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Theory of Relativistic Supermultiplets. II. Periodicities in Hadron Spectroscopy*

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The theory for *all* fundamental particles proposed earlier is further discussed and the higher hadron supermultiplets and the corresponding mass levels of the various spins and parities have been calculated. Each supermultiplet is designated by the principal quantum numbers N_0 and N in the form $[N_0,N]$, where N_0 and N refer to the dimension numbers of the groups $SO(3,2)$ and $U(3,1)$, respectively. The generators of the groups $SO(3,2)$ and $U(3.1)$ constitute the algebraic basis of the fundamental wave equation. The results of the theory and the experimental data on hadron masses are found to be in excellent accord. The parities of the particles are obtained unambiguously, and the theory predicts 2^- and higher-spin mesons in addition to known ones. In the extreme limit of the parameters ρ and λ where $\rho=0$, $\lambda=0$, the [5,4], [5,6], [5,10], [5,15] levels each describe the photon, while the level [5,20] yields the graviton as a massless 2^+ particle. The limit $\rho=0$, $\lambda=0$ for the baryon and the lepton supermultiplets $[4_b,N]$, $[4_l,N]$, respectively, when coupled together implies the existence, at the lowest levels $[4_b,4]$ and $[4_l,4]$, of two $s=\frac{1}{2}$ two-component neutrinos. Higher-spin massive and massless leptons are also predicted.

I. INTRODUCTION

IN a previous paper,[1] to be referred to as I, a wave equation describing the energy levels of the free hadron was proposed. A preliminary numerical analysis led to the identification of the first eight free baryon levels in the mass relations

$$\frac{1}{M_\Sigma}-\frac{1}{M_{\Xi^0}}=\frac{1}{M_{\Delta^-}}-\frac{1}{M_{Y^+}} \qquad (1.1)$$

and

$$\frac{1}{4}\left(\frac{1}{M_n}-\frac{1}{M_\Sigma}\right)=\frac{1}{M_\Delta}-\frac{1}{M_{\Xi^-}} \qquad (1.2)$$

of the supermultiplets [4,6] and [4,1], [4,4], respectively, where each supermultiplet is designated by its principal quantum numbers N_0 [dimension number of $SO(3,2)$] and N [dimension number of $U(3,1)$] in the form $[N_0,N]$. It has been pointed out in I that the only physically relevant representations of $SO(3,2)$ are for $N_0=4$, 5, and 10 dimensions, while N of $U(3,1)$ ranges over 1, 4, 6, 10, 15, 20, ... However, there are two sets of 4×4 matrices which form the bases for the four-dimensional representations of the group $SO(3,2)$. These two sets are $\frac{1}{2}i\gamma_\mu$, $\frac{1}{2}\sigma_{\mu\nu}$ and $\frac{1}{2}i\gamma_5\gamma_\mu$, $\frac{1}{2}\sigma_{\mu\nu}$, both of which satisfy the commutation relations of the $SO(3,2)$. These two representations when combined can form the basis for the four-dimensional representation of $SU(2,2)$. Thus in the latter sense the two representations of $SO(3,2)$, to be referred to as B and L representations, respectively, are related within the $SU(2,2)$. The L representation $(\frac{1}{2}i\gamma_5\gamma_\mu,\frac{1}{2}\sigma_{\mu\nu})$ of $SO(3,2)$ will be applied to lepton classification, where we shall see that the parameter ρ in the lepton wave equation

$$(\tau_{\mu\nu}\gamma_5\gamma^\mu p^\nu - imc)\Psi_l=0 \qquad (1.3)$$

will have to be restricted to $\rho<1$. The matrices $\tau_{\mu\nu}$, as in the baryon wave equation

$$(\tau_{\mu\nu}\gamma^\mu p^\nu - iMc)\Psi_B=0\,, \qquad (1.4)$$

are defined as the linear combination of the generators $\Gamma_{\mu\nu}+\rho g_{\mu\nu}$ and $J_{\mu\nu}$ of the group $U(3,1)$,

$$\tau_{\mu\nu}=(\Gamma_{\mu\nu}+\rho g_{\mu\nu}+\lambda J_{\mu\nu})/\rho\,, \qquad (1.5)$$

where the range of the parameter λ for the leptons differs from its range of values for the baryons.

In this paper we shall continue with further investigation of the hadron wave equation

$$(\tau_{\mu\nu}\zeta^\mu p^\nu - iMc)\Psi_H=0\,, \qquad (1.6)$$

where the four matrices ζ_μ belong to $N_0=4$ (for baryons) or to $N_0=5$, 10 (for mesons) of the $SO(3,2)$ representations.

The quantized field theory of the present model implies that the spin and statistics connection of the elementary systems can be discussed within the framework of $SO(3,2)$-symmetry breaking. Each member of the supermultiplets $[4,N]_B$ and $[4,N]_l$ will obey the Fermi-Dirac statistics, while each member of the supermultiplets $[5,N]$ and $[10,N]$ will obey the Bose-Einstein statistics. The relation between breaking of the $SO(3,2)$ symmetry and the spin and statistics connection is based on the observation that the Dirac and Kemmer wave equations break the $SO(3,2)$ symmetry at a rate of p/Mc, where p is the momentum of the particle. This property, in our theory, is taken over and generalized further by including the group $U(3,1)$ (which has no double-valued representations) to induce a level structure to all the states of the hadron.

Like the Fermi-Dirac systems, the mesons also are of the two-prong type, i.e., both $[5,N]$ and $[10,N]$ are meson supermultiplets obeying the same symmetry laws, as contrasted to $[4,N]_l$ and $[4,N]_B$, which have different discrete symmetries. Both five- and ten-dimensional representations of $SO(3,2)$ obey the

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[1] B. Kurşunoğlu, Phys. Rev. D 1, 1115 (1970).

Kemmer-Duffin algebra

$$\beta_\mu\beta_\nu\beta_\rho+\beta_\rho\beta_\nu\beta_\mu=-(g_{\mu\nu}\beta_\rho+g_{\nu\rho}\beta_\mu)\,, \tag{1.7}$$

just as both γ_μ and $\gamma_5\gamma_\mu$ representations of $SO(3,2)$ obey the Dirac algebra

$$\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu=-2g_{\mu\nu}\,. \tag{1.8}$$

Operating with $p^\mu p^\nu p^\rho$ on both sides of (1.7), we obtain

$$(\beta^\nu p_\nu)[(\beta^\mu p_\mu)^2+p^2]=0\,.$$

Hence, using the Kemmer equation

$$(\beta^\mu p_\mu-imc)\phi=0\,, \tag{1.9}$$

we obtain

$$(p^2-m^2c^2)\phi=0\,,$$

which in momentum space implies

$$p^2=m^2c^2\,. \tag{1.10}$$

The same procedure applies for the Dirac algebra and leads to the statement (1.10). We thus see that $N_0=4_l$, $N_0=4_B$ and $N_0=5$, 10 are the only physically relevant representations of the group $SO(3,2)$.

Finally, we remark that within the group $SO(3,2)$ there exists a possibility of obtaining mesons whose discrete symmetries differ from $[5,N]$ and $[10,N]$. For example, the algebra (1.7) can be satisfied by taking $i\beta_5\beta_\mu$ in place of β_μ, where

$$\beta_5\beta_\mu+\beta_\mu\beta_5=0\,,\quad \beta_5\beta_{\mu\nu}=\beta_{\mu\nu}\beta_5\,,$$
$$\beta_{\mu\nu}=-i(\beta_\mu\beta_\nu-\beta_\nu\beta_\mu)\,. \tag{1.11}$$

Thus, as can easily be verified, the matrices $\beta_5\beta_\mu$ and $\beta_{\mu\nu}$ belong to the Lie algebra of $SO(3,2)$.

II. SUPERMULTIPLET [4,10]

The energy levels of the supermultiplets [4,4] and [4,6] were discussed in great detail in I. The method of I was useful only for the sake of a general presentation of the theory and also for a systematic spin decomposition of the wave equation. Here we shall introduce a much shorter technique which yields the same results and is based on the space-time symmetry classification of the wave functions leading to coupled sets of equations for each spin in a supermultiplet. For the definition of the various symbols and quantities the reader should consult I. In order to illustrate the method we shall recalculate the levels for the [4,6].

