Reformulation of the Dirac-Fierz-Pauli Equation for Spin 3/2

S. TANI
Physics Department, Marquette University, Milwaukee, Wisconsin 53233
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We study the $(1,\frac{1}{2})\oplus(\frac{1}{2},1)$ irreducible representation of the homogeneous Lorentz group thoroughly. Though our starting point is the original version of the spinor equation due to Dirac and Fierz, we resort to an algebraic method and derive all necessary transformation functions and projection operators. Consequently, we can see a physical significance of each mathematical operation immediately. There are two kinds of particles in this representation: One has mass m and spin $\frac{3}{2}$ in the rest system, the other mass $2m$ and spin $\frac{1}{2}$. A projection operator to a subspace for each type of particle is Lorentz invariant. There are four γ matrices which transform like components of a four-vector. The inverse of the operator which defines the equation of motion for a free particle will define the (Lorentz-invariant) propagator. Unfortunately, it is hard to carry out the canonical quantization, and we cannot derive the same propagator by a canonical method. The algebraic method suggested here will be very useful when we extend the theory by adding a $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ representation, as is done by Pauli and Fierz, Gupta, and others.

I. INTRODUCTION

In studying higher spin, we have to investigate two aspects: (1) Lorentz covariance and (2) canonical quantization. In the case of the Dirac equation for spin $\frac{1}{2}$, not only each aspect but also their relationship is well understood. The algebra in the Dirac equation can be formulated in the neatest form by using the two commuting sets of Pauli matrices

$$\sigma_i \quad\text{and}\quad \rho_i \quad (i=1,2,3).$$

It is the purpose of this paper to study the algebra in the $(1,\frac{1}{2})\oplus(\frac{1}{2},1)$ representation of the Lorentz group, and to break it down to the same level as was done for the Dirac equation. We start from Dirac and Fierz's spinor theory.[1,2]

We are motivated by the following considerations. To begin, we must carry out at least the following stages in order to establish a theory of higher-spin particles.

(1) We identify an irreducible representation of the Lorentz group to which the wave function belongs.

(2) We find the set of four γ matrices, if it exists, and other tensor operators. [We shall call a tensor constructed without use of the energy momentum p_μ an "algebraic tensor." In this terminology step (2) is to find algebraic tensors of various characters.]

(3) Using an algebraic tensor and p_μ, we write down the equation of motion and construct a projection operator to a physical state, if there is any redundant component. Subsequently, we derive a propagator for a physical particle.

(4) We introduce the Feynman-Dyson series in the presence of an interaction.

Now, as a result of the extensive studies made so far, we understand all theories clearly with respect to stage (1). A few words should be added at this point, however. All theories can be divided into two categories, depending on whether (a) a single irreducible representation or (b) a superposition (direct sum) is used. With respect to category (b), the reader is referred to a recent paper by Capri[3]; here we shall limit ourselves to (a). The discussion is continued in the next paragraph, and a review of previous work is given.

As shown by Joos,[4] by Weinberg,[5] and by Weaver, Hammer, and Good,[6] and in applications made by others, there are certain advantages in using an irreducible representation $(s,0)\oplus(0,s)$ in order to describe a particle of spin s. The values of mass and spin are unique in this scheme, and there is no redundant component. On the other hand, we cannot derive the four γ matrices, and the form of a propagator is rather complicated. The latter point is related, perhaps, to the lack of a linear equation of motion. In view of the urgent need to settle the problem of higher spin, and in view of the fact that the theory of interaction in the $(s,0)\oplus(0,s)$ scheme has not yet achieved final success, let us turn to another possibility. When specialized to spin $\frac{3}{2}$, there is only one available representation besides $(\frac{3}{2},0)\oplus(0,\frac{3}{2})$, namely, $(1,\frac{1}{2})\oplus(\frac{1}{2},1)$. The algebraic work for $(\frac{3}{2},0)\oplus(0,\frac{3}{2})$ has been summarized by Schay, Song, and Good.[7] With regard to $(1,\frac{1}{2})\oplus(\frac{1}{2},1)$, we must recall that the 12-component wave function in the Dirac-Fierz theory is the basis for that representation. Therefore, we shall undertake a thorough study of the algebra in this case. We can develop a matrix algebra which is very handy in deriving transformation functions, algebraic tensors, and projection operators of various kinds. Consequently, we can find a physical meaning of each algebraic operation straightforwardly. The 12-space with which we start can be divided into two subspaces: (i) an 8-space which describes a physical particle with mass m and

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[3] A. Z. Capri, Phys. Rev. **178**, 2427 (1969).
[4] H. Joos, Fortschr. Physik **10**, 65 (1962).
[5] S. Weinberg, Phys. Rev. **133**, B1318 (1964); **134**, B882 (1964); **181**, 1893 (1969).
[6] D. L. Weaver, C. L. Hammer, and R. H. Good, Jr., Phys. Rev. **135**, B241 (1964).
[7] D. Shay, H. S. Song, and R. H. Good, Jr., Nuovo Cimento Suppl. **3**, 455 (1965).

spin $\frac{3}{2}$ in the rest system, and (ii) a 4-space which describes an unphysical particle with mass $2m$ and spin $\frac{1}{2}$ in the rest system. A projection operator to each subspace is Lorentz invariant.

The reader's attention should be called also to the second version of the spinor theory, which was first studied by Fierz and Pauli.[8] Gupta[9] and others studied an interaction in this scheme. The 12-space of the $(1,\frac{1}{2})\oplus(\frac{1}{2},1)$ representation is extended to a 16-space by the addition of one $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ representation. Recently, Capri[3] discussed this case as a special example of his general theory. It is generally accepted that the quantization can be carried out with less difficulty in Fierz and Pauli's version, than in the original version. However, we stay with the original version, because a thorough study of the algebra of the irreducible representation $(1,\frac{1}{2})\oplus(\frac{1}{2},1)$ has not yet been carried out at the level presented here. One can also point out that the present work prepares for an extension into the Fierz-Pauli theory.

