}(q_1+q_2)\,,\quad \Delta=p_2-p_1=q_1-q_2\,,\\ s=(P+Q)^2,\quad \bar{s}=(P-Q)^2,\quad t=\Delta^2,\\ \nu=P\cdot Q\,,\quad q_1{}^2=u_1\,,\quad q_2{}^2=u_2\,,\quad w=Q\cdot\Delta\,,\\ P^2=\mu^2-\tfrac{1}{4}t=\sigma^2\,,\quad Q^2=\tfrac{1}{2}(u_1+u_2)-t\,.\end{gathered}\quad(2.3)$$

We shall choose $q_1{}^2=q_2{}^2=u$. The relation between u and ν is a parabola whose parameters depend on the frame of reference we are working in. In the Breit frame ($\mathbf{P}=0$), it becomes

$$u=\nu^2/\sigma^2-\mathbf{Q}^2\,,\quad w=-\mathbf{Q}\cdot\boldsymbol{\Delta}\,,\quad(2.4)$$

and in the rest frame ($\Delta=0$, $\mathbf{P}=0$),

$$u=\nu^2/\mu^2\,,\quad w=0\,.\quad(2.5)$$

One can introduce a covariant parametrization for this parabola with the help of the following definitions:

$$\begin{aligned}q_{1\mu}'&=q_{1\mu}-\frac{(P\cdot q_1)}{\sigma^2}P_\mu\,,\quad q_i'P=0\,,\\ q_{2\mu}'&=q_{2\mu}-\frac{(P\cdot q_2)}{\sigma^2}P_\mu\,,\qquad x=\frac{\nu}{\sigma^2}\,.\end{aligned}\quad(2.6)$$

Then the equations for the parabolas (2.4) and (2.5) become

$$\nu=x\sigma^2\,,\quad u=x^2\sigma^2+q'^2\,,\quad w=Q'\Delta\,,\quad(2.4')$$

where $q'^2=q_1'^2=q_2'^2$, and

$$\nu=\mu^2x\,,\quad u=\mu^2x^2\,.\quad(2.5')$$

Afterwards we shall use this parabola for the extrapolation program, as suggested in Ref. 1. Among the advantages of this path are the strong selection rules. Moreover, we shall be interested in combining our sum rule with static $SU(4)$ algebra and, as is well known, the most remarkable results of this model are derived just in the rest frame.[10]

[9] Rigorously, one should take noncovariant contributions to $T_{\mu\nu}{}^{\alpha\beta}$ into account. It can be shown, however, that such noncovariant terms drop out in the Ward identity (2.11). Therefore, we shall consider in the following the covariant part of (2.1) only.

[10] B. W. Lee, Phys. Rev. Letters **14**, 676 (1965); R. F. Dashen and M. Gell-Mann, Phys. Letters **17**, 142 (1965).

Low-Energy Theorem

Let us now in this kinematical framework proceed with the derivation of the low-energy theorem. The isospin decomposition of $T_{\mu\nu}{}^{\alpha\beta}$ is

$$T_{\mu\nu}{}^{\alpha\beta}=T_{\mu\nu}{}^{(1)}\delta_{\alpha\sigma}\delta_{\beta\tau}+T_{\mu\nu}{}^{(2)}\delta_{\alpha\beta}\delta_{\sigma\tau}+T_{\mu\nu}{}^{(3)}\delta_{\alpha\tau}\delta_{\beta\sigma}. \quad (2.7)$$

From now on we shall be interested only in the antisymmetric amplitude defined by

$$T_{\mu\nu}=T_{\mu\nu}{}^{(3)}-T_{\mu\nu}{}^{(1)}. \quad (2.8)$$

The Lorentz decomposition in the basis (P,q_1',q_2') is

$$\begin{aligned}T_{\mu\nu}=AP_\mu P_\nu+B_1P_\mu q_{1\nu}'+B_2P_\mu q_{2\nu}'+B_3q_{1\mu}'P_\nu\\+B_4q_{2\mu}'P_\nu+C_1q_{1\mu}'q_{1\nu}'+C_2q_{1\mu}'q_{2\nu}'\\+C_3q_{2\mu}'q_{1\nu}'+C_4q_{2\mu}'q_{2\nu}'+Dg_{\mu\nu}. \quad (2.9)\end{aligned}$$

We assume a similar decomposition for $t_{\mu\nu}$ in a, b_i, c_i, and d. Note that the invariant amplitudes are functions of covariant variables

$$x,\ q'^2,\ \sigma^2. \quad (2.10)$$

From local $SU(2)$ current algebra follows the Ward identity

$$q_{1\mu}T_{\mu\nu}=4F_\pi(t)P_\nu, \quad (2.11)$$

where F_π is the pion electromagnetic form factor (Appendix A). From (2.11) and the crossing properties, the following Ward identities can be obtained for the invariant amplitudes:

$$\begin{aligned}x\sigma^2A+q'^2B_2+(q'^2+2\sigma^2-2\mu^2)B_1+xD\\=4F_\pi(4\mu^2-4\sigma^2), \quad (2.12a)\end{aligned}$$

$$x\sigma^2B_1+q'^2C_1+(q'^2+2\sigma^2-2\mu^2)C_3+D=0, \quad (2.12b)$$

$$x\sigma^2B_2+q'^2C_2+(q'^2+2\sigma^2-2\mu^2)C_4=0. \quad (2.12c)$$

If we vary x, keeping q'^2 and σ^2 fixed, we change the "mass" accompanying the vector currents. We shall be interested in the soft-"mass" theorem, which means $q'^2\to0$, $x\to0$. However, the functions A, B_i, C_i, and D can be singular there, owing to the presence of the one-pion intermediate state. Therefore, we must first separate these pion contributions from the invariant amplitudes. They can be calculated in many different ways, but in fact one only wants to subtract the part of A, B_i, C_i, and D singular at $x=0$, $q'^2=0$, and this procedure need not be unique. The only requirement is that the difference of the full A, B_i, C_i, and D and their pion contributions calculated in some way is regular at $x=q'^2=0$. Here we follow one possible line. It is easily checked that other ways lead to the same final low-energy theorem. We separate

$$A=A^\pi+\tilde{A}, \quad (2.13)$$

where

$$A^\pi=\frac{1}{\pi}\int\frac{dx''a^\pi(x'',q'^2,\sigma^2)}{x''-x}, \quad (2.14)$$

and similarly for the other invariant amplitudes. (a^π is the one-pion contribution to a.) For instance, one finds

$$A^\pi(x,q'^2,\sigma^2)=\frac{F_\pi^2(u_\pi)(2+x_\pi)^2}{\sigma^2\epsilon}\frac{x}{x^2-x_\pi^2}, \quad (2.15)$$

where

$$x_\pi=-1+\epsilon,\quad \epsilon=\sigma^{-1}(2\mu^2-\sigma^2-q'^2)^{1/2},\quad u_\pi=x_\pi^2\sigma^2+q'^2.$$

After introducing these contributions into (2.12), we get

$$\begin{aligned}[(2+x_\pi)^2/\epsilon]F_\pi^2(u_\pi)+x\sigma^2\tilde{A}+q'^2\tilde{B}_2+(q'^2+2\sigma^2-2\mu^2)\tilde{B}_1\\+xD=4F_\pi[4\mu^2-4\sigma^2], \quad (2.16a)\end{aligned}$$

$$x\sigma^2\tilde{B}_1+q'^2\tilde{C}_1+(q'^2+2\sigma^2-2\mu^2)\tilde{C}_3+D=0, \quad (2.16b)$$

$$x\sigma^2\tilde{B}_2+q'^2\tilde{C}_2+(q'^2+2\sigma^2-2\mu^2)\tilde{C}_4=0. \quad (2.16c)$$

Since $D^\pi=0$, we used $D=\tilde{D}$.