The wave equation for the supermultiplet [4,6], using the definitions of $\Gamma_{\mu\nu}$, $J_{\mu\nu}$ and also the wave function

$$\Psi_{\alpha a}=-\tfrac{1}{2}Q^{\mu\nu}{}_a\Psi_{\alpha[\mu\nu]}\,,$$

can be written as

$$[(\rho-1)\gamma^\mu p_\mu-imc\rho]\Psi_{[\mu\nu]}$$
$$+(1+i\lambda)(\gamma_\mu p^\rho\Psi_{[\rho\nu]}-\gamma_\nu p^\rho\Psi_{[\rho\mu]})$$
$$+(1-i\lambda)(p_\mu\gamma^\rho\Psi_{[\rho\nu]}-p_\nu\gamma^\rho\Psi_{[\rho\mu]})=0\,, \tag{2.1}$$

where a square bracket around the indices implies antisymmetry, while a curly bracket implies symmetry under permutations of the respective indices, and where the Dirac index α of the wave function has been suppressed. The two $\frac{1}{2}^+$ wave functions of the [4,6] can be represented in the form

$$\eta_1=\gamma^\mu\gamma^\nu\Psi_{[\mu\nu]}\,,\quad \eta_2=\frac{1}{p}\gamma^\mu p^\nu\Psi_{[\mu\nu]}\,. \tag{2.2}$$

Introducing the definitions (2.2) in (2.1), we obtain the coupled equations

$$[(\rho+2i\lambda-3)\gamma^\mu p_\mu-imc\rho]\eta_1-4(\rho+2i\lambda)p\eta_2=0\,, \tag{2.3}$$

$$[(\rho+2i\lambda-1)\gamma^\mu p_\mu+imc\rho]\eta_2-(1-i\lambda)p\eta_1=0\,. \tag{2.4}$$

Hence, eliminating η_1 or η_2, we find that both η_1 and η_2 satisfy the wave equation

$$[\gamma^\mu p_\mu-i\mathfrak{M}c]\eta_1=[\gamma^\mu p_\mu-i\mathfrak{M}c]\eta_2=0\,, \tag{2.5}$$

with

$$\mathfrak{M}c=[p^2(\rho^2+4\lambda^2+3)-m^2c^2\rho^2]/2mc\rho\,,$$

from which, as in I, we obtain the mass spectrum

$$m\rho/M(Z,B)=ZB+B[\rho^2+4(1+\lambda^2)]^{1/2}\,,$$
$$p^2=M^2c^2 \tag{2.6}$$

where

$$Z=\pm1 \quad\text{and}\quad B=\pm1 \text{ (baryon number)}.$$

The Z quantum number appears in all the baryon multiplets. The two $\frac{3}{2}^+$ wave functions

$$\eta_{1\mu}=\gamma^\nu\Psi_{\mu\nu}{}^+\,,\quad \eta_{2\mu}=(1/p)p^\nu\Psi_{\mu\nu}{}^+ \tag{2.7}$$

obey the coupled wave equations

$$[(\rho+i\lambda)\gamma^\mu p_\mu-imc\rho]\eta_{2\mu}+(1-i\lambda)p\eta_{1\mu}=0\,, \tag{2.8}$$

$$[(\rho+i\lambda-2)\gamma^\mu p_\mu+imc\rho]\eta_{1\mu}+2(\rho+i\lambda)p\eta_{2\mu}=0\,, \tag{2.9}$$

where we used the fact that the wave functions $\eta_{1\mu}$ and $\eta_{2\mu}$, as follows from the projection operators (6.2) of I, satisfy the relations

$$\gamma^\mu\eta_{1\mu}=0\,,\quad p^\mu\eta_{1\mu}=0\,,\quad \gamma^\mu\eta_{2\mu}=0\,,\quad p^\mu\eta_{2\mu}=0\,. \tag{2.10}$$

Equations (2.8) and (2.9) lead to

$$(\gamma^\mu p_\mu-i\mathfrak{M}c)\eta_{1\mu}=(\gamma^\mu p_\mu-i\mathfrak{M}c)\eta_{2\mu}=0\,, \tag{2.11}$$

where now

$$\mathfrak{M}=[m^2c^2\rho^2-p^2(\rho^2+\lambda^2)]/2mc\rho\,,$$

and the corresponding mass spectrum is

$$m\rho/M(Z,B)=ZB+B(\rho^2+\lambda^2+1)^{1/2}\,. \tag{2.12}$$

For the supermultiplet [4,10], using the definitions (A3.1) and (A3.2) of I for $\Gamma_{\mu\nu}$ and $J_{\mu\nu}$ of I, the wave equation can be split up in the form

$$(\gamma^\mu p_\mu-imc)\Psi_{5\mu}=0 \tag{2.13}$$

and

$$[(\rho-1)\gamma^\mu p_\mu - imc\rho]\Psi_{[\mu\nu]} + (1+i\lambda)(\gamma_\mu p^\rho \Psi_{[\rho\nu]} - \gamma_\nu p^\rho \Psi_{[\rho\mu]}) + (1-i\lambda)(p_\mu \gamma^\rho \Psi_{[\rho\nu]} - p_\nu \gamma^\rho \Psi_{[\rho\mu]}) = 0, \quad (2.14)$$

which is the same as the wave equation of the supermultiplet [4,6]. The four spin-$\frac{1}{2}$ wave functions of [4,10] can be defined as

$$\zeta_1 = \gamma^\mu \gamma^\nu \Psi_{[\mu\nu]}, \quad \zeta_2 = (1/p)\gamma^\mu p^\nu \Psi_{[\mu\nu]}, \quad \zeta_3 = \gamma^\mu \Psi_{5\mu}, \quad \zeta_4 = (1/p)p^\mu \Psi_{5\mu}, \quad (2.15)$$

where ζ_3 and ζ_4 are the wave functions[2] of a pair of $\frac{1}{2}^-$ particles and they satisfy the coupled set of equations

$$(\gamma^\mu p_\mu + imc)\zeta_3 + 2p\zeta_4 = 0, \quad (2.16)$$

$$(\gamma^\mu p_\mu - imc)\zeta_4 = 0. \quad (2.17)$$

From (2.14) we obtain the same mass spectrum (2.6) of the supermultiplet [4,6]. Equations (2.16) and (2.17) yield the mass m for the remaining two $\frac{1}{2}^-$ particles of [4,10]. In deriving Eqs. (2.16)–(2.18), we used the relations

$$\gamma^\mu(\gamma^\nu p_\nu) = -(\gamma^\nu p_\nu)\gamma^\mu - 2p^\mu,$$
$$\gamma^\mu\gamma^\nu(\gamma^\rho p_\rho) = (\gamma^\rho p_\rho)\gamma^\mu\gamma^\nu + 2(p^\mu\gamma^\nu - \gamma^\nu p^\mu).$$

The three $\frac{3}{2}$ states of [4,10] are represented by

$$\zeta_{1\mu} = \gamma^\nu \Psi_{[\mu\nu]}^+, \quad \zeta_{2\mu} = (1/p)p^\nu \Psi_{[\mu\nu]}^+, \quad \zeta_{3\mu} = \Psi_{5\mu}^+, \quad (2.18)$$

where $\zeta_{3\mu}$ represents a $\frac{3}{2}^-$ particle and where, as in the case of [4,6], the superscript (+) of Ψ signifies the action of the projection operator (6.2) of I on $\Psi_{[\mu\nu]}$ for the state of spin $\frac{3}{2}$, and each of (2.18) satisfies the relations of the type (2.10). The wave functions $\zeta_{1\mu}$, $\zeta_{2\mu}$ obey the same set of coupled equations as in [4,6] and the wave function $\zeta_{3\mu}$ satisfies the equation

$$(\gamma^\mu p_\mu - imc)\zeta_{3\mu} = 0. \quad (2.19)$$

Hence the mass of the $\frac{3}{2}^-$ particle is just m.

From the discussion in I of the [4,4] and [4,6] mass spectra, it is clear that the parameters ρ and λ may, for the supermultiplet [4,10], assume a different set of values from [4,4] and [4,6]. In I, it was also shown that the mass formulas (1.1) and (1.2) could be fitted with various other combinations of $s=\frac{1}{2}$ and $s=\frac{3}{2}$. All these fits correspond, of course, to different sets of values for the parameters ρ and λ. Thus the pair of $\frac{1}{2}^+$ and the pair of $\frac{3}{2}^+$ of [4,10] mass levels are given by (7.17) of I and therefore they can fit a mass relation of the type (1.1) with mass values different from those used in [4,6]. Furthermore, the parameters ρ and λ also assume values different from that of [4,6]. The remaining pair of $\frac{1}{2}^-$ and also the $\frac{3}{2}^-$ are of equal mass, represented by the parameter m. All of these masses of [4,10], as follows from (7.17) of I, can be expressed in a single expression

$$m\rho/M(Z,s) = ZB + B\{\rho^2 + Z^2(\lambda^2+1) \times [19/4 - s(s+1)]\}^{1/2}, \quad (2.20)$$

where $s=\frac{1}{2}, \frac{3}{2}$, B $(=\pm 1)$ is the baryon number, and the Z quantum number assumes the values -1, 0, 1. The mass formula (2.20) differs from (7.17) of I in the appearance of Z^2 in the square root, and also in the fact that the $Z=0$ and $s=\frac{1}{2}$ is doubly degenerate, while $Z=0$, $s=\frac{3}{2}$ is a singlet.