The Dirac-Fierz-Pauli theory is superior in that there exists the set of four γ matrices which transforms like a four-vector so that we can choose an interaction term of any tensor character, while in the $(\frac{3}{2},0)\oplus(0,\frac{3}{2})$ scheme the lowest rank of an algebraic tensor is three. Thus, on defining the propagator as an inverse of the operator which defines the equation of motion, we can formally write down the Feynman-Dyson series in the presence of an interaction. However, owing to a conflict with a canonical quantization, we cannot go very far with the present method in tackling the long-standing difficulty concerning an interaction. Instead, we shall take a critical viewpoint and present both advantageous and disadvantageous aspects, with intent to facilitate a further investigation. In particular, it is strongly suggested that the Fierz-Pauli theory be studied with a method similar to the present one. As for the recent trends in the study of an interaction, the reader is referred to Wightman.[10]

Finally, I would like to emphasize some of the advantages presented in the present approach. Let us recall that there were two main currents in the study of a higher spin. Various types of linear equation had been proposed by the early 1950's, including those of Dirac,[1] Fierz,[2] Pauli,[8] Rarita and Schwinger,[11] Kramers, Belinfante, and Lubanski,[12] Bargmann and Wigner,[13] Bhabha,[14] and Harish-Chandra.[15] They were summarized by Corson[16] and by Umezawa.[17] However, an alternative approach based on group theory has been vigorously advanced in the past decade and a half. In contrast to the earlier workers who searched for a physical meaning of a solution to an equation of specific type, Foldy[18] demonstrated that a fruitful and wider view develops when one inverts the procedure. Namely, he searched for a mathematical expression which serves as a representation of a basic physical assumption, and thus he initiated a new trend. When what Foldy called a synthetic approach is tried in our problem, it will be suggested that we derive the generating operators of the Lorentz transformation using the familiar spin operators. If we do this, the algebra of the representation reduces to a more tractable form since only well-studied spin operators appear in the later stages. We have found an answer to the problem in a general form, which is simple in principle. We need to keep in mind only that a multiple product of spin operators demands a more involved manipulation, as the magnitude of spin becomes larger. Hence, the program for us to work out is to start from Dirac and Fierz's spinor equation and to develop an algebraic technique to handle the generating operators of a Lorentz transformation.

In addition to the references mentioned already, the spin-$\frac{3}{2}$ Foldy-Wouthyusen transformations for the Rarita-Schwinger and for the Bargmann-Wigner equations were worked out by Bryden[19] and by Pursey,[20] respectively. Subsequently, Pursey[21] made a detailed examination of the representations of the Lorentz group. The Hamiltonian formalism in the $(s,0)\oplus(0,s)$ scheme was reviewed by Nelson and Good,[22] which covers the work done by their group, by Mathews,[23] and by others. For actual applications of a boost, we refer to a review by Pais.[24] We no not attempt a complete list of recent references, but we suggest, as sources of further information for recent works, the articles by Tung,[25] by Scadron and Jones,[26] and by Chang.[27] A homogeneous Lorentz transformation will be referred to simply as a Lorentz transformation throughout this paper.

In Sec. II, we discuss a representation of the Lorentz group and construct the generating operators of a

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[27] S. J. Chang, Phys. Rev. **161**, 1316 (1967).

Lorentz transformation as well as the four γ matrices. In Sec. III, we discuss the equation of motion as well as the projection operator to a physical state. In Sec. IV, we derive the subsidiary condition and the propagator of a physical particle. In Sec. V, we discuss the formal theory of interaction. Section VI is designated for further discussions of the algebra. Section VII deals with quantization and with some disparity which appears in the inner helicity states $h=\pm\frac{1}{2}$ in our scheme. On the other hand, such a disparity never appears in the outer helicity states $h=\pm\frac{3}{2}$. The conclusions of our study are given in Sec. VIII.

II. REPRESENTATION OF LORENTZ GROUP

The original Dirac-Fierz equation for spin $\frac{3}{2}$ consists of a pair of equations:

$$p^i\sigma_i{}^{\dot\mu\alpha}b_{\lambda\alpha}{}^{\dot\nu}=ma_\lambda{}^{\dot\mu\dot\nu}\,, \tag{2.1}$$

$$p_i\sigma^i{}_{\mu\dot\alpha}a_\nu{}^{\dot\lambda\dot\alpha}=mb_{\mu\nu}{}^{\dot\lambda}\,, \tag{2.2}$$

which deal with a pair of symmetric spinors, $a_\lambda{}^{\dot\mu\dot\nu}$ and $b_{\mu\nu}{}^{\dot\lambda}$. In these equations the left-hand side is not necessarily symmetric under an interchange of μ with ν. Hence, each of the variables λ, μ, and ν takes a value 1 or 2 independently of others. Then, both (2.1) and (2.2) stand for eight independent equations, and there are 16 equations altogether. When the left-hand side is symmetrized, we derive 12 equations, which have the form

$$\tfrac{1}{2}p^i(\sigma_i{}^{\dot\mu\alpha}b_{\lambda\alpha}{}^{\dot\nu}+\sigma_i{}^{\dot\nu\alpha}b_{\lambda\alpha}{}^{\dot\mu})=ma_\lambda{}^{\dot\mu\dot\nu}\,, \tag{2.3}$$

$$\tfrac{1}{2}p_i(\sigma^i{}_{\mu\dot\alpha}a_\nu{}^{\dot\lambda\dot\alpha}+\sigma^i{}_{\nu\dot\alpha}a_\mu{}^{\dot\lambda\dot\alpha})=mb_{\mu\nu}{}^{\dot\lambda}\,. \tag{2.4}$$

As we shall show in Sec. III, these 12 equations serve as the equation of motion. When the left-hand side is antisymmetrized, we derive four equations, which have the form

$$p^i(\sigma_i{}^{\dot1\alpha}b_{\lambda\alpha}{}^{\dot2}-\sigma_i{}^{\dot2\alpha}b_{\lambda\alpha}{}^{\dot1})=0\,, \tag{2.5}$$

$$p_i(\sigma^i{}_{1\dot\alpha}a_2{}^{\dot\lambda\dot\alpha}-\sigma^i{}_{2\dot\alpha}a_1{}^{\dot\lambda\dot\alpha})=0\quad(\lambda=1,\,2)\,. \tag{2.6}$$

We shall find later that the subsidiary conditions follow from these four equations.