The expression (2.15) for A_π shows that it has a singularity at $x=x_\pi$. In the collinear configuration, $q'^2=0$ ($\sigma^2=\mu^2$), $x_\pi=0$, and $\epsilon=1$, so the singularity of A_π in this case is pushed to the origin. However, after substitution in the Ward identities, these singularities cancel and we get equations with regular coefficients for regular functions. Taking the derivative of (2.16) with respect to q'^2 and putting $x=q'^2=0$, we get

$$\tilde{B}_1(0,0,\mu^2)+\tilde{B}_2(0,0,\mu^2)=-8F_\pi(0)F_\pi'(0). \quad (2.17)$$

Differentiation of (2.16b) and (2.16c) with respect to x gives

$$\left.\frac{\partial D}{\partial x}\right|_{x=q'^2=0}=-\mu^2\tilde{B}_1(0,0,\mu^2), \quad (2.18)$$

$$\tilde{B}_2(0,0,\mu^2)=0. \quad (2.19)$$

Combining these equations, the low-energy theorem finally reads

$$\left.\frac{\partial D}{\partial x}\right|_{x=q'^2=0}=8\mu^2F_\pi'(0). \quad (2.20)$$

Extrapolation

Let us proceed to the extrapolation. We define

$$F(x,q'^2,\sigma^2)=[1-(\mu^2/m_\rho^2)x^2]^2D(x,q'^2,\sigma^2). \quad (2.21)$$

We prefer working with F rather than with D itself, because we thus avoid complications arising from the ρ pole in D. $F(x,q'^2,\sigma^2)$ does not have this pole. In a way completely analogous to that of Ref. 1 it is seen that

$$F(m_\rho/\mu,0,\mu^2)=\lim_{x\to m_\rho/\mu}\left(1-\frac{\mu^2}{m_\rho^2}x^2\right)^2D=\frac{f_\rho^2}{m_\rho^4}D^{\rho\pi},$$
$$x=m_\rho/\mu. \quad (2.22)$$

$D^{\rho\pi}$ is the analog of D, with the vector currents, however, replaced by ρ-meson sources. We also know from (2.20)

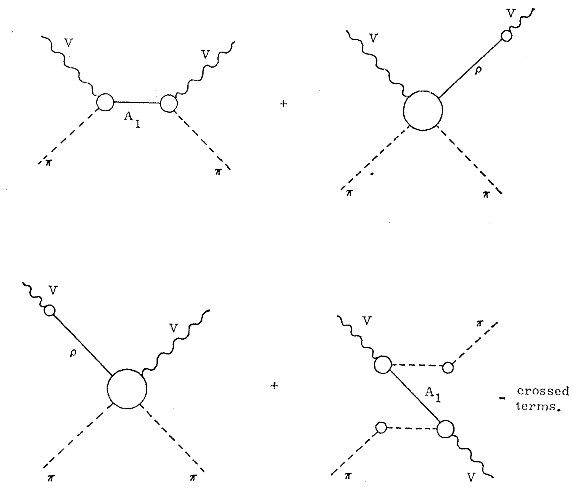

FIG. 1. Lowest-mass contributions to the sum rule (2.34).

that

$$\left.\frac{\partial F}{\partial x}\right|_{q'^2=x=0}=8\mu^2F_\pi'(0)\,. \tag{2.23}$$

In order to estimate the asymptotic behavior of F, we assume the validity of Bjorken's argument

$$T_{\mu\nu}{}^{\alpha\beta}\underset{x\to\infty}{\sim}-\frac{1}{\mu x}\int dy\, e^{iq\cdot y}\delta(y_0)$$
$$\times\langle p,\sigma|[V_\mu{}^\alpha(y),V_\nu{}^\beta(0)]|p,\tau\rangle\,. \tag{2.24}$$

From this we obtain in the rest frame

$$D^{\alpha\beta}g_{ij}\sim-(\mu x)^{-1}\langle p,\sigma|[V_i{}^\alpha,V_j{}^\beta]|p,\tau\rangle\,, \tag{2.25}$$

which implies, if we use field algebra or the quark model for the equal-time commutator,

$$D\sim-4c/x,\quad\begin{cases}c=1\text{ in quark model}\\c=0\text{ in field algebra}\end{cases} \tag{2.26}$$

or

$$F\sim-4cx^3. \tag{2.26$'$}$$

Using crossing $[F(x)=-F(-x)]$, we have

$$G(x)\equiv F(x)+4cx^3\mu^4/m_\rho{}^4\sim x\,. \tag{2.27}$$

This enables us to write a twice-subtracted dispersion relation for $G(x)$; we take $q'^2=0$, and as subtraction points we choose $x=\pm m_\rho/\mu$. For $G'(0)$ we then obtain the representation

$$G'(0)=\frac{\mu}{m_\rho}G\left(\frac{m_\rho}{\mu}\right)-\frac{1}{\pi}\left(\frac{m_\rho}{\mu}\right)^2\int\frac{dx\,\mathrm{Im}G(x)}{x^2[x^2-(m_\rho/\mu)^2]}\,. \tag{2.28}$$

Using (2.22) and (2.23), we get the sum rule

$$\frac{m_\rho{}^2}{m_\rho{}^3}D^{\rho\pi}(\mathrm{thr})=-8m_\rho{}^2\mu^2F_\pi'(0)+4c\mu^2$$
$$-\frac{2}{\pi}\int_{\substack{0\\x\neq m_\rho/\mu}}^\infty\frac{(\mu^2x^2-m_\rho{}^2)d(x)dx}{x^2}\,. \tag{2.29}$$

Comparing with (2.9), one sees that

$$d^{\alpha\beta}g_{ij}=t_{ij}{}^{\alpha\beta}\,. \tag{2.30}$$

Using completeness,

$$t_{ij}{}^{\alpha\beta}=\tfrac{1}{2}(2\pi)^4\sum_n\delta(p+q-p_n)$$
$$\times\langle p,\sigma|V_i{}^\alpha|p_n,\gamma\rangle\langle p_n,\gamma|V_j{}^\beta|p,\pi\rangle-\text{c.t.}\,, \tag{2.31}$$

where by c.t. we have denoted the crossed terms. The pion states are at rest now, because we took $q'^2=0$.

The integration in (2.25) in the variable x is equivalent to integration along the parabola (2.5′) in the νu plane. For such a path, it has been suggested that the contributions from semidisconnected graphs can be important. In fact, following Fubini and Furlan, we can decompose

$$t_{ij}{}^{\alpha\beta}=t_{ij}{}^{\alpha\beta}{}_{,\mathrm{I}}+t_{ij}{}^{\alpha\beta}{}_{,\mathrm{II}}+t_{ij}{}^{\alpha\beta}{}_{,\mathrm{II'}}+t_{ij}{}^{\alpha\beta}{}_{,\mathrm{III}}+t_{ij}{}^{\alpha\beta}{}_{,\mathrm{IV}}\,, \tag{2.32}$$