III. SUPERMULTIPLET [4,15]

In order to compare our model with the current theory and experiments and to derive further conclusions from it, we shall derive the mass spectra for the [4,15] and also for one of the [4,20]. The wave equation for the [4,15], as follows from the definitions (A4.1) and (A4.2) of I for the generators $\Gamma_{\mu\nu}$ and $J_{\mu\nu}$, respectively, can be written in the form

$$[(\rho-\tfrac{1}{2})\gamma^\mu p_\mu - imc\rho]\Psi_{[ab]} - \tfrac{1}{2}g_{ac}g_{bd}\gamma^\mu p_\mu \Psi_{[cd]} - \tfrac{1}{4}(1+i\lambda)\gamma^\mu p^\nu (J_{\mu\rho})_{ab}(J_\nu{}^\rho)_{cd}\Psi_{[cd]} - \tfrac{1}{4}(1-i\lambda)\gamma^\mu p^\nu (J_{\nu\rho})_{ab}(J_\mu{}^\rho)_{cd}\Psi_{[cd]} = 0, \quad (3.1)$$

where the Dirac spinor index is suppressed and where g_{ab} $(a, b=1, \ldots, 6)$ are the elements of the matrix Γ_5 defined by (A2.20) of I. In the derivation of (3.1), we used the relations

$$\tfrac{1}{2}J_{\mu\nu ab}J^{\mu\nu}{}_{cd} = \delta_{ad}\delta_{bc} - \delta_{ac}\delta_{bd} + g_{ad}g_{bc} - g_{ac}g_{bd} \quad (3.2)$$

between the generators $(J_{\mu\nu})_{ab}$ of the $N=6$ representation of $U(3,1)$. The relations (3.2), as will be seen, play a fundamental role in predicting negative-parity particles.

Now, on multiplying (3.1) through with the coefficients $(J_{\mu\nu})_{ab}$ and summing over a and b $(=1, 2, \ldots, 6)$ and using the relations

$$\Gamma_5 J_{\mu\nu} = J_{\mu\nu}\Gamma_5, \quad \tfrac{1}{4}(J_{\mu\nu})_{ab}(J_{\rho\sigma})_{ab} = g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma}, \quad (3.3)$$

we obtain the wave equation for the supermultiplet [4,6] given by (2.1). Hence we see that the [4,15] contains a mass spectrum of the same form as the spectrum of [4,6]. Thus the [4,6] subspectrum of the [4,15] consists of a pair of $\frac{1}{2}^+$ and a pair of $\frac{3}{2}^+$ particles described by the 24-component wave function

$$\eta_{[\mu\nu]} = \tfrac{1}{2}i \operatorname{Tr}(J_{\mu\nu}\Psi), \quad (3.4)$$

where Ψ represents the matrix form of the wave function $\Psi_{[ab]}$. The sum rule for the mass spectrum of $\eta_{[\mu\nu]}$ is the same as (1.1), but the parameters ρ, λ, and m may

[2] The quantity $\frac{1}{2}\sigma^{\lambda\omega}\Psi_{[\lambda\omega]}$, with $\lambda, \omega=1, 2, \ldots, 5$, is a scalar under $SO(3,1)$ transformations and a four-dimensional spinor under the $SL(2,C)$ transformations, where $\sigma_{\lambda\omega} = -\frac{1}{2}i(\gamma_\lambda\gamma_\omega - \gamma_\omega\gamma_\lambda)$ are the ten generators of the group $SO(3,2)$, and where $\{\gamma_\lambda, \gamma_\omega\} = -2g_{\lambda\omega}$. The $g_{\lambda\omega}$ is the metric tensor of the group $SO(3,2)$, $g_{55}=g_{44}=1$, $g_{5\mu}=0$, $g_{4j}=0$, $g_{jk}=-\delta_{jk}$, where $7=1, \ldots, 4$ and $j, k=1, 2, 3$. The γ_λ are the five Dirac matrices γ_μ and γ_5. Hence we see that the states $\Psi_{5\mu}$ describe negative-parity particles.

assume different values from that of the [4,6] and therefore the corresponding particle masses are also different from the [4,6] supermultiplet.

The remaining members of [4,15], as indicated by the reduction (A4.3) in I of the $\Psi_{[ab]}$, are described by the wave function

$$\phi_{\{\mu\nu\}}=\tfrac{1}{2}i\,\mathrm{Tr}(\Gamma_{\mu\nu}\Gamma_5\Psi)\,, \tag{3.5}$$

where $(\Gamma_{\mu\nu})_{\{ab\}}$ refer to the generators of the [4,6], and where

$$\phi_\mu{}^\mu=0\,. \tag{3.6}$$

The corresponding wave equation can be obtained by multiplying (3.1) through with the coefficients $(\Gamma_{\mu\nu})_{\{a'b\}}g_{a'a}$ and summing over a, b, and a' as

$$(\gamma^\rho p_\rho-imc)\phi_{\{\mu\nu\}}=0\,. \tag{3.7}$$

The appearance of the Γ_5 $(=g_{ab})$ in the definition (3.5) of the wave function $\phi_{\{\mu\nu\}}$ implies, clearly, that it describes negative-parity particles. This can more readily be seen in terms of space-time transformation properties of the wave function. Thus, using the definitions (see Appendix A of I)

$$\tfrac{1}{2}ig_{ab}Q_{\mu\nu a}Q_{\rho\sigma b}=\epsilon_{\mu\nu\rho\sigma}\,,\quad g_{ab}=(i/8)\epsilon^{\mu\nu\rho\sigma}Q_{\mu\nu a}Q_{\rho\sigma b}\,, \tag{3.8}$$

the wave function $\phi_{\{\mu\nu\}}$ as defined by (3.5) can be written as

$$\begin{aligned}\phi_{\{\mu\nu\}}&=\tfrac{1}{2}i(\Gamma_{\mu\nu})_{a'b}g_{a'a}\Psi_{ab}\\&=\tfrac{1}{4}\epsilon^{\gamma\delta\rho\sigma}(g_{\gamma\nu}\Psi_{[[\mu\delta],[\rho\sigma]]}+g_{\gamma\mu}\Psi_{[[\nu\delta],[\rho\sigma]]})\,,\end{aligned} \tag{3.9}$$

where

$$\Psi_{[[\mu\nu],[\rho\sigma]]}=\tfrac{1}{4}(Q_{\mu\nu a}Q_{\rho\sigma b}-Q_{\mu\nu b}Q_{\rho\sigma a})\Psi_{[ab]}\,. \tag{3.10}$$

The wave function $\phi_{\{\mu\nu\}}$ can be decomposed, according to its spin content, into the wave functions

$$\phi_1=(1/p^2)p^\mu p^\nu\phi_{\{\mu\nu\}}\,,\quad \phi_2=(1/p)\gamma^\mu p^\nu\phi_{\{\mu\nu\}} \tag{3.11}$$

for a pair of $\frac{1}{2}^-$ particles, and

$$\phi_{1\mu}=(1/p)p^\nu\phi_{\{\mu\nu\}}\,,\quad \phi_{2\mu}=\gamma^\nu\phi_{\{\mu\nu\}}\,, \tag{3.12}$$

with

$$\begin{aligned}p^\mu\phi_{1\mu}&=p\phi_1\,, & \gamma^\mu\phi_{1\mu}&=\phi_2\,,\\ p^\mu\phi_{2\mu}&=-p\phi_2\,, & \gamma^\mu\phi_{2\mu}&=0\end{aligned}$$

for a pair of $\frac{3}{2}^-$ particles, and

$$\zeta_\mu=\phi_{1\mu}-(p_\mu/p)\phi_1\,,\quad p^\mu\zeta_\mu=0 \tag{3.13}$$

for a $\frac{5}{2}^-$ particle. A further reason for the presence of negative-parity particles in the [4,15] is contained in the structure of the Pauli-Lubansky operator

$$\begin{aligned}W_{AB}/p^2=\tfrac{3}{4}\delta_{AB}-\tfrac{1}{4}(J_{\mu\nu})_A(J^{\mu\nu})_B\\ -\tfrac{1}{2}i\gamma^\mu\gamma^\nu(J_{\mu\nu})_{[AB]}\\ -(i/p^2)\gamma^\rho p_\rho(J_{\mu\nu})_{[AB]}\gamma^\mu p^\nu\,,\end{aligned} \tag{3.14}$$

where the subscripts A and B are used in place of $[ab]$ and $[cd]$, respectively, and where we used the relations

$$\begin{gathered}(J_{\mu\nu})_{[AC]}(J^{\mu\nu})_{[CB]}=-(J_{\mu\nu})_A(J^{\mu\nu})_B\,,\\ \delta_{AB}=\delta_{ac}\delta_{bd}-\delta_{ad}\delta_{bc}\,,\\ (J_{\mu\rho})_{[AC]}(J_\nu{}^\rho)_{[CB]}=-i(J_{\mu\nu})_{[AB]}-\tfrac{1}{4}g_{\mu\nu}(J_{\rho\sigma})_A(J^{\rho\sigma})_B\,.\end{gathered}$$