It is not so complicated to convert the above spinor equations into more conventional matrix equations. A symmetric spinor of a single kind (dotted or undotted) carries a representation of the rotation group, which is, so to speak, built in.[28] The square root of a binomial coefficient is the correct normalization. Thus, a normalized element of the canonical representation[29] $D(\frac{1}{2}n)$ (in which the z component of the angular momentum is diagonal) is provided by

$$[n!/m!(n-m)!]^{1/2}u_1{}^m u_2{}^{n-m}\quad(m=0,\,1,\,\ldots,\,n)\,, \tag{2.7}$$

where we denote an undotted spinor of the first rank by (u_1,u_2). A similar formula holds for a dotted spinor. In this way, we easily identify the irreducible representation of the rotation group to which the given spinor belongs. Subsequently, we can easily identify an irreducible representation of the Lorentz group by considering a direct product of two representations of the rotation group. Thus we recognize that the symmetric spinors a and b belong to a direct product of spin 1, spin $\frac{1}{2}$, and ρ spin,

$$(a_\lambda{}^{\dot\mu\dot\nu})\oplus(b_{\mu\nu}{}^{\dot\lambda})=(\text{spin }1)\otimes(\text{spin }\tfrac{1}{2})\otimes(\rho\text{ spin})\,. \tag{2.8}$$

The spin-1 and -$\frac{1}{2}$ operators will be denoted by Σ and σ, respectively. These matrices satisfy the comutation relations

$$[\Sigma_i,\Sigma_j]=i\epsilon_{ijk}\Sigma_k\,, \tag{2.9}$$

$$[\sigma_i,\sigma_j]=i\epsilon_{ijk}\sigma_k\,, \tag{2.10}$$

$$[\rho_i,\rho_j]=2i\epsilon_{ijk}\rho_k\,, \tag{2.11}$$

$$[\sigma_i,\Sigma_j]=[\rho_i,\Sigma_j]=[\rho_i,\sigma_j]=0\,. \tag{2.12}$$

In addition, the ρ operators have the familiar properties of Pauli matrices:

$$\rho_i{}^2=1\,, \tag{2.13}$$

$$\rho_i\rho_j=i\epsilon_{ijk}\rho_k\,. \tag{2.14}$$

On the other hand, for the σ operators the relations (2.13) and (2.14) are replaced by

$$\sigma_i{}^2=\tfrac{1}{4}\,, \tag{2.15}$$

$$\sigma_i\sigma_j=i\tfrac{1}{2}\epsilon_{ijk}\sigma_k\,. \tag{2.16}$$

In addition, we need the Duffin relation for the spin-1 operators

$$\Sigma_i\Sigma_j\Sigma_k+\Sigma_k\Sigma_j\Sigma_i=\Sigma_i\delta_{jk}+\Sigma_k\delta_{ij}\,. \tag{2.17}$$

The crucial step is to construct a generator of a rotation according to

$$J_i=\Sigma_i+\sigma_i \tag{2.18}$$

and a generator of a boost according to

$$K_i=\rho_1(\sigma_i-\Sigma_i)\,. \tag{2.19}$$

We can immediately verify the commutation relations satisfied by these generating operators:

$$[J_i,J_j]=i\epsilon_{ijk}J_k\,, \tag{2.20}$$

$$[K_i,K_j]=i\epsilon_{ijk}J_k\,, \tag{2.21}$$

$$[K_i,J_j]=[J_i,K_j]=i\epsilon_{ijk}K_k\,. \tag{2.22}$$

Next, we show that the four operators defined by

$$\gamma_0=\rho_3(\tfrac{1}{2}+\boldsymbol{\sigma}\cdot\boldsymbol{\Sigma}) \tag{2.23}$$

and

$$\begin{aligned}\gamma_i&=-[\gamma_0,K_i]\\&=i\rho_2[\tfrac{1}{2}\Sigma_i-\sigma_i+\Sigma_i\boldsymbol{\sigma}\cdot\boldsymbol{\Sigma}+\boldsymbol{\sigma}\cdot\boldsymbol{\Sigma}\Sigma_i]\quad(i=1,\,2,\,3)\end{aligned} \tag{2.24}$$

transform under a Lorentz transformation like the four

[28] E. M. Corson, Ref. 16, pp. 45 and 46; M. Hamermesh, *Group Theory* (Addison-Wesley, Reading, Mass., 1962), p. 353; B. L. van der Waerden, *Die Gruppentheoretische Methode in der Quantenmechanik* (Springer, Berlin, 1932), Secs. 17 and 20.

[29] See, e.g., L. I. Schiff, 3rd ed., *Quantum Mechanics* (McGraw-Hill, New York, 1968), p. 203.

components of a vector. Their property under a rotation is an obvious consequence of the commutation relations for spin 1 and spin $\frac{1}{2}$ [Eqs. (2.9) and (2.10)] and the definition (2.18) of a rotation. For a boost, we need to show that

$$[\gamma_i, K_j] = \gamma_0 \delta_{ij}. \tag{2.25}$$

For this purpose, we derive from the Duffin relation (2.17)

$$\Sigma_i \mathbf{\Sigma}\cdot\boldsymbol{\sigma}\Sigma_j + \Sigma_j \mathbf{\Sigma}\cdot\boldsymbol{\sigma}\Sigma_i = \sigma_i\Sigma_j + \sigma_j\Sigma_i \tag{2.26}$$

and

$$\mathbf{\Sigma}\cdot\boldsymbol{\sigma}\Sigma_i\Sigma_j + \Sigma_j\Sigma_i\mathbf{\Sigma}\cdot\boldsymbol{\sigma} = \Sigma_j\sigma_i + \mathbf{\Sigma}\cdot\boldsymbol{\sigma}\delta_{ij}. \tag{2.27}$$

Using the properties of spin-$\frac{1}{2}$ operators, (2.15) and (2.16), we find

$$-[\gamma_i, K_j] = \rho_3(\tfrac{1}{2}\delta_{ij} - 2\Sigma_j\sigma_i - \Sigma_i\sigma_j + \Sigma_i\mathbf{\Sigma}\cdot\boldsymbol{\sigma}\Sigma_j + \Sigma_j\mathbf{\Sigma}\cdot\boldsymbol{\sigma}\Sigma_i + \mathbf{\Sigma}\cdot\boldsymbol{\sigma}\Sigma_i\Sigma_j + \Sigma_j\Sigma_i\mathbf{\Sigma}\cdot\boldsymbol{\sigma}). \tag{2.25$'$}$$

However, when (2.26) and (2.27) are used on the right-hand side of (2.25′), (2.25) follows immediately. One immediate consequence is a formula for transforming the operator

$$\gamma_\mu p^\mu = \gamma_0 p_0 - \boldsymbol{\gamma}\cdot\mathbf{p}$$

into the corresponding form in the rest system, which takes the form

$$\Omega^{-1}\gamma_\mu p^\mu \Omega = (\operatorname{sgn} p_0)(p_0{}^2 - \mathbf{p}^2)^{1/2}\gamma_0, \tag{2.28}$$

where we set

$$\Omega = \exp[-\tanh^{-1}(|\mathbf{p}|/p_0)\cdot\mathbf{p}\cdot\mathbf{K}/|\mathbf{p}|]. \tag{2.29}$$

The Hermitian conjugate of the wave function Ψ will be denoted by Ψ^*. When we introduce a conjugate

$$\Psi^\dagger = \Psi^*\rho_3, \tag{2.30}$$

it transforms inversely under a Lorentz transformation

$$\Psi^\dagger \to \Psi'^\dagger = \Psi^\dagger\Omega^{-1}, \tag{2.30$'$}$$

while the wave function undergoes a Lorentz transformation

$$\Psi \to \Psi' = \Omega\Psi. \tag{2.30$''$}$$

A conjugate of this type is well known in the case of the Dirac equation, and may be called the Pauli conjugate. Note that a pure rotation will be represented by a unitary transformation. For instance, the transformation for a rotation by an angle ϕ around the axis $\mathbf{n}$ (a unit vector) is

$$\Omega_{\text{rot}} = \exp(i\phi\mathbf{J}\cdot\mathbf{n}). \tag{2.31}$$

Finally, the transformations under various inversion operations will be defined as follows.