where

$$\begin{aligned}
t_{ij}{}^{\alpha\beta}{}_{,\mathrm{I}}&=\tfrac{1}{2}(2\pi)^4\sum_{n\gamma}\delta(p+q-p_n)\langle p,\sigma|V_i{}^\alpha|p_n,\gamma\rangle_{\mathrm{con}}\\
&\qquad\times\langle p_n,\gamma|V_j{}^\beta|p,\tau\rangle_{\mathrm{con}}-\text{c.t.}\,,\\
t_{ij}{}^{\alpha\beta}{}_{,\mathrm{II}}&=\tfrac{1}{2}(2\pi)^4\sum_{n\gamma}\delta(p_n-q)\langle 0|V_i{}^\alpha|p_n,\gamma\rangle\\
&\qquad\times\langle p_n,\gamma;\,p,\sigma|V_j{}^\beta|p,\tau\rangle_{\mathrm{con}}-\text{c.t.}\,,\\
t_{ij}{}^{\alpha\beta}{}_{,\mathrm{II'}}&=\tfrac{1}{2}(2\pi)^4\sum_{n\gamma}\delta(p_n-q)\langle p,\sigma|V_i{}^\alpha|p,\tau;\,p_n,\gamma\rangle_{\mathrm{con}}\\
&\qquad\times\langle p_n,\gamma|V_j{}^\beta|0\rangle-\text{c.t.}\,,\\
t_{ij}{}^{\alpha\beta}{}_{,\mathrm{III}}&=\tfrac{1}{2}(2\pi)^4\sum_{n\gamma}\delta(p-q-p_n)\langle 0|V_i{}^\alpha|p,\tau;\,p_n,\gamma\rangle\\
&\qquad\times\langle p_n,\gamma;\,p,\sigma|V_j{}^\beta|0\rangle-\text{c.t.}\,,\\
t_{ij}{}^{\alpha\beta}{}_{,\mathrm{IV}}&=\tfrac{1}{2}(2\pi)^4\sum_{n\gamma}\delta(p_n-q)\langle 0|V_i{}^\alpha|p_n,\gamma\rangle\\
&\qquad\times\langle p_n,\gamma|V_j{}^\beta|0\rangle-\text{c.t.}
\end{aligned} \tag{2.33}$$

Here $\langle p,\sigma|V_i{}^\alpha|p,\tau\rangle_{\mathrm{con}}$ denotes the connected part of this matrix element. We see that because of rest-frame selection rules we can have 1^+ contributions to $t_{ij}{}^{\alpha\beta}{}_{,\mathrm{I}}$ and $t_{ij}{}^{\alpha\beta}{}_{,\mathrm{III}}$ and 1^- to $t_{ij}{}^{\alpha\beta}{}_{,\mathrm{II}}$. The fully disconnected contributions can be subtracted as usual, and from now on we shall omit them from the discussions.

Writing explicitly, the A_1 contribution to $t_{ij}{}^{\alpha\beta}{}_{,\mathrm{I}}$ and

$t_{ij}{}^{\alpha\beta}{}_{,\mathrm{III}}$, sum rule (2.25) finally becomes (see Fig. 1)

$$-(f_\rho{}^2\mu/m_\rho{}^3)D^{\rho\pi}(\mathrm{thr})=\frac{\mu}{m_A}\{[m_\rho{}^2/(m_A-\mu)^2-1]\times G^2[(m_A-\mu)^2]-[m_\rho{}^2/(m_A+\mu)^2-1]\times G^2[(m_A+\mu)^2]\}-8m_\rho{}^2\mu^2F_\pi{}'(0)+4c\mu^2-\frac{2}{\pi}\int'\frac{d^S(x)dx}{x^2(\mu^2x^2-m_\rho{}^2)},\quad(2.34)$$

where $d^S(x)$ is obtained from $d(x)$ by making the substitution

$$V_i{}^\alpha(x)\to(\Box+m_\rho{}^2)V_i{}^\alpha(x).\quad(2.35)$$

The form factors G of the matrix element $\langle\pi|V|A_1\rangle$ are defined in Appendix A.

Let us stress that we now have an on-mass-shell expression for the $\rho\pi$ scattering matrix, despite the fact that the low-energy theorem, which we have extrapolated, was valid for a weak quantity and not for the off-mass-shell $\rho\pi$ scattering matrix. Of course, the low-energy theorem (2.20) can be converted with the help of vector-meson dominance (or the field-algebra model) into a low-energy theorem for $\rho\pi$ scattering:

$$\left.\frac{\partial D^{\rho\pi}}{\partial x}\right|_{x=0}=\frac{8\mu^2m_\rho{}^4}{f_\rho{}^2}F_\pi{}'(0).\quad(2.36)$$

Here the assumption of the field-algebra model is crucial. Relation (2.34), however, is more general and is valid for any model in which the space-space commutator exists, e.g., the quark model. Moreover, the latter relation is on mass shell. It is interesting to compare our results with the results of a linear extrapolation. Using (2.36), we have

$$D^{\rho\pi}\left(\frac{m_\rho}{\mu}\right)=D^{\rho\pi}(0)+\left.\frac{\partial D^{\rho\pi}}{\partial x}\right|_{x=0}\times\frac{m_\rho}{\mu}=\frac{8\mu m_\rho{}^5}{f_\rho{}^2}F_\pi{}'(0).\quad(2.37)$$

Specializing (2.29) for field algebra, we recover (2.37) plus an explicit expression for the corrections, among which is an important A_1 contribution.

It is interesting to remember what the exact current-algebra input for our sum rule is. We wrote a dispersion relation for F, connecting the values of F at three different points. One was determined by $SU(4)$ or field algebra, another by local $SU(2)$.

Saturation

It is a general feature in current algebra that despite the different mathematical assumptions involved, the covariant dispersion approach is equivalent to the non-covariant saturation. We want to show that this is also true in this case. First of all, let us assume the equal-time commutator of electric dipoles, which follows from local $SU(2)$,

$$[D_i{}^\alpha,D_j{}^\beta]=i\epsilon^{\alpha\beta\gamma}Q_{ij}{}^\gamma,\quad(2.38)$$

where

$$D_i{}^\alpha=\int d^3x\,x_iV_0{}^\alpha(\mathbf{x},0),\qquad Q_{ij}{}^\gamma=\int d^3x\,x_ix_jV_0{}^\gamma(\mathbf{x},0).\quad(2.39)$$

Sandwiching (2.38) between pion states at rest and inserting a complete set of intermediate states, we obtain[11]

$$\frac{1}{\pi}\int\frac{dx\,d(x)}{\mu x^2}=8\mu F_\pi{}'(0).\quad(2.40)$$

We used the usual substitution[12]

$$\langle p\cdot\sigma|D_i{}^\alpha|p_n,\gamma\rangle=(2\pi)^3i\frac{\delta(\mathbf{p}-\mathbf{p}_n)}{E_n-E}\langle p,\sigma|V_i{}^\alpha|p_n,\gamma\rangle\quad\text{for}\quad E_n\neq E,\quad(2.41)$$

and introduced the integration over x as in Ref. 13.

One of the features of sum rule (2.10) is that, because we are working in the forward direction, the ρ contributions from classes II and II′ (2.33) to d are both singular. Fortunately, the sum of $d^\rho{}_{\mathrm{II}}$ and $d^\rho{}_{\mathrm{II}'}$ is not singular, but because of this interplay the function $D(x,0)$ has a double pole at $x=m_\rho/\mu$. Consequently, $d^\rho{}_{\mathrm{II}}+d^\rho{}_{\mathrm{II}'}$ will be proportional to $\delta'(x-m_\rho/\mu)$ and on the left-hand side of (2.40) will appear (besides the $\rho\pi$ scattering matrix) the derivative $\partial D^{\rho\pi}/\partial x$ at threshold. Such a derivative would require very careful handling, whereas knowledge of it is of little use. It is clearly desirable to get expressions which do not involve this derivative. This we can achieve by using the algebra of space components of vector charges

$$[V_i{}^\alpha,V_j{}^\beta]=c(i\epsilon_{\alpha\beta\gamma}\delta_{ij}V_0{}^\gamma+i\delta_{\alpha\beta}\epsilon_{ijk}A_k{}^0).\quad(2.42)$$

[11] One has to be careful, because the matrix element of the operator $Q_{ij}{}^\gamma$ depends essentially on the normalization. For general normalization, $\langle p|p'\rangle=(2\pi)^3N(E)N(E')\delta^3(p-p')$, one obtains for the right-hand side of (2.40)

$$4F_\pi{}'(0)N(\mu)N(\mu)-\frac{2F_\pi(0)}{\mu}N(\mu)\left.\frac{\partial N(E')}{\partial E'}\right|_{E'=\mu}.$$

The second term of this expression appears also as a pion contribution to the dispersion integral, so that the sum rule does not depend on the normalization. For $N(E)$ constant, the normalization-dependent terms do not appear, as one would expect. Particularly the one-pion contribution to the left-hand side vanishes, as suggested by the selection rules.