The second term on the right-hand side of (3.14) is defined by (3.2). Hence we see that the presence of the latter allows the presence of the negative-parity particles. This observation is, of course, compatible also with the structure of the wave equation (3.1). Equation (3.1), because of its second term, can be multiplied through by $g_{a'b}$ to see the negative-parity structure in all the terms.

From (3.7) and the definitions (3.11)–(3.13) of the wave functions we obtain the coupled set of wave equations

$$\begin{aligned}&(\gamma^\mu p_\mu-imc)\phi_1=0\,, && (\gamma^\mu p_\mu+imc)\phi_2+2p\phi_1=0\,,\\ &(\gamma^\mu p_\mu-imc)\phi_{1\rho}=0\,, && (\gamma^\mu p_\mu+imc)\phi_{2\rho}+2p\phi_{1\rho}=0\,,\\ &(\gamma^\mu p_\mu-imc)\zeta_\rho=0\,.\end{aligned} \tag{3.15}$$

Hence we see that all of the above five negative-parity particles have equal masses represented by m.

IV. SUPERMULTIPLET [4,20]

It has been pointed out in I that the group $U(3,1)$ has two 20-dimensional representations defined by $\Psi_{\{ab\}}$ $(\Psi_{\{aa\}}=0)$ and $\Psi_{[abc]}$. Here we shall discuss the symmetric wave function $\Psi_{\{ab\}}$ and defer the fully antisymmetric wave function $\Psi_{[abc]}$ or its dual representation $\frac{1}{6}\epsilon_{abcdef}\Psi_{[def]}$ to the future publications in this series. The fully antisymmetric tensor ϵ_{abcdef}, as for $\epsilon_{\mu\nu\rho\sigma}$, has only one nonvanishing component, where the indices $a, b, \ldots, f$ $(=1, 2, \ldots, 6)$ have different values.

The wave equation of the [4,20], as follows from (A5.1) and (A5.2) of I, can be written as

$$\begin{aligned}&[(\rho+\tfrac{1}{2})\gamma^\mu p_\mu-imc\rho]\Psi_{\{ab\}}-\tfrac{1}{3}g_{ab}\gamma^\mu p_\mu\zeta\\ &\quad-\tfrac{1}{2}(1+i\lambda)\gamma^\rho p^\sigma(\Gamma_{\rho\lambda})_{\{ab\}}\phi_\sigma{}^\lambda\\ &\quad-\tfrac{1}{2}(1-i\lambda)\gamma^\rho p^\sigma(\Gamma_{\sigma\lambda})_{\{ab\}}\phi_\rho{}^\lambda\\ &\quad\pm(1/\sqrt{3})[g_{ab}\gamma^\rho p^\sigma\phi_{\rho\sigma}+(\Gamma_{\rho\sigma})_{\{ab\}}\gamma^\rho p^\sigma\zeta]=0\,,\end{aligned} \tag{4.1}$$

where

$$\zeta=\tfrac{1}{2}i\,\mathrm{Tr}(\Gamma_5\Psi)\,,\quad \phi_{\mu\nu}=\tfrac{1}{2}\,\mathrm{Tr}(\Gamma_{\mu\nu}\Psi)\,,\quad \phi_\mu{}^\mu=0\,, \tag{4.2}$$

and where we used the relations

$$\tfrac{1}{2}(\Gamma_{\mu\nu})_{\{ab\}}(\Gamma^{\mu\nu})_{\{cd\}}=\delta_{ad}\delta_{bc}+\delta_{ac}\delta_{bd}-\tfrac{1}{3}\delta_{ab}\delta_{cd}-\tfrac{2}{3}g_{ab}g_{cd}\,. \tag{4.3}$$

It must be observed that, because of the noncompactness of the group $U(3,1)$, the wave function ζ does not vanish, viz.,

$$\zeta=\tfrac{1}{2}g_{ab}\Psi_{\{ab\}}=\tfrac{1}{2}(\Psi_{jj}-\Psi^{jj})=\tfrac{1}{8}\epsilon^{\mu\nu\rho\sigma}\Psi_{\{[\mu\nu],[\rho\sigma]\}}\,, \tag{4.4}$$

where

$$\Psi^{jj}=\Psi^{11}+\Psi^{22}+\Psi^{33}=(\Psi_{jj})^*,$$

so that Ψ^{jj} is just the complex conjugate of Ψ_{jj}. In all the above relations the Dirac spinor indices have been suppressed. The ζ is the wave function of a $\frac{1}{2}^-$ particle, where $\Psi_{\{[\mu\nu],[\rho\sigma]\}}$, the space-time tensor-spinor form of the [4,20] wave function, is defined by

$$\Psi_{\{[\mu\nu],[\rho\sigma]\}}=-\tfrac{1}{4}(Q_{\mu\nu a}Q_{\rho\sigma b}+Q_{\mu\nu b}Q_{\rho\sigma a})\Psi_{\{ab\}},$$
$$g^{\mu\rho}g^{\nu\sigma}\Psi_{\{[\mu\nu],[\rho\sigma]\}}=0. \tag{4.5}$$

Hence, by contracting, we obtain

$$g^{\nu\sigma}\Psi_{\{[\mu\nu],[\rho\sigma]\}}=\tfrac{1}{2}(\Gamma_{\mu\nu})_{\{ab\}}\Psi_{\{ab\}}, \tag{4.6}$$

which is the same as (4.2). By using the definition (4.6) in (4.1) and noting that $g_{ab}(\Gamma_{\mu\nu})_{\{ab\}}=0$, we obtain a wave equation for the reduced wave functions $\phi_{\{\mu\nu\}}$ and ζ,

$$\begin{aligned}&[(\rho+\tfrac{1}{2})\gamma^\rho p_\rho-imc\rho]\phi_{\{\mu\nu\}}\\&\quad-(1+i\lambda)p(\gamma_\mu\phi_{1\nu}+\gamma_\nu\phi_{1\mu}-\tfrac{1}{2}g_{\mu\nu}\phi_2)\\&\quad-(1-i\lambda)(p_\mu\phi_{2\nu}+p_\nu\phi_{2\mu}-\tfrac{1}{2}g_{\mu\nu}p\phi_2)\\&\quad\pm(2/\sqrt{3})(\gamma_\mu p_\nu+\gamma_\nu p_\mu-\tfrac{1}{2}g_{\mu\nu}\gamma^\rho p_\rho)\zeta=0,\end{aligned} \tag{4.7}$$

where

$$\begin{aligned}\phi_{1\mu}&=(1/p)p^\rho\phi_{\mu\rho}, & \phi_{2\mu}&=\gamma^\rho\phi_{\mu\rho},\\ \phi_1&=(1/p)p^\mu\phi_{1\mu}, & \phi_2&=(1/p)p^\mu\phi_{2\mu},\end{aligned} \tag{4.8}$$

and where we used the relations

$$\tfrac{1}{4}(\Gamma_{\mu\nu})_{ab}(\Gamma_{\rho\sigma})_{cd}=g_{\mu\sigma}g_{\nu\rho}+g_{\mu\rho}g_{\nu\sigma}-\tfrac{1}{2}g_{\mu\nu}g_{\rho\sigma}. \tag{4.9}$$