(a) space inversion:

$$\Psi' = \rho_3\Psi; \tag{2.32}$$

(b) time inversion:

$$\Psi' = \rho_2\Psi; \tag{2.33}$$

(c) space-time inversion:

$$\Psi' = \rho_1\Psi. \tag{2.34}$$

III. EQUATION OF MOTION

The suggested form of the equation of motion is

$$(\gamma_0 p_0 - \boldsymbol{\gamma}\cdot\mathbf{p} - m)\Psi = 0. \tag{3.1}$$

On writing down all 12 lines of (3.1) and comparing them with the 12 spinor equations (2.3) and (2.4), we shall find that those two sets of equations are identical. Such a comparison is tedious but straightforward. All necessary instructions for writing down matrices and spinors have been given at the beginning of Sec. II.

We shall transform (3.1) into the rest system in order to derive the individual eignefunctions. We set

$$\Psi = \Omega\Phi, \tag{3.2}$$

and denote the wave function in the rest system by Φ. On applying (2.28) and (2.29), we have

$$[(\operatorname{sgn} p_0)(p_0{}^2 - \mathbf{p}^2)^{1/2}\gamma_0 - m]\Phi = 0. \tag{3.3}$$

An eigenvalue of the operator $\boldsymbol{\sigma}\cdot\mathbf{\Sigma}$ is equal to $\frac{1}{2}$ or -1, depending on whether the total spin J is equal to $\frac{3}{2}$ or $\frac{1}{2}$. We introduce two projection operators projecting onto these subspaces:

$$\epsilon_{(3/2)} = \tfrac{2}{3}(1 + \boldsymbol{\sigma}\cdot\mathbf{\Sigma}) \quad \text{for} \quad J = \tfrac{3}{2}, \tag{3.4}$$

$$\epsilon_{(1/2)} = \tfrac{1}{3}(1 - 2\boldsymbol{\sigma}\cdot\mathbf{\Sigma}) \quad \text{for} \quad J = \tfrac{1}{2}. \tag{3.5}$$

Remembering the definition of γ_0, (2.23), we also have

$$\gamma_0{}^2\epsilon_{(3/2)} = \epsilon_{(3/2)} \tag{3.6}$$

and

$$\gamma_0{}^2\epsilon_{(1/2)} = \tfrac{1}{4}\epsilon_{(1/2)}. \tag{3.7}$$

Then, we derive from (3.3)

$$(p_0{}^2 - \mathbf{p}^2 - m^2)\epsilon_{(3/2)}\Phi = 0 \tag{3.8}$$

and

$$(p_0{}^2 - \mathbf{p}^2 - 4m^2)\epsilon_{(1/2)}\Phi = 0. \tag{3.9}$$

Thus, if the spin J in the rest system is equal to $\frac{3}{2}$ ($\frac{1}{2}$), the mass defined by a positive number $(p_0{}^2 - \mathbf{p}^2)^{1/2}$ is equal to m ($2m$). We define a particle (antiparticle) contingent upon the condition that the rest-system wave function be an eigenstate of ρ_3 with an eigenvalue 1 (-1). If we define the mass operator $\mathfrak{M}$ (with a signature) by

$$\mathfrak{M} = (\operatorname{sgn} p_0)(p_0{}^2 - \mathbf{p}^2)^{1/2}, \tag{3.10}$$

we may state that the eigenvalue of the *mass operator* for the particle with spin $\frac{1}{2}$ is $-2m$.

We should note that

$$\boldsymbol{\sigma}\cdot\mathbf{\Sigma} = 2\gamma_0{}^2 - \tfrac{3}{2}. \tag{3.11}$$

Then, the projection operators introduced in (3.4) and

(3.5) can be rewritten as

$$\epsilon_{(3/2)}=\tfrac{1}{3}(4\gamma_0{}^2-1) \tag{3.12}$$

and

$$\epsilon_{(1/2)}=\tfrac{1}{3}(4-4\gamma_0{}^2). \tag{3.13}$$

When these projection operators are transformed into a moving system, we derive the corresponding projection operators

$$\Lambda_{(3/2)}=\Omega\epsilon_{(3/2)}\Omega^{-1}=\tfrac{1}{3}[4(p^\mu\gamma_\mu)^2(p^\mu p_\mu)^{-1}-1] \tag{3.14}$$

and

$$\Lambda_{(1/2)}=\Omega\epsilon_{(1/2)}\Omega^{-1}=\tfrac{4}{3}[1-(p^\mu\gamma_\mu)^2(p^\mu p_\mu)^{-1}]. \tag{3.15}$$

IV. SUBSIDIARY CONDITION AND PROPAGATOR

The subsidiary condition to eliminate an unphysical particle (with mass $2m$) is given by

$$\Lambda_{(1/2)}\Psi=0 \tag{4.1}$$

or

$$\Psi=\Lambda_{(3/2)}\Psi. \tag{4.2}$$

The algebra required in these formulas was discussed in Sec. III. In order to interpret them physically, let us note that (4.1) can be written as

$$[p^\mu p_\mu-(p^\mu\gamma_\mu)^2]\Psi=0, \tag{4.3}$$

so long as the mass squared $p_0{}^2-\mathbf{p}^2$ does not vanish. When we eliminate $(\gamma_\mu p^\mu)$ using the equation of motion (3.1), we derive from (4.3)

$$[p^\mu p_\mu-m^2]\Psi=0. \tag{4.4}$$

Equation (4.4) shows explicitly that the mass squared for a free particle should be equal to m^2, if the wave function is acceptable as physically significant. The Lorentz invariance of the subsidiary condition is obvious. When we write down the 12 lines of (4.3) explicitly, we shall find that these 12 linear equations follow from the spinor subsidiary condition, (2.5), and (2.6). Again the actual comparison is tedious but we only have to follow the instructions given at the beginning of Sec. II.