[12] L. Maiani and G. Preparata, Fortschr. Physik **17**, 537 (1969).

[13] G. Furlan and C. Rossetti, Lectures given at the International Seminar on Elementary Particle Theory, Varna, Bulgaria, 1968 (unpublished); G. Furlan, in *Acta Physics Austrica*, edited by Paul Urban (Springer Verlag, Vienna, 1969), Suppl. VI.

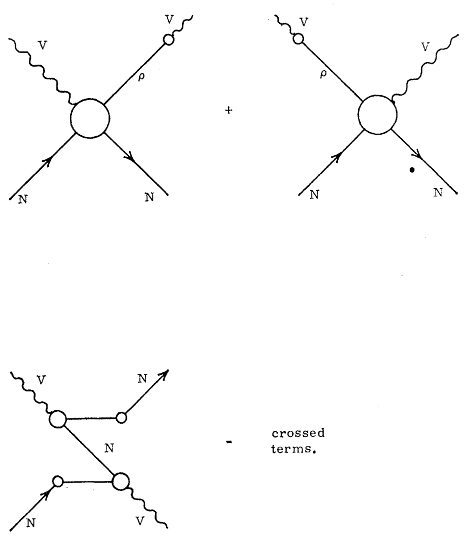

FIG. 2. Lowest-mass contributions to the sum rule (2.47).

In the usual way, one obtains

$$\frac{1}{\pi}\int dx\, d(x) = 4c. \tag{2.43}$$

Combining (2.40) and (2.43), we find

$$\frac{1}{\pi}\int \frac{dx}{\mu x^2} d(x)(m_\rho^2 - \mu^2 x^2) = 8\mu m_\rho^2 F_\pi'(0) - 4\mu c\,, \tag{2.44}$$

which, after evaluating the ρ contribution to d_{II}, is equivalent to (2.29). Note that, unlike (2.40) and (2.43), the final sum rule does not have the $\partial D^{\rho\pi}/\partial x$ term.

In an analogous way, a sum rule can be derived for vector-meson–nucleon scattering. We define again $T_{\mu\nu}{}^{\alpha\beta}$ as in (2.1), where $|p,\sigma\rangle$ now stands for a nucleon state. The isospin decomposition is

$$T_{\mu\nu}{}^{\alpha\beta} = i\epsilon_{\alpha\beta\gamma}\tau^\gamma T_{\mu\nu}{}^- + \delta_{\alpha\beta} T_{\mu\nu}{}^+. \tag{2.45}$$

In the rest frame,

$$T_{ij}{}^- = g_{ij}\bar{D}. \tag{2.46}$$

The bar denotes that this D has the nucleon external states. For $D^{\rho N}$, we then obtain

$$\begin{aligned} -(f_\rho^2/m_\rho^3)D^{\rho N} &= -\tfrac{1}{2}c + \tfrac{1}{2}m_\rho^2[2F_1^{V\prime}(0) - F_2^V(0)] \\ &+ (m_\rho^2/8m^2)[F_1^V(0) + 2mF_2^V(0)]^2 \\ &+ (1 - m_\rho^2/4m^2)\tfrac{1}{2}[F_1^V(4m^2) + 2mF_2^V(4m^2)]^2 \\ &- \tfrac{2}{3}[1 - m_\rho^2/(m - m_{**})^2] g_{**}^2 \\ &- \frac{2}{\pi m}\int' \frac{\bar{d}^S(x)dx}{x^2(m^2x^2 - m_\rho^2)}. \end{aligned} \tag{2.47}$$

The form factors are defined in Appendix A. The spectral function $\bar{d}^S(x)$ has again a decomposition into classes as in (2.33). $\bar{d}_{\text{I}}$ has contributions from $\frac{1}{2}$ and $\frac{3}{2}$ states (spin and isospin) with negative parity and baryon number 1; $\bar{d}_{\text{II}}$, from meson states with spin-parity 1^- and isospin 1; $\bar{d}_{\text{III}}$, from $\frac{1}{2}$ and $\frac{3}{2}$ states with negative parity and baryon number -1. The lowest-mass contributions[14] (see Fig. 2) are the $N^{**}(1518)$ and the antinucleon, in classes I and III, respectively, which have been written explicitly in (2.47).

In the saturation approach, the term $-\frac{1}{2}c$ and the terms involving the nucleon form factors for vanishing momentum transfer come from the right-hand side of the equal-time commutators. In the dispersion approach, $-\frac{1}{2}c$ appears because of the application of the Bjorken limit, the others from the low-energy theorem for $\partial\bar{D}/\partial x$.

It is amusing to note that the terms

$$(m_\rho^2/8m^2)[F_1^V(0) + 2mF_2^V(0)]^2 + (1 - m_\rho^2/4m^2)\tfrac{1}{2}[F_1^V(4m^2) + 2mF_2^V(4m^2)]^2,$$

which come, respectively, from the nucleon Born contribution to $\partial\bar{D}/\partial x|_{x=0}$ and from the class-III antinucleon contribution, add up, after using ρ dominance for $F_{1,2}$, to give the full nucleon Born term at threshold. This means that the remaining terms on the right-hand side of (2.47) are the difference between the current-algebra prediction and the lowest-order Feynman graph for $D^{\rho N}(\text{thr})$.

III. TENSOR CURRENTS

In general, it is very interesting to have sum rules from different algebras involving related spectral functions. As we pointed out in the Introduction, a necessary condition for achieving this is to consider algebras with the same quantum numbers. In our case one is then naturally led to a study of the tensor currents—the divergences of which indeed have the quantum numbers of the electric dipoles—and the space components of the vector currents.

The quark model suggests for the tensor charges the following equal-time commutation relations:

$$[\mathcal{T}_{0i}{}^\alpha, \mathcal{T}_{0j}{}^\beta] = -i\epsilon_{\alpha\beta\gamma}\delta_{ij}V_0{}^\gamma - i\epsilon_{ijk}\delta_{\alpha\beta}A_k{}^0, \tag{3.1}$$

where, in the quark model,

$$\mathcal{T}_{\mu\nu}{}^\alpha = \int d^3x\, \bar{\psi}(x)\gamma_\mu\gamma_\nu\tfrac{1}{2}\tau^\alpha\psi(x) = \int d^3x\, \mathcal{T}_{\mu\nu}{}^\alpha(x). \tag{3.2}$$

Saturating between pions at rest, one obtains the sum rule

$$\frac{1}{\pi}\int \frac{d^\tau(x)dx}{\mu x^2} = -4\mu\,, \tag{3.3}$$

[14] D. P. Sidhu and M. Dresden, Phys. Rev. Letters **23**, 447 (1969). In this reference an analogous sum rule has been derived. The class-II and class-III type contributions have not been considered.

where

$$d^\tau{}_{ij}{}^{\alpha\beta}=\tfrac{1}{2}(2\pi)^4\sum_{n,\gamma}\delta(p+q-p_n)\langle p,\sigma|\mathfrak{D}_i{}^\alpha(0)|n,\gamma\rangle \times\langle n,\gamma|\mathfrak{D}_j{}^\beta(0)|p,\tau\rangle-\text{c.t.}, \quad (3.4)$$

$$\mathfrak{D}_i{}^\alpha=\partial_\mu\mathcal{T}_{\mu i}{}^\alpha.$$

The function d^τ is defined in terms of $d^\tau{}_{ij}{}^{\alpha\beta}$ with the help of definitions analogous to (2.7)–(2.9).