Operating on the (4.7) by $p^\mu p^\nu$, $\gamma^\mu p^\nu$ and also on the (4.1) by g_{ab}, we obtain the coupled set of equations

$$[(\rho-\tfrac{3}{2}-2i\lambda)\gamma^\mu p_\mu-imc\rho]\phi_1+(2i\lambda-1)p\phi_2 \pm\sqrt{3}\gamma^\mu p_\mu\zeta=0, \tag{4.10}$$

$$[(\rho-\tfrac{1}{2}-2i\lambda)\gamma^\mu p_\mu+imc\rho]\phi_2+2(\rho-\tfrac{5}{2}-3i\lambda)p\phi_1 \pm3\sqrt{3}p\zeta=0, \tag{4.11}$$

$$[(\rho-\tfrac{1}{2})\gamma^\mu p_\mu-imc\rho]\zeta\pm\sqrt{3}p\phi_2=0, \tag{4.12}$$

which describe two $\frac{1}{2}^+$ (ϕ_1,ϕ_2) and one $\frac{1}{2}^-$ (ζ) particles. From (4.10)–(4.12) it follows that the three wave functions ϕ_1, ϕ_2, and ζ satisfy the wave equation

$$(\gamma^\mu p_\mu-iMc)\phi=0, \tag{4.13}$$

where $\phi\equiv\phi_1$, ϕ_2 or ζ, and where

$$\begin{aligned}&M/m\rho=-(g+x^2)/(h+fx^2),\\&f=\tfrac{3}{2}-\rho,\quad g=-(\rho^2-3\rho+57/4+8\lambda^2),\\&h=(\rho-\tfrac{1}{2})(8\lambda^2+\rho^2-4\rho+23/4)+3(\rho+\tfrac{1}{2}),\\&x=mc\rho/p.\end{aligned} \tag{4.14}$$

From (4.13) and (4.14) we obtain the cubic equations

$$x^3\pm fx^2+gx\pm h=0, \tag{4.15}$$

where the + and − signs of f and h refer to particles and antiparticles, respectively. Thus, if we let x_1, x_2, and x_3 represent the roots of the cubic equation (4.15) with the + sign, then $-x_1$, $-x_2$, and $-x_3$ are the roots of (4.15) with the − sign.

The roots x_1, x_2, and x_3 satisfy the linear relation

$$x_1+x_2+x_3=-f. \tag{4.16}$$

Hence we obtain the "sum rule"

$$\frac{1}{M_1}+\frac{1}{M_2}+\frac{1}{M_3}=\frac{1}{m}-\frac{3}{2m\rho}, \tag{4.17}$$

where, as will be shown, the M_1 and M_2 refer to negative masses, and where m and ρ on the right-hand side will be eliminated in favor of the masses for the $\frac{3}{2}^+$, $\frac{3}{2}^-$, and $\frac{5}{2}^-$ particles.

The ± signs in Eqs. (4.1), (4.7), and (4.10)–(4.12) originate from the $N=20$ representation of the $\Gamma_{\mu\nu}$ as given by (A5.1) of I, where it has been pointed out that the + sign (or − sign) can be transformed into − sign (or + sign) by a parity transformation on the two pairs of indices of the $(\Gamma_{\mu\nu})_{\{\{ab\},\{cd\}\}}$ affected by $(\Gamma_{44})_{\{ab\}}(\Gamma_{44})_{\{cd\}}$. It is interesting that the commutation relations of the group $U(3,1)$ contain, for the $N=20$ representations, those solutions in its structure which make the emergence of a negative-parity state ζ along with a pair of positive-parity states ϕ_1 and ϕ_2, as seen from (4.10) and (4.11), a Lorentz-invariant fact.

Now from the theory of cubic equations it follows that the three roots of (4.15) are given by

$$x=4(-\tfrac{1}{3}a)^{1/2}(\tfrac{1}{2}-Z^2)\cos(\tfrac{1}{3}Z\pi+\tfrac{1}{3}\phi)-\tfrac{1}{3}Bf, \tag{4.18}$$

where $Z=0,\pm1$, $B=\pm1$,

$$x=(m\rho/M)B\begin{pmatrix}-1\\-1\\1\end{pmatrix},$$

and

$$a=-4(\tfrac{1}{3}f^2+2\lambda^2+3),$$
$$b=B[8(1+\lambda^2)-(16/27)f(f^2+9\lambda^2)],$$
$$\cos\phi=\frac{(\sqrt{27})b}{2a\sqrt{(-a)}},\quad \Delta=4a^3+27b^2,$$
$$\begin{aligned}-(\tfrac{1}{4})^4\Delta=&1+6(1+\lambda^2)+(21/4)(1+\lambda^2)^2+8(1+\lambda^2)^3\\&+f(1+\lambda^2)(f^2+9\lambda^2)+f^2(\lambda^4+12\lambda^2+9+f^2),\end{aligned}$$

and therefore

$$-\Delta>0, \tag{4.19}$$

so that the cubic equations have three pairs of real and unequal roots. The functions y $(=Bx^3+fx^2+Bgx+h)$ assume their maximum (minimum) and minimum (maximum) values at the points

$$\chi_1=-\tfrac{1}{3}[f-\sqrt{(3a)}]\quad(>0)$$

and

$$\chi_2=-\tfrac{1}{3}[f+\sqrt{(-3a)}]\quad(<0),$$

respectively. At the points χ_1 and χ_2, the second derivative y'' for $B=1$ satisfies the inequalities $y''(\chi_1)>0$ and $y''(\chi_2)<0$. Therefore, because of the relations $|\chi_2|>|\chi_1|$, $h>0$ of the three roots of $y=0$ (with $B=1$), the two of them are negative and one is positive, while for the case $B=-1$ the two roots are positive and one is negative. Hence we see that we have three positive and three negative roots corresponding to baryons and antibaryons, respectively. These roots are as given by (4.18).

The higher-spin reduction of Eq. (4.7), after using the projection operator (8.7) of I for the spin $\frac{3}{2}$, consists of the coupled set of equations

$$[(\rho-\tfrac{1}{2}-i\lambda)\gamma^\nu p_\nu-imc\rho]\phi_{1\mu}{}^+-(1-i\lambda)p\phi_{2\mu}{}^+=0, \quad (4.20)$$

$$[(\rho+\tfrac{3}{2}-i\lambda)\gamma^\nu p_\nu+imc\rho]\phi_{2\mu}{}^+ +2(\rho-\tfrac{5}{2}-3i\lambda)p\phi_{1\mu}{}^+=0, \quad (4.21)$$

which describe two $\frac{3}{2}^+$ particles. The mass spectrum corresponding to (4.20) and (4.21) is given by

$$m\rho/M=ZB+B[\tfrac{3}{2}+s(s+1)+\rho(\rho-1)+5\lambda^2]^{1/2}, \quad (4.22)$$

where $s=\frac{3}{2}$ and $Z=\pm1$, $B=\pm1$. Hence, putting $M=M_4$ for $Z=-1$, $B=1$ and $M=M_5$ for $Z=1$, $B=1$, we can write the relation

$$\frac{2}{m\rho}=\frac{1}{M_5}-\frac{1}{M_4}. \quad (4.23)$$

On combining (4.17) and (4.23) we obtain

$$\frac{1}{M_1}+\frac{1}{M_2}+\frac{1}{M_3}=\frac{3}{4}\left(\frac{1}{M_4}-\frac{1}{M_5}\right)+\frac{1}{m}, \quad (4.24)$$

where m will be eliminated with the use of the mass relations for the other members of the [4,20].