We then turn to the equation of motion in the presence of an interaction. All formulas dealing with an interaction will remain on a rather formal basis. As for their detail, some further work is necessary. It is our purpose to explore a possibility rather than a detail. Let us now try an equation

$$(\gamma_\mu p^\mu-m-J)\Psi=0. \tag{4.5}$$

What is essential is the fact that the two projection operators $\Lambda_{(3/2)}$ and $\Lambda_{(1/2)}$ are defined completely on the basis of the algebraic properties of the four γ matrices. Those equations which define them read

$$\Lambda_{(3/2)}{}^2=\Lambda_{(3/2)}, \tag{4.6}$$

$$\Lambda_{(1/2)}{}^2=\Lambda_{(1/2)}, \tag{4.7}$$

$$\Lambda_{(3/2)}\Lambda_{(1/2)}=\Lambda_{(1/2)}\Lambda_{(3/2)}=0, \tag{4.8}$$

$$\Lambda_{(3/2)}+\Lambda_{(1/2)}=I. \tag{4.9}$$

They follow from

$$4(\gamma_\mu A^\mu)^4-5(A_\mu A^\mu)(\gamma_\mu A^\mu)^2+(A_\mu A^\mu)^2=0. \tag{4.10}$$

The crucial point is that Eq. (4.10) is valid for any A_μ, as long as these four quantities are effectively c numbers. In other words, the validity of (4.10) is guaranteed under the conditions

$$[A_\mu,A_\nu]=0, \tag{4.11}$$

$$[A_\mu,\gamma_\nu]=0. \tag{4.12}$$

Therefore, the definition of the projection operators (4.6)–(4.9) remains valid if we use momentum-energy operators in configuration representation,

$$p^\mu=i\partial/\partial x_\mu. \tag{4.13}$$

Let us now go back to Eq. (4.5). If we assume that there is no off-diagonal part to link the two different mass values in the interaction operator J,

$$J=\Lambda_{(3/2)}J\Lambda_{(3/2)}+\Lambda_{(1/2)}J\Lambda_{(1/2)}, \tag{4.14}$$

then we can satisfy simultaneously Eqs. (4.5) and (4.1). It looks possible to have a subsidiary condition which involves an interaction operator. However, the above method of introducing an interaction is the *simplest* choice. We shall sacrifice the generality in this paper to postulate (4.14), (4.5), and (4.1). When the latter two equations are combined, they lead to

$$[p^\mu p_\mu-(m+\mathbf{J})^2]\Phi=0, \tag{4.15}$$

which indicates that there is a change in mass due to the interaction.

When we adopt the above approach, the propagator of the physical particle should be represented in momentum space by

$$S(p)=\Lambda_{(3/2)}(p^\mu\gamma_\mu-m)^{-1}\Lambda_{(3/2)}. \tag{4.16}$$

Using the relation

$$(p^\mu\gamma_\mu)^2\Lambda_{(3/2)}=(p^\mu p_\mu)\Lambda_{(3/2)}, \tag{4.17}$$

which follows from (3.14) and (4.10), we rewrite (4.16) in the form

$$\begin{aligned}S(p)&=\Lambda_{(3/2)}(p^\mu\gamma_\mu+m)(p^\mu p_\mu-m^2)^{-1}\Lambda_{(3/2)}\\&=\tfrac{1}{3}(p^\mu\gamma_\mu+m)[4(p^\mu\gamma_\mu)^2(p^\mu p_\mu)^{-1}-1]\\&\qquad\times(p^\mu p_\mu-m^2)^{-1}.\end{aligned} \tag{4.18}$$

V. FORMAL THEORY OF INTERACTION

By transforming its momentum representation (4.18), we derive the propagator in configuration space,

$$\begin{aligned}S(x-x')=(3m^2)^{-1}&[-i(\partial^\mu\gamma_\mu)+m][4(\partial^\mu\gamma_\mu)^2-\Box]\\&\times[D_0(x-x')-D_m(x-x')].\end{aligned} \tag{5.1}$$

On taking advantage of the algebraic property of the four γ matrices shown in (4.10), we can derive the following two relations. First, the propagator satisfies the subsidiary condition, since we have

$$[(\partial^\mu\gamma_\mu)^2-\Box]S(x-x')=0. \tag{5.2}$$

Secondly, applying the operator which defines the free-particle equation of motion, we have

$$-[i(\partial^\mu\gamma_\mu)+m]S(x-x') =\tfrac{1}{3}[\Box-4(\partial^\mu\gamma_\mu)^2]D_0(x-x'). \tag{5.3}$$

Here the right-hand side is the projection operator to the physical subspace in the configuration representation

$$\Lambda_{(3/2)}(x-x') =\tfrac{1}{3}[\delta^{(4)}(x-x')-4(\partial^\mu\gamma_\mu)^2D_0(x-x')], \tag{5.4}$$

whose corresponding momentum representation was given by (3.14).

The projection operator (5.4) is nonlocal, because a D_0 function appears in the second term of the right-hand side. (I believe that the standing-wave D_0 function ought to be used in order to establish symmetry between the future and the past; however, no further study will be made in this paper concerning this point.) Since we define the propagator using the formula (4.16),

$$S(p)=\Lambda_{(3/2)}(p^\mu\gamma_\mu-m)^{-1}\Lambda_{(3/2)},$$

we need to construct projection operators which select an eigenvalue of the operator

$$\Gamma=(\gamma_\mu p^\mu)^2/p_\mu p^\mu. \tag{5.5}$$

This operator is Lorentz invariant and "normalized" in the sense that its eigenvalue is identical with the corresponding eigenvalue of $\gamma_0{}^2$ [see (3.12)–(3.15)]. The "normalization factor" $(p_\mu p^\mu)^{-1}$ which appears in (3.14) is the origin of the D_0 function. Therefore, we cannot avoid a nonlocality of this nature in a multimass theory.

Pursey constructed a projection operator on the basis of group theory.[30] In the case of the Bargmann-Wigner equation, Aurilia and Umezawa recently discussed a projection operator for the same purpose.[31] As will be shown in Sec. VI, the operator $(\gamma_\mu p^\mu)^2$ is a linear function of the Pauli-Lubanski invariant. Consequently, each eigenvalue of Γ is uniquely associated with some eigenvalue of the rest-system spin. It is worth remembering that the eigenvalue spectrum of γ_0 determines crucially the details of a multispin theory, as we have shown above. We shall take up this point in Sec. VII.