To have a physical interpretation for the matrix elements of the tensor current, we shall assume one formulation[15] of the frequently used PCTC hypothesis[2]:

$$\mathfrak{D}_\mu{}^\alpha(x)=i\Gamma V_\mu{}^\alpha(x). \quad (3.5)$$

If ρ dominance is to be assumed for the matrix elements of both $\mathfrak{D}_\mu{}^\alpha$ and $V_\mu{}^\alpha$,

$$\begin{aligned}\langle a|\mathfrak{D}_\mu{}^\alpha(0)|b\rangle&=f_\tau g_{ab\rho}/(m_\rho{}^2-t), \quad t\approx m_\rho\\ \langle a|V_\mu{}^\alpha(0)|b\rangle&=f_\rho g_{ab\rho}/(m_\rho{}^2-t), \quad t\approx m_\rho\end{aligned} \quad (3.6)$$

then one would have

$$\Gamma=f_\tau/f_\rho. \quad (3.7)$$

Using PCTC, our sum rule (3.3) becomes

$$\frac{1}{\pi}\int\frac{d(x)dx}{\mu x^2}=4\mu\Gamma^{-2}. \quad (3.8)$$

Note that now the same spectral function d appears as in Sec. II. Indeed, comparing (2.40) and (3.8), one has the exact relationship (i.e., no saturation hypothesis is made)

$$\tfrac{1}{2}\Gamma^{-2}=F_\pi{}'(0). \quad (3.9)$$

In obtaining (3.9), we in fact compared two sum rules involving poorly defined quantities which are present because of the double ρ pole (see Sec. II). If one wants to be careful, (3.9) can be combined with the $SU(4)$ sum rule (2.43), which gives

$$\frac{1}{\pi}\int\frac{dx}{\mu x^2}d(x)(m_\rho{}^2-\mu^2x^2)=4\mu m_\rho{}^2\Gamma^{-2}. \quad (3.10)$$

Eliminating the left-hand sides of (3.10) and (2.44), which are well defined, the relation (3.9) is confirmed.

Sandwiching the tensor algebra between nucleon states, one can derive in a similar way

$$\frac{1}{\pi}\int\frac{dx}{mx^2}\bar{d}(x)(m_\rho{}^2-m^2x^2)=-\tfrac{1}{2}m_\rho{}^2\Gamma^{-2}+\tfrac{1}{2}c. \quad (3.11)$$

Here the well-known ambiguity in the evaluation of the matrix elements of tensor currents between nucleons is encountered. In this case some care must be taken in the evaluation of the nucleon contribution to class I. As is well known, from the PCTC assumption would follow a singularity for matrix elements of tensor currents at vanishing momentum transfer. However, this can be avoided by assuming different masses for external and internal nucleons. As pointed out in Ref. 15, one has the freedom to make such a choice because it is part of the definition of the tensor currents. If we take this mass difference into account we find, because of the selection rules, a vanishing nucleon contribution. For tensor currents between pion states this problem was not present, because d^π was identically equal to zero. In the case of dipole currents one has no ambiguity for either pions or nucleons. In fact, these contributions can uniquely be shown to be vanishing.

Combination of (3.11) with (2.47) gives rise to one more saturation-independent relationship:

$$\tfrac{1}{2}\Gamma^{-2}=F_1{}^{V\prime}(0)+\frac{F_1{}^V(0)}{8m^2}+\frac{F_2{}^V(0)}{2m}. \quad (3.12)$$

This relation was obtained by Ademollo, Gatto, Longhi, and Veneziano.[15] As we have done in this paper, these authors combined the algebra of tensor currents with the local $SU(2)$. However, their derivation was given in the infinite-momentum frame. It is not surprising that (3.12) and the relation of Ref. 15 coincide. Since no saturation approximation was made, the frame dependence does not influence the final result. Meanwhile it is interesting to note that, while in the infinite-momentum frame, the terms in (3.12) have their origin partly in the nucleon terms and partly in the right-hand side of the equal-time commutation relations, in the rest frame they all come from the latter part.

Of course, Γ can be determined numerically by using the known values of the measured quantities on the right-hand sides of (3.9) and (3.12), but it seems more interesting to eliminate Γ from these relations. We then find

$$\tfrac{1}{6}\langle r_\pi{}^2\rangle=\tfrac{1}{6}\langle r_N{}^2\rangle_V+\frac{1}{8m^2}+\frac{\mu_p{}^A-\mu_n{}^A}{4m^2}, \quad (3.13)$$

where we introduced the mean-square pion and nucleon radii and the anomalous magnetic moments of proton and neutron. Taking for $\frac{1}{6}\langle r_N{}^2\rangle_V$ and $(\mu_p{}^A-\mu_n{}^A)$ the known values $1.36/m_\rho{}^2$ and 3.7,[16] relation (3.13) predicts

$$\langle r_\pi{}^2\rangle^{1/2}=0.9\text{ fm}. \quad (3.14)$$

This must be compared with the experimental data $\langle r_\pi{}^2\rangle^{1/2}=0.80\pm0.10$,[17] 0.86 ± 0.14,[18] and ≤ 0.9 fm.[19]

[15] M. Ademollo, R. Gatto, G. Longhi, and G. Veneziano, Phys. Rev. **153**, 1623 (1967).

[16] J. R. Dunning, Jr., K. W. Chew, A. A. Cone, G. Hartwig, N. F. Ramsey, J. K. Walker, and R. Wilson, Phys. Rev. **141**, 1286 (1966).

[17] C. W. Akerlof, W. W. Ash, K. Berkelman, A. C. Lichtenstein, A. Romanaskas, and R. H. Siemon, Phys. Rev. **163**, 1482 (1967).

[18] C. Mistretta, D. Imrie, J. A. Apel, R. Budnitz, L. Carrol, M. Goitein, K. Hanson, and R. Wilson, Phys. Rev. Letters **20**, 1523 (1968).

[19] M. M. Block, I. Kenyon, J. Keren, D. Koetke, P. Malhotra, R. Walker, and H. Winzeler, Phys. Rev. **169**, 1074 (1968).