The wave functions of the remaining two $\frac{3}{2}^-$ and three $\frac{5}{2}^-$ baryons can be constructed from

$$\begin{aligned}\zeta_{\{\mu\nu\}}&=\tfrac{1}{2}i\,\mathrm{Tr}(\Gamma_{\mu\nu}\Gamma_5\Psi^+),\\ \zeta_{[\mu\nu]}&=\tfrac{1}{2}i\,\mathrm{Tr}(J_{\mu\nu}\Gamma_5\Psi^+),\end{aligned} \quad (4.25)$$

where the superscript $+$ signifies the absence of spin-$\frac{1}{2}$ states in the wave functions (4.25). Now using the relations

$$\mathrm{Tr}(\Gamma_{\mu\nu}\Gamma_5\Gamma_{\rho\sigma})=\mathrm{Tr}(J_{\mu\nu}\Gamma_5\Gamma_{\mu\nu})=0 \quad (4.26)$$

and the wave equation (4.1) of the [4,20], we find that the wave functions (4.25) satisfy the equation

$$[(\rho+\tfrac{1}{2})\gamma^\sigma p_\sigma-imc\rho]\zeta_{\{\mu\nu\}} =[(\rho+\tfrac{1}{2})\gamma^\sigma p_\sigma-imc\rho]\zeta_{[\mu\nu]}=0. \quad (4.27)$$

The $\Gamma_{\mu\nu}$ and $J_{\mu\nu}$ in (4.25) and (4.26) refer to $N=6$ representation of $U(3,1)$. From (4.27) it follows that the two $\frac{3}{2}^-$ and the three $\frac{5}{2}^-$ baryons described by the wave functions (4.25) have equal masses given by

$$M_6=Bm\rho/(\rho+\tfrac{1}{2}). \quad (4.28)$$

It is interesting to observe that the mass (4.28) was obtained in I for the spin singlet $\frac{3}{2}^+$ of the supermultiplet [4,4]. This kind of periodicity in the baryon spectrum is one more novel aspect of the present theory. Now, on eliminating the parameters ρ and m between (4.23), (4.24), and (4.28) and remembering that M_1 and M_2 in (4.17) referred to negative mass, we obtain the sum rule of the supermultiplet [4,20] in the form

$$\frac{1}{M_1}+\frac{1}{M_2}-\frac{1}{M_3}=\frac{1}{M_5}-\frac{1}{M_4}-\frac{1}{M_6}, \quad (4.29)$$

where M_1, M_2, and M_3 are spin $\frac{1}{2}$, M_4 and M_5 spin $\frac{3}{2}$, M_6 both spin $\frac{3}{2}$ and spin $\frac{5}{2}$, and, furthermore, M_3 and M_6 refer to the masses of the negative-parity baryons of the [4,20]. All of the masses appearing in (4.29) are now positive.

V. DISCUSSION OF BARYON SPECTRUM

The mass levels of the supermultiplets [4,N] depend on the baryon number B, the spin s, and also the space-time quantum number Z. The dependence on Z, for some supermultiplets, is of the form BZ. Therefore, a different classification can be based on a quantum number $\mathcal{A}$ defined by

$$B+Z=\mathcal{A}. \quad (5.1)$$

We may thus eliminate the baryon number B in favor of $\mathcal{A}$ and Z. For the supermultiplets calculated so far, the number $\mathcal{A}$ assumes the values 0, 1, and 2 for the baryons and 0, -1, and -2 for the antibaryons. The additive character of $\mathcal{A}$ as defined by (5.1) is similar to the hypercharge quantum number of $SU(3)$, but it is quite clear that $\mathcal{A}$ is not related to the hypercharge.

Each member of a supermultiplet with fixed parity $\mathcal{P}$ $(=\pm1)$ can be depicted in a space spanned by the quantum numbers $\mathcal{A}$, Z, and s. The numbers $\mathcal{A}$ and Z belong to the category of external quantum numbers like the spin and parity of the Poincaré group. The four numbers $\mathcal{A}$, Z, s, and $\mathcal{P}$, because of the absence of a definition for the electric charge, do not completely specify the baryon. We need to discover the internal quantum numbers connecting different supermultiplets. In other words, the members of an "internal supermultiplet" consisting of particles with the same spin and parity lie in different space-time supermultiplets described by the four numbers $\mathcal{A}$, Z, s, and $\mathcal{P}$. For example, the mass relations

$$m\rho/M=ZB+B[(\rho-\tfrac{1}{2})^2+3(\lambda^2+1)]^{1/2},$$
$$m\rho/M=ZB+B[\rho^2+4(\lambda^2+1)]^{1/2}$$

for the $s=\frac{1}{2}$ members of the [4,4] and [4,6], respectively, do indicate an internal structure in their dependence on λ^2+1 and on ρ. The same type of ρ and λ^2+1 dependence arises for the pair of $\frac{3}{2}^+$ part of the

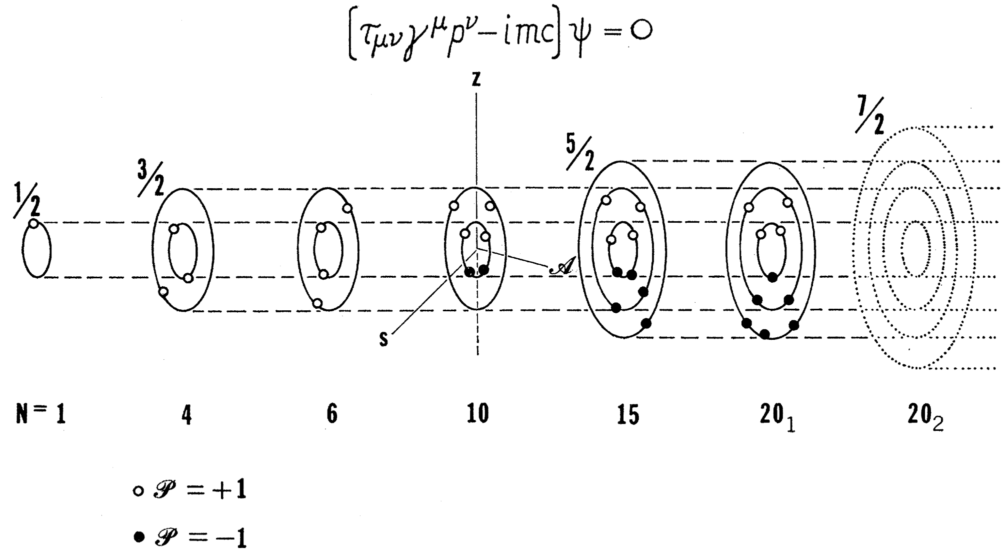

FIG. 1. Baryon supermultiplets. The planes perpendicular to the axes contain baryons of different spins and parities, forming the space-time supermultiplets. A hyperplane through the central axis may be constructed in such a way that it contains baryons of equal spin and parity, hence defining an internal supermultiplet.

[4,20] as

$$m\rho/M = ZB + B[(\rho-\tfrac{1}{2})^2 + 5(1+\lambda^2)]^{1/2}.$$

However, it is not, as yet, clear enough to deduce some new "numbers" from them.

Figure 1 constitutes a diagrammatic illustration of the space-time and internal structure of the baryons. The perpendicular intersections of the coaxial "cylinders" for fixed internal quantum numbers yield concentric "spherical" shells. Each spherical shell contains baryons of fixed spin angular momentum with the appropriate values of $\mathcal{A}$, Z, and $\mathcal{P}$. Particles of negative parity are placed on the plane $Z=0$, but not all particles with $Z=0$ have negative parity. Thus the space-time supermultiplets result from the perpendicular intersection of the coaxial cylinders by a plane specified by fixed internal quantum numbers, and no two members of a supermultiplet assume the same values of all the four numbers $\mathcal{A}$, Z, s, and $\mathcal{P}$ and carry the same electric charge. In the latter sense, in the space spanned by $\mathcal{A}$, Z, and s (together with the parity and electric charge), the baryon classification behaves like the electron configuration in the atoms determined by the Pauli exclusion principle operating in the position and spin spaces. In the present context, the many-particle behavior arises from the assignments of different values (not different space-time coordinates) of the quantum numbers $\mathcal{A}$, Z, s, $\mathcal{P}$, and electric charge, etc., to the various particle states described by the wave function of a supermultiplet. *Thus a "super-exclusion principle" in the above sense may be the basis for the proliferation of the strong-interaction resonance states. Therefore, in the space of* $\mathcal{A}$, Z, s, $\mathcal{P}$, *and electric charge all the baryons are identical particles and no two baryons can occupy the same state. Thus, if the quantum numbers of one baryon are interchanged as a group with those of another, the baryon wave function must change sign.*

An "internal supermultiplet" lies in a plane containing the axis of the coaxial cylinders and the baryons of equal spin and parity. It is quite conceivable that different internal supermultiplets lie on planes oriented with respect to one another with fixed angles between them. Such relationships between these internal planes is expected to emerge from the theory as a result of the representations of some internal group.

Another regularity to be observed refers to the recurrence of the low-lying mass structures at the higher mass levels. Thus the [4,1] mass level recurs in the supermultiplet [4,10] for the pair of $\frac{1}{2}^-$ and also for the pair $\frac{3}{2}^-$. Furthermore, the mass relation of the $s=\frac{3}{2}$ member of the [4,4] recurs for the mass of the $s=\frac{3}{2}$ and $s=\frac{5}{2}$ members of the [4,20], while the entire mass structure of the [4,6] recurs in the mass spectra of the [4,10] and [4,15]. This type of recurrence may be used to describe decay processes of higher-spin states into the lower-spin states which lie in different supermultiplets. Thus the intersupermultiplet transitions are induced according to the above recurrence rules. The above interpretation of the theory had also the expectations pointed out there have constituted the bases for the choice of this paper's title.