When we assume (4.5) and (4.14) to introduce an interaction, we can choose an interaction operator $\mathbf{J}$ of any tensorial character. Let us denote an external field whose transformation property under a Lorentz transformation is of scalar, vector, or other tensorial character by

$$V(x),\quad V_\mu(x),\quad \text{etc.}$$

The interaction operator $\mathbf{J}$ in the respective cases will be given by a nonlocal operator

$$\mathbf{J}_s=\int d^4x\,\Lambda_{(3/2)}(x'-x)V(x)\Lambda_{(3/2)}(x-x''),$$

$$\mathbf{J}_v=\int d^4x\,\Lambda_{(3/2)}(x'-x)\gamma_\mu V^\mu(x)\Lambda_{(3/2)}(x-x''),$$

etc. When solved by a perturbation method, a solution $\Psi(x)^{\text{out}}$ to the equation of motion with an external-field interaction takes the form of the Feynman-Dyson series,

$$\Psi^{\text{out}}(x)=\Psi^{\text{in}}(x) +\int d^4x'd^4x''\,S(x-x')J(x'-x'')\Psi^{\text{in}}(x'') +\int d^4x'd^4x''d^4x'''d^4x^{iv}\,S(x-x')J(x'-x'') \times S(x''-x''')J(x'''-x^{\text{iv}})\Psi^{\text{in}}(x^{\text{iv}})+\cdots. \tag{5.6}$$

Here $\Psi(x)^{\text{in}}$ is a solution of the free-particle equation in the corresponding state of spin. Both $\Psi(x)^{\text{out}}$ and $\Psi(x)^{\text{in}}$ satisfy the subsidiary condition as a consequence of (4.3) and (5.2). More rigorously, we ought to examine the convergence of the perturbation expansion as well as the possibility of making its analytical continuation. In the case of potential scattering, these two points are firmly established under a certain condition which eliminates a long-range or strongly singular interaction.[32] In view of such a well-established status of potential scattering, we may justify the use of a perturbation series provided that a due restriction be made in the choice of functional form of an external field. The perturbation series presented by (5.6) meets all requirements, as far as the Lorentz covariance is concerned, imposed by Weinberg on the S-matrix theory.[5] However, there is a serious disparity between our definition of propagator and canonical quantization which we shall discuss in Sec. VII. On the other hand, the nonlocality discussed in the two preceding paragraphs should not be considered as a source of inconsistency. Instead, we must modify conventional field theory to cover a multispin theory. If we extend our pure $(1,\frac{1}{2})\oplus(\frac{1}{2},1)$ formalism by the addition of one $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ representation, the eigenvalue spectrum of γ_0 will change, as shown by Pauli and Fierz,[8] Gupta,[9] Capri,[3] and others. It may well be that we have less disparity with canonical quantization in a modified theory.

[30] See Ref. 21, pp. 167 and 168.

[31] A. Aurilia and H. Umezawa, Phys. Rev. **182**, 1682 (1969).

[32] S. Tani, Ann. Phys. (N. Y.) **37**, 411, 451 (1966); Phys. Rev. **174**, 2054 (1968).

VI. FURTHER DISCUSSION OF ALGEBRA

A. Algebra of 12-Space

As is well known,[13] the generating operators of a Lorentz transformation, K_i and J_i, (2.20) and (2.21), transform under a Lorentz transformation like components of an antisymmetric tensor of the second rank:

$$(I_{0k},I_{ij})\equiv(K_k,i\epsilon_{ijm}J_m). \tag{6.1}$$

Then, on defining a pseudovector

$$w_m=\tfrac{1}{2}\epsilon_{mnpq}p_nI_{pq}, \tag{6.2}$$

we can derive the Pauli-Lubanski invariant

$$\Pi=w_mw^m. \tag{6.3}$$

This quantity represents the mass squared times the magnitude of spin in the rest system. After some algebra, we find

$$\Pi=4(p^\mu\gamma_\mu)^2-\tfrac{1}{4}p^\mu p_\mu. \tag{6.4}$$

Consequently, the projection operators $\Lambda_{(3/2)}$ and $\Lambda_{(1/2)}$, (3.14) and (3.15), can be rewritten in terms of the Pauli-Lubanski invariant. Then Eqs. (4.1)–(4.4), which have been interpreted in terms of the mass, can be reinterpreted in terms of the rest-system spin.[31]

As noted already by Bhabha[14] and Harish-Chandra,[15] the relation between the six generating operators (6.1) and the four γ matrices cannot be so simple (shown by $\neq$) as

$$I_{mn}\neq\gamma_m\gamma_n-\gamma_n\gamma_m,$$

but must be derived by means of a contraction of a higher rank tensor,

$$I_{mn}=\gamma_a\cdots\gamma_m\gamma_b\cdots\gamma_n\gamma_c\cdots-\gamma_a\cdots\gamma_n\gamma_b\cdots\gamma_m\gamma_c\cdots. \tag{6.5}$$

There are $12^2=144$ elements of the algebra. A systematic study of the entire algebra generated by the four γ matrices will be left for a future work.

B. Generality of Method

It is well known that there are two commuting sets of rotation operators,

$$A_i=\tfrac{1}{2}(J_i+K_i), \tag{6.6a}$$

$$B_i=\tfrac{1}{2}(J_i-K_i), \tag{6.6b}$$

and the magnitudes j,j' of these spin operators characterize an irreducible representation of the Lorentz group. Ordinarily, we start from the commutation relations (2.20)–(2.22) among the generating operators and derive the commuting sets (6.6a) and (6.6b) subsequently. Now let us try the inverse problem: Derive the six generating operators which satisfy the commutation relations (2.20)–(2.22) by starting from the two spins, $\sigma_i=A_i$ and $\Sigma_i=B_i$. The answer will be given by (2.18) and (2.19), provided that σ_i and Σ_i are not of the same magnitude. In the excluded case, we have a self-dual or, in other words, a genuinely tensor representation.