IV. $\pi\rho$ SYSTEM

We should now like to draw some conclusions from the $\rho\pi$ sum rule, as obtained in Sec. II from the assumption of $SU(4)$ and local $SU(2)$. As already pointed out, it is a general feature of rest-frame saturation of current algebra that an important contribution to the sum rule comes from the scattering matrix at threshold for particles bearing the quantum numbers of the currents on the external particles. This brings an immediate advantage when the particles are mesons. Using the fact that currents have the quantum numbers of mesons, one can get additional information by exchanging the role of currents and external particles.[20] In our case the quantity $D^{\rho\pi}$ appearing in (2.34) will also appear if we sandwich the equal-time commutator of axial charges, as given by the $SU(2)\times SU(2)$ algebra, between ρ states at rest. Using for axial charges and divergences of axial-vector currents

$$[Q_5^\alpha,Q_5^\beta]=i\epsilon^{\alpha\beta\gamma}Q^\gamma \tag{4.1}$$

and

$$[D_5^\alpha,D_5^\beta]=0 \tag{4.2}$$

we find, after steps similar to those made in Ref. 1,

$$\frac{f_\pi^2}{m_\rho}D^{\rho\pi}(\text{thr})=4\mu-\frac{1}{\pi}\int\frac{dx\,r(x)}{\mu m_\rho^2x^2}(\mu^2-m_\rho^2x^2)\,, \tag{4.3}$$

where

$$r_{\lambda\lambda'}{}^{\alpha\beta}=\tfrac{1}{2}\int dx\,e^{iqx}\langle\rho(\lambda,\sigma)|[D_5^\alpha(x),D_5^\beta(0)]|\rho(\lambda',\tau)\rangle$$
$$=-\delta_{\lambda\lambda'}(r^{(1)}\delta_{\alpha\sigma}\delta_{\beta\tau}+r^{(2)}\delta_{\alpha\beta}\delta_{\sigma\tau}+r^{(3)}\delta_{\alpha\tau}\delta_{\beta\sigma})\,, \tag{4.4}$$
$$r=r^{(3)}-r^{(1)}\,.$$

Note that neglect of the dispersion integral in (4.3) leads to the Weinberg[21] formula for $\pi\rho$ scattering. It is interesting to look at the nature of the corrections to this formula. One could expect an important contribution from the $A_1(1^+)$ meson in class I and class III. Taking them explicitly into account, we obtain

$$\begin{aligned}(f_\pi^2/m_\rho)D^{\rho\pi}(\text{thr})&=4\mu\\&+(\mu m_Am_\rho)^{-1}\{[1-\mu^2/(m_A-m_\rho)^2]\beta^2[(m_A-m_\rho)^2]\\&-[1-\mu^2/(m_A+m_\rho)^2]\beta^2[(m_A+m_\rho)^2]\}\\&-\frac{1}{\pi}\int\frac{dx\,r^S(x)}{\mu m_\rho^2x^2(\mu^2-m_\rho^2x^2)}\,.\end{aligned} \tag{4.5}$$

The β's are defined in Appendix A. A weak form of partial conservation of axial-vector current (PCAC) would suggest that β is damped by a factor μ^2. This would be in agreement with the validity of the Weinberg formula.

We can now combine (4.5) and (2.34) to eliminate the unknown $D^{\rho\pi}(\text{thr})$:

$$\begin{aligned}m_\rho^2F_\pi'(0)=\tfrac{1}{2}c&+\frac{1}{2m_\rho^2}\left(\frac{f^2}{f_\pi}\right)-\frac{1}{8\mu m_A}\left\{\left[1-\frac{m_\rho^2}{(m_A-\mu)^2}\right]G^2[(m_A-\mu)^2]-\left[1-\frac{m_\rho^2}{(m_A+\mu)^2}\right]G^2[(m_A+\mu)^2]\right\}\\&+\frac{1}{8\mu^2m_Am_\rho^3}\left(\frac{f_\rho}{f_\pi}\right)^2\left\{\left[1-\frac{\mu^2}{(m_A-m_\rho)^2}\right]\beta^2[(m_A-m_\rho)^2]-\left[1-\frac{\mu^2}{(m_A+m_\rho)^2}\right]\beta^2[(m_A+m_\rho)^2]\right\}\\&-\frac{1}{4\pi\mu^2}\int_0^{\infty'}\frac{dx\,d^S(x)}{x^2(\mu^2x^2-m_\rho^2)}-\frac{f_\rho^2}{4\pi\mu^2m_\rho^4f_\pi^2}\int_0^{\infty'}\frac{dx\,r^S(x)}{x^2(\mu^2-m_\rho^2x^2)}\,.\end{aligned} \tag{4.6}$$

This is an exact sum rule for the pion mean-square radius, which is valid in different models for the equal-time commutator $[V_i^\alpha,V_j^\beta]$. We recall the origin of the various terms. $F_\pi'(0)$ came from the local $SU(2)$ low-energy theorem or, equivalently, from the right-hand side of the electric dipoles commutation relation. The first term on the right-hand side of (4.6) arose from application of the Bjorken limit in the extrapolation procedure for the above-mentioned low-energy theorem or, equivalently, from the $[V_i^\alpha,V_j^\beta]$ commutator. The origin of the second term is in the usual low-energy theorem for $\rho\pi$ scattering. The remaining terms are the lowest mass contributions to the class-I and class-III spectral functions. If we neglect the latter and put $c=0$ (field algebra), we find

$$F_\pi'(0)=\frac{1}{2m_\rho^4}\left(\frac{f_\rho}{f_\pi}\right)^2. \tag{4.7}$$

With the help of KSRF,[22]

$$F_\pi'(0)=1/m_\rho^2\,. \tag{4.8}$$

This is the ρ-dominance prediction for $F_\pi'(0)$, as could be expected. The quark model ($c=1$) gives, under the same assumptions,

$$F_\pi'(0)=1.5/m_\rho^2\,. \tag{4.9}$$

[20] F. J. Gilman and H. J. Schnitzer, Phys. Rev. **150**, 1362 (1966).
[21] S. Weinberg, Phys. Rev. Letters **17**, 616 (1966).
[22] K. Kawarabayashi and M. Suzuki, Phys. Rev. Letters **16**, 255 (1966); Riazuddin and Fayyazuddin, Phys. Rev. **147**, 1071 (1966), hereafter referred to as KSRF.

In the rest of this section we shall try to estimate the corrections to these results. First of all, let us neglect the continuum integrals and take the hint coming from the Weinberg formula by neglecting the β contribution in (4.5) and thus in (4.6). Indeed, since we are making ρ-meson mass extrapolation, we can safely neglect second-order terms in the pion mass extrapolation, which is supposed to be smooth. The remaining contribution in (4.6) is then determined by G. In evaluating G, we shall use current-algebra sum rules for three-point functions.

The $SU(2)$ sum rules (see Appendix B) are

$$\tfrac{1}{2}[G((m_A-\mu)^2)+G((m_A+\mu)^2)]=f_A/f_\pi\,, \quad (4.10)$$

$$\frac{1}{2}\left[\frac{G((m_A-\mu)^2)}{m_A-\mu}+\frac{G((m_A+\mu)^2)}{m_A+\mu}\right]=\frac{f_A}{f_\pi m_A}, \quad (4.11)$$

with corrections already neglected.

From these equations it follows that

$$G((m_A-\mu)^2)=\frac{f_A}{f_\pi}\left(1-\frac{\mu}{m_A}\right),$$
$$G((m_A+\mu)^2)=\frac{f_A}{f_\pi}\left(1+\frac{\mu}{m_A}\right). \quad (4.12)$$

We can insert now these values in (4.6). We shall express all quantities in terms of the parameter η, defined by

$$\eta=f_\rho/\sqrt{2}m_\rho f_\pi\,. \quad (4.13)$$

Experimentally, $\eta^2=1.25\pm0.15$.[23] The KSRF relation would give $\eta^2=1$. f_A can be expressed in terms of η using Weinberg's first sum rule.[24]

We finally get

$$m_\rho^2F_\pi'=\eta^2+\tfrac{1}{2}c+(\eta^2-\tfrac{1}{2})\,. \quad (4.14)$$

$\eta^2-\frac{1}{2}$ is the contribution of the terms with G, or

$$m_\rho^2F_\pi'=2\eta^2-\tfrac{1}{2}+\tfrac{1}{2}c\,. \quad (4.15)$$

Numerically, this yields the following.