VI. MESON SPECTRUM

In accordance with the discussion in I, the wave functions of the massive integral-spin supermultiplets can be represented as

$$\begin{array}{cccccc} \phi_{\lambda}\,, & \phi_{\lambda\mu}\,, & \phi_{\lambda a}\,, & \phi_{\lambda[\omega\eta]}\,, & \phi_{\lambda[ab]}\,, & \phi_{\lambda\{ab\}}\,, \\ N=1\,, & N=4\,, & N=6\,, & N=10\,, & N=15\,, & N=20\,, \end{array} \tag{6.1a}$$

$$\begin{array}{cccc} \phi_{\lambda[abc]}\,, & \phi_{\lambda[[\eta\omega],[\xi\zeta]]}\,, & \phi_{\lambda\{abc\}}\,, & \phi_{\lambda\{[\eta\omega],[\xi\zeta]\}}\,,\ldots, \\ N=20\,, & N=45\,, & N=50\,, & N=55\,, \end{array} \tag{6.1b}$$

with

$$\phi_{\lambda\{aa\}}=0\,,\quad \phi_{\lambda\{abb\}}=0\,,$$

where the subscript λ is acted on by the $SO(3,2)$ and its subgroup of homogeneous Lorentz group transformations and ranges from 1 to 5 for $N_0=5$, or from 1 to 10 for $N_0=10$ representations of the $SO(3,2)$. All the other indices are as defined in I. Thus the mesons are classified according to five- and ten-dimensional representations of the group $SO(3,2)$. Both classifications contain $0^{\pm}$, $1^{\pm}$, $2^{\pm}$, ..., mesons. The properties of these two sequences of mesons and their dynamical and symmetry differences may lead to some new information on these particles.

Appendix A 7 of I contains a brief discussion of the group $SO(3,2)$ and its integral-spin representation in terms of the β matrices. The β matrices for $N_0=5$ representation of $SO(3,2)$, in terms of tensor notation, can be represented by

$$(\beta_\mu)_5{}^\nu=i\delta_\mu{}^\nu\,,\quad (\beta_\mu)_\nu{}^5=ig_{\mu\nu}\,,\quad (\beta_\mu)_\rho{}^\sigma=0\,, \tag{6.2}$$

where

$$\mu,\,\nu,\,\rho,\,\sigma=1,\,2,\,3,\,4.$$

These relations aid considerably in the discussion of the meson wave equation

$$(\tau_{\mu\nu}\beta^\mu p^\nu-imc)\phi=0 \tag{6.3}$$

describing the bare supermultiplets [5,N] and [10,N]. The total angular momentum operators of these supermultiplets can be represented as

$$G_{\mu\nu}=L_{\mu\nu}+\beta_{\mu\nu}+J_{\mu\nu}\,, \tag{6.4}$$

where, as before, $L_{\mu\nu}=x_\mu p_\nu-x_\nu p_\mu$ and $J_{\mu\nu}$ are the six generators of $SU(3,1)$. The $\beta_{\mu\nu}$ are the spin matrices in the five- or ten-dimensional space of the $SO(3,2)$ transformations. The operators $G_{\mu\nu}$ commute with the $\tau_{\mu\nu}\beta^\mu p^\nu$. Under a Lorentz transformation of the coordinates $x^\mu\longrightarrow\Lambda_\nu{}^\mu x^\nu$, the wave function of the meson supermultiplet transforms according to

$$\phi(x)\longrightarrow S(\Lambda)\phi(\Lambda^{-1}x)=\phi'(x')\,, \tag{6.5}$$

where the nonunitary operator $S(\Lambda)$ is defined by

$$S(\Lambda)=\exp[-\tfrac{1}{2}if^{\mu\nu}(\beta_{\mu\nu}+J_{\mu\nu})]\,, \tag{6.6}$$

and it acts on both the $SO(3,2)$ and $U(3,1)$ indices of ϕ. The Lorentz matrix is as defined by Eqs. (2.5) and (2.6) of I. The Lorentz invariance of the wave equation (6.3) further requires the transformation rules

$$S(\Lambda^{-1})\tau_{\mu\nu}\beta^\nu S(\Lambda)=\Lambda_\mu{}^\rho\tau_{\rho\nu}\beta^\nu\,, \tag{6.7}$$

which entail the statement that the operator $\tau_{\mu\nu}\beta^\nu$ transforms as a vector. The statement (6.7) is valid also under the unitary representations of the Poincaré group. As in I, it is easy to show that the wave equation (6.3) is, for all the supermultiplets [5,N] and [10,N], Poincaré invariant, and therefore the Poincaré group is implemented unitarily.

The supermultiplet [5,1] is described by the Kemmer wave equation

$$(\beta^\mu p_\mu-imc)\phi=0\,. \tag{6.8}$$

The discussion in Appendix A 7 of I on the composite structure of its spin shows that the meson of (6.8) is of 0^- type. Equation (6.8) is invariant under the reducible Lorentz transformations $D^{(\frac{1}{2},\frac{1}{2})}+D^{(0,0)}$ generated by the $\beta_{\mu\nu}$ which leave $\beta^\lambda\phi_\lambda=\beta_5\phi_5+\beta^\mu\phi_\mu$ also invariant.

By using the definitions (6.2) in (6.8), we obtain the coupled equations

$$p^\mu\phi_\mu-mc\phi_5=0\,,\quad p_\mu\phi_5-mc\phi_\mu=0\,. \tag{6.9}$$

Hence

$$(p^2-m^2c^2)\phi_5=0\,,\quad (p^2-m^2c^2)p^\mu\phi_\mu=0\,.$$

The Pauli-Lubansky operator for Eq. (6.8) is given by

$$W/p^2=2[1+(1/p^2)(\beta^\mu p_\mu)^2]\,,\quad W^2=2Wp^2$$

which can be used to construct spin-0 and spin-1 projection operators

$$\Gamma_0=\tfrac{1}{2}(2-W/p^2)\,,\quad \Gamma_1=\tfrac{1}{2}W/p^2. \tag{6.10}$$

Thus from $\Gamma_0(\beta^\mu p_\mu)=\beta^\mu p_\mu$, $\Gamma_1(\beta^\mu p_\mu)=0$ it follows that $\phi(s=1)=\Gamma_1\phi=0$. Furthermore, the corresponding current density is given by

$$J_\mu=-i\phi\beta_\mu\phi=\phi_5{}^\dagger\phi_\mu+\phi_\mu{}^\dagger\phi_5 = \frac{i\hbar}{mc}\left(\phi_5{}^\dagger\frac{\partial\phi_5}{\partial x^\mu}-\frac{\partial\phi_5{}^\dagger}{\partial x^\mu}\phi_5\right). \tag{6.11}$$

Hence Eq. (6.8) corresponds to a 0^- meson described by a wave function ϕ_λ ($=\phi_5$, ϕ_μ).

VII. SUPERMULTIPLET [5,4]

From the meson wave equation (6.3) and the definitions (6.2) we obtain the coupled equations

$$(\tau_{\mu\nu})_\rho{}^\sigma p^\nu\phi_\sigma{}^\mu-mc\phi_{5\rho}=0\,, \tag{7.1}$$

$$(\tau_{\mu\nu})_{\rho}{}^{\sigma}p^{\nu}\phi_{5\sigma}-mc\phi_{\mu\rho}=0. \tag{7.2}$$

In general, to obtain the mass levels corresponding to the various spins within a supermultiplet, one uses the spin projection operators. For the [4,5] the corresponding projection operators are given by

$$\Gamma_0=\frac{1}{12}\left(2-\frac{W}{p^2}\right)\left(6-\frac{W}{p^2}\right), \tag{7.3}$$

$$\Gamma_1=-\frac{1}{8}\frac{W}{p^2}\left(6-\frac{W}{p^2}\right), \tag{7.4}$$

$$\Gamma_2=\frac{1}{24}\frac{W}{p^2}\left(\frac{W}{p^2}-2\right), \tag{7.5}$$

where

$$p^2=p_\mu p^\mu=p_4{}^2-\mathbf{p}^2,\quad \Gamma_0+\Gamma_1+\Gamma_2=1,$$
$$(W/p^2)^3=8(W/p^2)^2-12(W/p^2),$$

and W is the Pauli-Lubansky operator of the supermultiplet [5,4] defined by

$$W=W_\mu W^\mu,\quad W^\mu=\tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}S_{\rho\sigma}p_\nu,\quad S_{\rho\sigma}=\beta_{\rho\sigma}+J_{\rho\sigma}.$$

The operator Γ_0 projects out the spins 1 and 2, and retains the spin 0. Similar actions are performed by the Γ_1 and Γ_2 as indicated by their subscripts 1 and 2, respectively. The spin matrices $S_{\mu\nu}$ are the sum of the spin matrices of the $SO(3,2)$ for $N_0=5$ and of the $U(3,1)$ for $N=4$, respectively. The $\beta_{\mu\nu}$ can be written as

$$\beta_{\mu\nu}=\Lambda_+M_{\mu\nu}, \tag{7.6}$$

where the 5×5 projection operators $\Lambda_\pm$ are defined by

$$\Lambda_\pm=\tfrac{1}{2}(1\pm\beta_5), \tag{7.7}$$

and where the 4×4 matrices $M_{\mu\nu}$ are of the same type as $J_{\mu\nu}$ defined by (8.2) of I. However, the $M_{\mu\nu}$ introduced in (7.6) commute with the $J_{\mu\nu}$ of the $U(3,1)$ for $N=4$.