The proof of the above statement proceeds as follows. First remember that we have introduced the chirality ρ_1 as a multiplicative factor in (2.19). When we use (2.19) on the left-hand side of (2.22) we find that

$$\rho_1^2=1. \tag{6.7}$$

Consequently, an eigenvalue of ρ_1 is equal to either 1 or -1. Secondly, once we assume a particular eigenvalue of the chirality, a representation which follows from (2.18) and (2.19) is identical with either the irreducible representation (j,j') or (j',j). In order to show the details, we introduce the projection operators $\lambda_\pm$ which select one of the two eigenvalues of ρ_1:

$$\lambda_\pm=\tfrac{1}{2}(1\pm\rho_1). \tag{6.8}$$

On substituting (2.18) and (2.19) into the right-hand side of (6.6), we find

$$A_i=\lambda_+\sigma_i+\lambda_-\Sigma_i, \tag{6.9a}$$

$$B_i=\lambda_+\Sigma_i+\lambda_-\sigma_i. \tag{6.9b}$$

This result shows that the two angular momenta A and B are assigned to σ or Σ according to

$$A=\sigma,\quad B=\Sigma\quad \text{for chirality}=1,$$
$$A=\Sigma,\quad B=\sigma\quad \text{for chirality}=-1.$$

An opposite choice between the two alternatives is possible, but such an arbitrariness does not affect the end result, as we shall see presently. The commutation relations (2.20)–(2.22) hold equally well when we replace ρ_1 by $-\rho_1$. Then, we can make an alternative choice of A_i and B_i:

$$A_i=\lambda_+\Sigma_i+\lambda_-\sigma_i, \tag{6.10a}$$

$$B_i=\lambda_+\sigma_i+\lambda_-\Sigma_i. \tag{6.10b}$$

Suppose we try to construct the entire representation space using the sum of the two subspaces, one in which (6.9) holds and the other in which (6.10) holds. By inspection of (6.9) and (6.10), we find that both irreducible representations (j,j') and (j',j) are realized once each for each eigenvalue of the chirality. The operators ρ_2 and ρ_3 transform ρ_1 into $-\rho_1$, and that means these operators can find their complete representation in our representation space. In conclusion, our representation is sufficient for all of our purposes and is identical with $(j,j')\oplus(j',j)$. Note that the definition of the reflection operators (2.35)–(2.37) is generally valid provided that j and j' are not equal to each other.

In the Dirac-Fierz-Pauli theory for a half-integral spin, the two angular momenta j and j' differ by $\frac{1}{2}$ from each other. Let us turn to spins higher than $\frac{3}{2}$. The Dirac-Fierz theory (the original spinor theory) for spin $\frac{5}{2}$ deals with the irreducible representation $(\frac{3}{2},1)$

$\oplus(1,\frac{3}{2})$, and we derived the four γ matrices in this case. Denoting a $\frac{3}{2}$ spin by $\boldsymbol{\Sigma}$ and a 1 spin by σ, the result is given by

$$\gamma_0=\rho_3[\tfrac{1}{3}(\boldsymbol{\Sigma}\cdot\boldsymbol{\sigma})^2+\tfrac{1}{2}\boldsymbol{\Sigma}\cdot\boldsymbol{\sigma}-\tfrac{1}{2}], \tag{6.11}$$

$$\gamma_i=i\rho_2\{\tfrac{1}{3}[\Sigma_i(\boldsymbol{\Sigma}\cdot\boldsymbol{\sigma})^2+(\boldsymbol{\Sigma}\cdot\boldsymbol{\sigma})^2\Sigma_i]+\tfrac{1}{3}[(\Sigma_i-\sigma_i)\boldsymbol{\Sigma}\cdot\boldsymbol{\sigma}+\boldsymbol{\Sigma}\cdot\boldsymbol{\sigma}(\Sigma_i-\sigma_i)]-\Sigma_i-\tfrac{1}{4}\sigma_i\}. \tag{6.12}$$

The eigenvalue spectrum of γ_0^2 consists of 1, $(\frac{2}{3})^2$, and 3^{-2}.

Finally, let us note that the formulas (2.18) and (2.19) for generating operators of a Lorentz transformation are useful when we take a direct product of two irreducible representations

$$[(j_1,j_2)\oplus(j_2,j_1)]\otimes[(j_3,j_4)\oplus(j_4,j_3)].$$

If we denote the generators for the first (second) representation by $J^{(1)}$ and $K^{(1)}$ ($J^{(2)}$ and $K^{(2)}$), the generators for the whole representation will be simply the sum

$$J_i=J_i^{(1)}+J_i^{(2)},\quad K_i=K_i^{(1)}+K_i^{(2)}. \tag{6.13}$$

The formulas (2.18) and (2.19) are useful also when we take a direct sum of two irreducible representations. For instance, suppose we try to extend our algebra to the Fierz-Pauli theory by adding a representation $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$. In each subspace, $(1,\frac{1}{2})\oplus(\frac{1}{2},1)$ or $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$, we use the formulas (2.18) and (2.19), using suitable spin operators for each case. It will turn out that, as far as a generator of a Lorentz transformation is concerned, we do not need any off-diagonal component which cross-links the two irreducible representations. The cross linkage between the two different irreducible representations will be realized by introducing a suitable off-diagonal component in the four γ matrices.

VII. HAMILTONIAN AND DISPARITY WITH CANONICAL QUANTIZATION

So far we have concentrated on the Lorentz covariance of the theory, but here let us turn to canonical quantization. Recall first the Dirac equation for spin $\frac{1}{2}$. In that case we can easily switch from the Lorentz covariant equation of motion,

$$(p_0\gamma_0-\mathbf{p}\cdot\boldsymbol{\gamma})\Psi=m\Psi, \tag{7.1}$$

to the Hamiltonian equation of motion,

$$p_0\Psi=(m\gamma_0+\mathbf{p}\cdot\boldsymbol{\alpha})\Psi. \tag{7.2}$$

The algebraic trick is just to take advantage of a simple property of γ_0, namely,

$$\gamma_0^2=\rho_3^2=1\quad[(\tfrac{1}{2},0)\oplus(0,\tfrac{1}{2})\text{ representation}]. \tag{7.3}$$

On the other hand, such a simple method does not work in the $(1,\frac{1}{2})\oplus(\frac{1}{2},1)$ representation, because there is a factor

$$g\equiv\tfrac{1}{2}+\boldsymbol{\Sigma}\cdot\boldsymbol{\sigma}$$

in γ_0 for this case. Because of that factor, the minimal equation for γ_0 takes a more complicated form

$$\gamma_0^4=(5/4)\gamma_0^2-\tfrac{1}{4}\quad[(1,\tfrac{1}{2})\oplus(\tfrac{1}{2},1)\text{ representation}]. \tag{7.4}$$

Then, we have to search for another method of defining the Hamiltonian.