Field algebra:

$$\langle r_\pi^2\rangle^{1/2}=0.77\text{ fm}\,,\qquad \eta^2=1\,,$$
$$=0.89\pm0.05\text{ fm}\,,\quad \eta^2=1.25\pm0.15\,.$$

However, one can derive $f_A=f_\rho$ from field algebra. This relationship together with Weinberg's first sum rule and the experimental relation $m_A=\sqrt{2}m_\rho$ gives $\eta=1$. So finally the field-algebra result is

$$\langle r_\pi^2\rangle^{1/2}=0.77\text{ fm}\,.$$

Quark model:

$$\langle r_\pi^2\rangle^{1/2}=0.89\text{ fm}\,,\qquad \eta^2=1\,,$$
$$=1.00\pm0.07\text{ fm}\,,\quad \eta^2=1.25\pm0.15\,.$$

[23] S. Okubo, Phys. Rev. **188**, 2300 (1969).
[24] S. Weinberg, Phys. Rev. Letters **18**, 507 (1967).

This has to be compared with the experimental values quoted at the end of Sec. III. We see that all of these values for both models are compatible with the present experimental data. This is not the case, for instance, for the uncorrected field-algebra value (4.8) (the ρ-dominance prediction), which is $\langle r_\pi^2\rangle^{1/2}=0.63$ fm.

V. DISCUSSION

In this paper we studied in some detail the rest-frame saturation of various commutators as suggested by higher symmetry and field-algebra models. We found a way of connecting the information of local $SU(2)$ algebra and space-space commutators in one dispersion sum rule. Such dispersion sum rules could be useful in comparing different models. In our example we find, after some approximations, that in order to make a clear distinction between field algebra and the quark model one would need slightly better measurements of the pion mean square radius and the parameter η measuring the deviation of the KSRF relation. In fact, we found that both are compatible with experiment.

For simplicity we restricted ourselves to the nonstrange components of the vector and tensor currents. It was realized[25] that the nonstrange $SU(4)$ algebra already gives the main $SU(6)$ results. One could also expect in our case a straightforward generalization.

In the derivation of our sum rules, assumptions were made about the convergence of certain dispersion integrals. They were supported by the Bjorken[7] criterion. In the saturation approach this amounted to assuming gentle properties of the operators involved, in such a way that it was possible to insert the completeness relation.

In extracting predictions from these sum rules, we assumed that they are approximately satisfied by lowest-mass states. Only in the case of tensor currents were we able to do this without making a saturation approximation. The results were not in disagreement with experiment.

We use fully throughout this work the advantages of the rest frame. As pointed out before, strong selection rules, due to parity and angular momentum conservation, eliminated many contributions. Of course, in principle, one could easily go out of the rest frame. More information would be obtained but the sum rules would be much more complicated.

One could expect that it would be possible to treat problems involving axial-vector currents in the same way as was done here for vector currents. The algebras of axial-vector currents are known to carry interesting information on axial-vector form factors, particularly on the F_A/F_V ratio.[9] However, a new problem appears. On the right-hand sides of the Ward identities (2.16) the retarded product of the axial-vector current and its divergence arises. Any progress would depend on the

[25] C. Ryan, Ann. Phys. (N. Y.) **38**, 1 (1969).

evaluation of these model-dependent terms.[26] In Ref. 5, the problem of these model-dependent terms was not present, because the local $SU(2)\times SU(2)$ was not used. Instead, the $SU(6)$ algebra was combined with the algebra of time derivatives of space components of the axial-vector current. The disadvantage of the latter method is the lack of detailed knowledge of the higher commutator.

ACKNOWLEDGMENTS

We are grateful to Professor G. Furlan for helpful discussions and for reading the manuscript. We also thank Dr. E. E. Radescu for many useful conversations. We thank Professor Abdus Salam and Professor P. Budini, and the International Atomic Energy Agency for hospitality at the International Centre for Theoretical Physics, Trieste. One of us (G. J. K.) acknowledges gratefully the award of a NATO Science Fellowship by the Netherlands Organization for the Advancement of Pure Research (ZWO).

APPENDIX A: FORM FACTORS AND CONVENTIONS

We used the metric defined by the tensor

$$g=\begin{bmatrix}1&&&\\&-1&&\\&&-1&\\&&&-1\end{bmatrix}. \tag{A1}$$

For the state normalization we adopted

$$\begin{aligned}\langle p_1|p_2\rangle&=2E\delta^3(\mathbf{p}_1-\mathbf{p}_2)(2\pi)^3\quad\text{for bosons}\\&=(E/m)\delta^3(\mathbf{p}_1-\mathbf{p}_2)(2\pi)^3\quad\text{for fermions}.\end{aligned}\tag{A2}$$

We present now the definitions of the form factors used in the text.

For vector currents:

$$\langle 0|V_\mu{}^\alpha|\rho(p,\beta,\lambda)\rangle=f_\rho\epsilon_\mu(p,\lambda)\delta_{\alpha\beta}\,,\tag{A3}$$

$$\langle\pi(p_1,\alpha)|V_\mu{}^\beta|\pi(p_2,\gamma)\rangle=i\epsilon_{\alpha\beta\gamma}(p_1+p_2)_\mu F_\pi(t)\,,\tag{A4}$$

$$\begin{aligned}\langle N(p_1)|V_\mu{}^\beta|N(p_2)\rangle=\bar{u}(p_1)\{&(F_1{}^V+2mF_2{}^V)\gamma_\mu\\&-(p_1+p_2)_\mu F_2{}^V\}\tfrac{1}{2}\tau^\beta u(p_1)\,,\end{aligned}\tag{A5}$$

where $F_1{}^V(0)=1$,

$$\begin{aligned}\langle\pi(p_1,\alpha)|V_\mu{}^\beta|A_1(p_2,\gamma,\lambda)\rangle=\epsilon_{\alpha\beta\gamma}\epsilon_\nu(p_2,\lambda)[&Gg_{\mu\nu}\\&+G_2p_{2\nu}(p_1+p_2)_\mu+G_3p_{2\nu}(p_1-p_2)_\mu].\end{aligned}\tag{A6}$$

In the rest frame,

$$\langle N(\alpha)|V_k{}^\beta|N^{**}(\gamma)\rangle=\bar{u}(\alpha)\sigma_k u_\beta(\gamma)\,.\tag{A7}$$

For axial currents and their divergences,

$$\langle 0|A_\mu{}^\alpha|A_1(p,\beta,\lambda)\rangle=f_A\epsilon_\mu(p,\lambda)\delta_{\alpha\beta}\,,\tag{A8}$$

$$\langle 0|A_\mu{}^\alpha|\pi(p,\beta)\rangle=if_\pi p_\mu\delta_{\alpha\beta}\,,\tag{A9}$$

$$\begin{aligned}\langle\rho(p_1,\alpha,\lambda_1)|D_5{}^\beta|A_1(p_2,\gamma,\lambda_2)\rangle\\=-\epsilon_{\alpha\beta\gamma}\epsilon_\nu(p_2,\lambda_2)\epsilon_\mu(p_1,\lambda_1)[\beta g_{\mu\nu}+\beta_2p_{2\mu}p_{1\nu}].\end{aligned}\tag{A10}$$

π, N, ρ, and A_1 denote that corresponding states are pion, nucleon, ρ-meson, and A_1-meson states, respectively. N^{**} denotes the $\frac{3}{2}^+$, $\frac{1}{2}$ (spin, parity, and isospin) resonance which occurs at 1518 MeV. p denotes momenta of states, α, β, and γ their Cartesian isospin components, and λ_1, λ_2 their polarization.

With our conventions PCAC reads $D_\alpha=f_\pi\mu^2\varphi_\alpha$ and the KSRF relation is $f_\rho=\sqrt{2}m_\rho f_\pi$.

The mean-square electromagnetic radius of pion and nucleon used in the text were defined by

$$\tfrac{1}{6}\langle r_\pi{}^2\rangle=\left.\frac{dF_\pi(t)}{dt}\right|_{t=0};\quad\tfrac{1}{6}\langle r_N{}^2\rangle_V=\left.\frac{dF_1{}^V(t)}{dt}\right|_{t=0}.\tag{A11}$$

APPENDIX B: $A_1\rho\pi$ SYSTEM

We should like to give some details in the derivation of the three-point function sum rules as used in Sec. IV. Let us assume the $SU(2)$ algebra

$$[Q_5{}^\alpha,V_i{}^\beta(0)]=i\epsilon^{\alpha\beta\gamma}A_i{}^\gamma(0).\tag{B1}$$

Sandwiching between vacuum and the A_1 state at rest

$$\begin{aligned}&(2\pi)^3\sum_n\delta(\mathbf{p}-\mathbf{p}_n)\langle p,\sigma|A_0{}^\alpha|n\rangle\langle n|V_i{}^\beta|0\rangle-\text{c.t.}\\&+(2\pi)^3\sum_n\delta(\mathbf{p}_n)\langle 0|A_0{}^\alpha|n\rangle\langle n;p,\sigma|V_i{}^\beta|0\rangle-\text{c.t.}\\&=if_A\epsilon_i(p)\epsilon^{\alpha\beta\sigma}.\end{aligned}\tag{B2}$$

The contributions are 1^- in the first term and in the crossed term of the second, and 0^- in the remaining terms. If one takes the ρ and π into account, one obtains

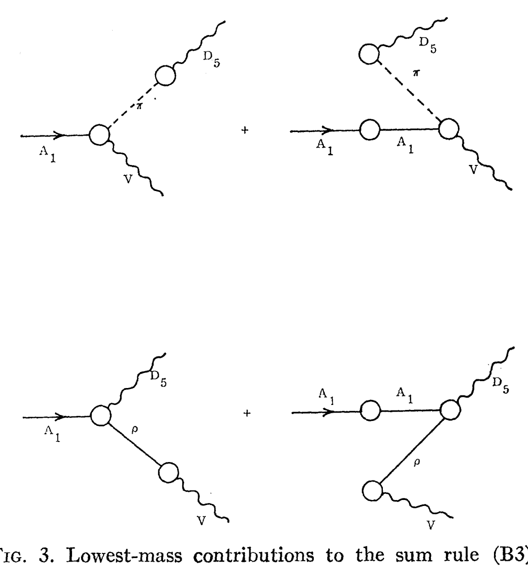

FIG. 3. Lowest-mass contributions to the sum rule (B3).

[26] As known, $SU(2)\times SU(2)$ and these model-dependent terms already determine F_A.

$$\tfrac{1}{2}[G((m_A-\mu)^2)+G((m_A+\mu)^2)]$$
$$=\frac{f_A}{f_\pi}+\frac{1}{2m_\rho}\frac{f_\rho}{f_\pi}\left[\frac{\beta((m_A-m_\rho)^2)}{m_A-m_\rho}-\frac{\beta((m_A+m_\rho)^2)}{m_A+m_\rho}\right]. \tag{B3}$$

The contributions are graphically depicted in Fig. 3. If the pion contribution is negelcted in the spirit of the arguments of Sec. IV, relation (4.10) is obtained.

Let us use

$$[Q_5{}^\alpha,D_i{}^\beta]=D_i{}^{5\gamma}, \tag{B4}$$

where $D_i{}^\beta$, $D_i{}^{5\gamma}$ are dipoles of vector and axial-vector currents defined as in (2.39).

Sandwiching again between an A_1 state at rest and the vacuum, one obtains

$$\frac{1}{2}\left[\frac{G((m_A-\mu)^2)}{m_A-\mu}+\frac{G((m_A+\mu)^2)}{m_A+\mu}\right]=\frac{f_A}{f_\pi m_A}$$
$$+\frac{f_\rho}{2m_\rho{}^2 f_\pi}\left[\frac{\beta((m_A-m_\rho)^2)}{m_A-m_\rho}+\frac{\beta((m_A+m_\rho)^2)}{m_A+m_\rho}\right]. \tag{B5}$$

PHYSICAL REVIEW D VOLUME 2, NUMBER 6 15 SEPTEMBER 1970

Set of Crossing-Symmetric Dynamical Equations and Infinitely Rising Regge Trajectories*

LOUIS A. P. BALÁZS†
Department of Physics, University of California, Los Angeles, California 90024
(Received 26 March 1970)

An equation based on unitarity and analyticity is written down. It is a relativistic generalization of a Schrödinger equation with a local energy-dependent potential of the Yukawa type, in contrast to the usual Mandelstam equations, which correspond to a nonlocal energy-independent potential. A prescription is given for our potential, which makes the amplitude exactly crossing symmetric. It is argued that a first guess at the potential might simply be to take the low-energy crossed-channel resonances of the Veneziano model. The cutoff on the number of resonances is then an undetermined parameter, which, however, merely serves to fix the energy scale in the limit of a vanishingly small pion mass. A determinantal approximation to our scheme (which differs from the determinantal approximation to the usual N/D equations) is then found to lead to Regge trajectories which rise indefinitely. The $I=1$ output resonances agree approximately with the input ones, average duality is found to be satisfied, and the P-wave scattering length comes out close to the Weinberg value. These results do not change much, at least at low engeries, even if a strong dose of inelasticity is introduced.

I. INTRODUCTION

An amplitude has been proposed by Veneziano which has linearly rising Regge trajectories, crossing, and duality.[1] Unfortunately, it depends on a number of arbitrary parameters. Indeed, this number becomes infinite if satellites are admitted. Since the Veneziano model does not satisfy unitarity, however, it might be hoped that this condition could be used to determine some, if not all, of these parameters. Now, in practice it is generally difficult to do this without sacrificing crossing. But it is always possible to vary the parameters of a unitary model until at least approximate partial crossing is achieved. One can then see how closely the resulting amplitude resembles the Veneziano model.

Numerous relativistic models based on unitarity and analyticity have been proposed over the years. Most of them are based on the Mandelstam representation with a finite number of subtractions[2] and differ only in their treatment of short-range effects. The most ambitious has been the strip approximation of Chew and Frautschi,[3] which satisfies crossing exactly, at least in principle. The difficulty with practically all of these models is that they do not lead to trajectories which rise very high. It is generally hoped that the addition of higher channels might change this (although doubts have recently been thrown even on this[4]). But this is difficult to implement in practice. In some sense, it is done in the multi-Regge integral equation approach,[5] which amounts to the inclusion of an infinite number of channels. However, it has been shown[6] that Regge trajectories are bounded by an $s^{1/2}$ behavior at infinity in this model, in contrast to their linear behavior in the Veneziano model. It might therefore be desirable to look at alternative models.

Two such schemes have recently been proposed.[7] Each

* Supported in part by the National Science Foundation.
† Present address: Department of Physics, Purdue University, Lafayette, Ind. 47907.
[1] G. Veneziano, Nuovo Cimento 57, 190 (1968).
[2] S. Mandelstam, Phys. Rev. 112, 1344 (1958).
[3] G. F. Chew and S. C. Frautschi, Phys. Rev. 124, 264 (1961).
[4] P. D. B. Collins and R. C. Johnson, Phys. Rev. 182, 1755 (1969).
[5] G. F. Chew, M. L. Goldberger, and F. E. Low, Phys. Rev. Letters 22, 208 (1969); I. G. Halliday and L. M. Saunders, Nuovo Cimento 60A, 115 (1969).
[6] C. I. Tan and J.-M. Wang, Phys. Rev. Letters 22, 1152 (1969).
[7] L. A. P. Balázs, Phys. Rev. 176, 1769 (1968).