When we go back to the spinor equations (2.1) and (2.2), we find eight equations among which each of the four quantities

$$p_0\alpha_\lambda{}^{\dot{1}\dot{2}}\quad\text{or}\quad p_0b_{12}{}^{\dot{\lambda}}\quad(\lambda=1,2) \tag{7.5}$$

is equated to two different expressions. On the other hand, each of these quantities is symmetric under the interchange of 1 with 2 (for a fixed value of λ), and the presence of two different expressions indicates a lack of that symmetry. After symmetrization between the two equations, however, each of the four quantities (7.5) is equated to a unique result. When these four equations are incorporated with another eight equations for eight quantities

$$p_0a_\mu{}^{\dot{\lambda}\dot{\lambda}}\quad\text{or}\quad p_0b_{\lambda\lambda}{}^{\dot{\mu}}\quad(\lambda,\mu=1,2), \tag{7.6}$$

we have 12 unique equations. They define the time derivative of wave function and constitute the Hamiltonian equations. By an antisymmetrization we derive four other equations from which time derivatives are eliminated, which will serve as the subsidiary conditions. The symmetrization and the antisymmetrization discussed here are similar operations to the derivation of the Lorentz-invariant equations of motion (2.3) and (2.4) and the subsidiary conditions (2.5) and (2.6). The only difference here lies in the fact that the operation has been applied to the time derivative, whereas it was applied to the mass term in Sec. II.

The Hamiltonian equation can be set in a closed form

$$p_0\Psi=[\gamma_0m+\rho_1\boldsymbol{\Sigma}\cdot\mathbf{p}]\Psi\equiv H\Psi. \tag{7.7}$$

There is a unitary transformation which diagonalizes the Hamiltonian (Foldy-Wouthuysen transformation). The diagonalized form of the Hamiltonian is given by

$$U^{-1}HU=\rho_3[(m^2+p^2)^{1/2}\epsilon_{(3/2)}-\tfrac{1}{2}m\epsilon_{(1/2)}]. \tag{7.8}$$

Thus, an energy eigenvalue is

$$p_0=\pm(m^2+p^2)^{1/2}$$

for the rest-system spin $\frac{3}{2}$, whereas

$$p_0=\mp\tfrac{1}{2}m$$

for the rest-system spin $\frac{1}{2}$. Since the details of the Foldy-Wouthuysen transformation are not directly relevant to the following discussion, we shall publish them elsewhere. In the case of the Dirac equation, there is a simple relationship[33] between the transformation function of a proper Lorentz transformation Ω (boost) and the Foldy-Wourthuysen transformation U:

$$\Omega=\tfrac{1}{2}(U+U^{-1})+\tfrac{1}{2}(U-U^{-1})\rho_3. \tag{7.9}$$

[33] H. Jehle and W. C. Parke, Phys. Rev. 137, B760 (1965).

A relation like Eq. (7.9) is definitely impossible in the Dirac-Fierz theory for spin $\frac{3}{2}$ if the rest-system spin is $\frac{1}{2}$. This is because an energy eigenvalue p_0 takes completely different values in the two cases which appear on different sides of an equation like (7.9): $p_0=\mp(4m^2+p^2)^{1/2}$ in the Lorentz-covariant theory, whereas $p_0=\mp\frac{1}{2}m$ in the Hamiltonian theory. If the rest-system spin is $\frac{3}{2}$, we have to take into account a subtle point, a distinction of helicity. Let us denote by

$$\Psi(h,\tfrac{3}{2},\pm)$$

a solution of the Hamiltonian equation with helicity h, rest-system spin $\frac{3}{2}$, and signature of energy $\pm$. We shall find, then, that any of them will satisfy the Lorentz-invariant equation of motion with an appropriate signature of p_0:

$$[\pm(m^2+p^2)^{1/2}\gamma_0-\boldsymbol{\gamma}\cdot\mathbf{p}-m]\Psi(h,\tfrac{3}{2},\pm)=0 \quad (\text{for } h=\pm\tfrac{3}{2},\pm\tfrac{1}{2}). \tag{7.10}$$

This means that a particular column in the Foldy-Wouthuysen transformation U is the same essentially as the corresponding column of the boost Ω, except for a normalization. When we are concerned with the outer helicity states, $h=\pm\frac{3}{2}$, the two columns being compared are identical up to the normalization. Then, as far as these helicity states are concerned, we suggest that there could be a relation between the two transformation functions:

$$\Omega\epsilon_{(3/2)}\overset{?}{=}\tfrac{1}{2}(U+U^{-1})\epsilon_{(3/2)}+\tfrac{1}{2}(U-U^{-1})\rho_3\epsilon_{(3/2)}. \tag{7.11}$$

However, when we turn to the inner helicity states, $h=\pm\frac{1}{2}$, the normalizations of the two columns taken from the two transformation functions are different from each other. Then, (7.11) fails in these helicity states.

The above difficulty as well as the one mentioned at the beginning of this section makes it impossible to derive a Lorentz-invariant propagator which agrees with canonical quantization of the $(1,\frac{1}{2})\oplus(\frac{1}{2},1)$ field. Thus, the Lorentz-invariant propagator introduced in Sec. IV must be considered as based on an *ad hoc* assumption. It is suggested, therefore, to examine the Pauli-Fierz theory, using an algebraic method of the same kind as presented in this paper.

VIII. CONCLUSIONS

We have studied the $(1,\frac{1}{2})\oplus(\frac{1}{2},1)$ representation of the homogeneous Lorentz group, which is equivalent to the Dirac-Fierz version of spinor theory. We have derived the following results.

(1) The algebra in this representation can be neatly handled by introducing the product space (2.8) which deals with spin $\frac{1}{2}$, spin 1, and the ρ spin. We construct the generating operators of a Lorentz transformation following (2.18) and (2.19). We derive the four γ matrices according to (2.23)–(2.25).

(2) As discussed in Sec. VI, the method introduced here is generally applicable to any irreducible representation unless the latter is self-dual.

(3) The Dirac-Fierz-Pauli theory is a multispin multimass theory. A projection operator to each mass value, which is Lorentz invariant, selects the corresponding spin value uniquely. We can express such a projection operator using the Pauli-Lubanski invariant. Though this statement was established only for spin $\frac{3}{2}$, strictly speaking, we conjecture its generality.

(4) We can define a Lorentz-invariant propagator by taking an operator inverse of the free-particle equation of motion. Subsequently, we can write down the Feynman-Dyson series in the presence of an interaction.

(5) However, it is hard to transform the Lorentz-invariant equation of motion into the Hamiltonian form, as discussed in Sec. VII. Accordingly, our Lorentz-invariant propagator ought to be considered as resulting from an *ad hoc* assumption.

It may well be that result (5) can be improved if we use the Fierz-Pauli theory instead of the original form of spinor theory. As discussed recently by Capri,[3] we can shift the spectrum of γ_0 by superposing a $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ representation. If the spectrum is changed from the present one, ±1, $\mp\frac{1}{2}$, into a more desirable set, ±1, 0, we may be able to remove some of the difficulties discussed in Sec. VII. Thus, it is very desirable to study the Fierz-Pauli version of spinor theory with the same algebraic method as developed in this paper.