Now, using the definitions (8.24) of I, the spin matrices of the [5,4] can be written as

$$\mathbf{S}=\tfrac{1}{2}\Lambda_-(\mathbf{T}_a+\mathbf{T}_b)+\tfrac{1}{2}\Lambda_+(\mathbf{T}_a+\mathbf{T}_b+\boldsymbol{\tau}_a+\boldsymbol{\tau}_b), \tag{7.8}$$

where the 4×4 matrices T_{aj} and T_{bj}, as in the definitions of τ_{aj} and τ_{bj} in terms of $M_{\mu\nu}$, are defined by

$$\mathbf{T}_a=\mathbf{M}'-i\mathbf{N}',\quad \mathbf{T}_b=\mathbf{M}'+i\mathbf{N}', \tag{7.9}$$

and they satisfy the commutation and anticommutation relations (8.25) of I. The two terms in the parentheses of (7.8), because of the completeness relations

$$\Lambda_++\Lambda_-=1,\quad \Lambda_+\Lambda_-=\Lambda_-\Lambda_+=0,$$

commute, and the first and second terms result from adding two and four one-half-spin angular momenta. Because of their transformation properties under parity operation, as discussed in Sec. VIII of I, the resultant spin angular momenta refer to the 0^-, 1^+ mesons. In fact, this spin and parity assignment is also evident from the transformation properties of the wave function $\phi_{\lambda\mu}$ ($\lambda=1,\ldots,5$, $\mu=1,\ldots,4$). Thus the parities of the [5,4] are fixed unambiguously.

However, only the component $\phi_{5\mu}$ of the wave function $\phi_{\lambda\mu}$ $(=\phi_{5\mu},\phi_{\nu\mu})$ generates a nonvanishing current, just as in the case of the [5,1] only the ϕ_5 part of ϕ_λ appeared in the current vector (6.11). Hence the $\phi_{\lambda\mu}$ describes one 0^- meson and one 1^+ meson.

On eliminating $\phi_{\mu\nu}$ from (7.1) and (7.2), we obtain an equation for the component $\phi_{5\mu}$ in the form

$$(\lambda^2+\rho^2+\rho+5/4)p^2\phi_{5\mu}+2(\lambda^2-2\rho+2)p_\mu p^\nu\phi_{5\nu}$$
$$-m^2c^2\rho^2\phi_{5\mu}=0. \tag{7.10}$$

Introducing the wave functions

$$u=(1/p)p^\rho\phi_{5\rho},\quad u_\mu=\phi_{5\mu}-(1/p)p_\mu u \tag{7.11}$$

for the 0^- and 1^+ mesons, respectively, we find that they satisfy the equations

$$(p^2-M_0{}^2c^2)u=0,\quad (p^2-M_1{}^2c^2)u_\mu=0, \tag{7.12}$$

where, as seen from the definition (7.11),

$$p^\mu u_\mu=0, \tag{7.13}$$

and where

$$(m\rho/M_0)^2=(\rho-\tfrac{3}{2})^2+3(1+\lambda^2),$$
$$(m\rho/M_1)^2=(\rho+\tfrac{1}{2})^2+1+\lambda^2. \tag{7.14}$$

Under a reflection of coordinates, the $SU(3,1)$ index μ of the wave function is acted on by the matrix F (see Sec. IV of I), while the $SO(3,2)$ index λ, as pointed out before, transforms by the action of β_0. Therefore, the space-parity operation on the wave function consists of writing

$$\mathcal{P}\phi=\beta_0F\phi(I_sx). \tag{7.15}$$

This means that the components $\phi_{5\mu}$ and $\phi_{\rho\mu}$ of the wave function $\phi_{\lambda\mu}$ transform like pseudoscalar$\otimes$(polar vector) and (axial vector)$\otimes$(polar vector), respectively, where we employed the symmetry properties of the wave function ϕ_λ for the [5,1], the Kemmer wave function.

Let us now introduce the dimension formula of the group $U(3,1)$ in the form

$$N=\sum(2J+1), \tag{7.16}$$

where $J=0$ for $N=1$, $J=\frac{3}{2}$ for $N=4$, $J=0$, 2 for $N=6$, $J=1$, 3 for $N=10$, $J=1$, 2, 3 for $N=15$, $J=\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, $\frac{7}{2}$ for $N=20$, and $J=0$, 2, 2, 4 for the other 20-dimensional representations of $U(3,1)$. This type of J structure was discussed by Flowers[3] for the group $U(4)$.

In terms of the J number, we can combine the mass formulas (7.14) into a general mass formula for the supermultiplet [5,4],

$$(m\rho/M)^2=[\rho+s(s+1)-J]^2$$
$$+[2J-s(s+1)](\lambda^2+1), \tag{7.17}$$

[3] B. A. Flowers, Proc. Roy. Soc. (London) **A212**, 249 (1952).

where $J=\frac{3}{2}$ and $s=0$, 1. However, in analogy with baryons, here also we can introduce the Z quantum number by writing (7.17) in the form

$$(m\rho/M)^2=(\rho-\tfrac{1}{2}+Z)^2 +[1+s(s+1)-2Z](\lambda^2+1), \quad (7.17')$$

where the meson masses are identified as ($s=0$, $Z=-1$) and ($s=1$, $Z=1$).

The conserved current vector of the [5,4] is given by

$$J_\mu=-i\bar{\phi}\beta^\nu\tau_{\nu\mu}\phi. \quad (7.18)$$

By using the relations (6.2), the definitions of the $\Gamma_{\mu\nu}$, $J_{\mu\nu}$ for $N=4$, and also by eliminating $\phi_{\mu\nu}$ in favor of $\phi_{5\mu}$, through the use of (7.1) and (7.2), we obtain

$$\begin{aligned} mc\rho^2J_\mu=&\tfrac{5}{2}\phi_{5\rho}{}^*p_\mu\phi_5{}^\rho+4(\phi_{5\mu}{}^*p^\rho\phi_{5\rho}+\phi_{5\rho}{}^*p^\rho\phi_{5\mu})\\ &+2\lambda^2(\phi_{5\rho}{}^*p_\mu\phi_5{}^\rho+\phi_{5\rho}{}^*p^\rho\phi_{5\mu}+\phi_{5\mu}{}^*p^\rho\phi_{5\rho})\\ &+2\rho^2\phi_{5\rho}{}^*p_\mu\phi_5{}^\rho+4\rho(\tfrac{1}{2}\phi_{5\rho}{}^*p_\mu\phi_5{}^\rho-\phi_{5\rho}{}^*p^\rho\phi_{5\mu}\\ &-\phi_{5\mu}{}^*p^\rho\phi_{5\rho}).\end{aligned}$$

Introducing the definitions (7.11) and (7.13) and then using (7.14), we find

$$\rho^2mcJ_\mu=\frac{2m^2}{M_0{}^2}u^*p_\mu u+\frac{2m^2}{M_1{}^2}u_\rho{}^*p_\mu u^\rho +2(2+\lambda^2-2\rho)(u_\mu{}^*pu+u^*pu_\mu),$$

where the last term can be written as

$$2(2+\lambda^2-2\rho)(u_\mu{}^*pu+u^*pu_\mu) =\left[\left(\frac{m\rho}{M_0}\right)^2-\left(\frac{m\rho}{M_1}\right)^2\right](u_\mu{}^*pu+u^*pu_\mu)=0.$$

Hence the current density vector becomes

$$J_\mu=\frac{2m}{c}\left(\frac{1}{M_0{}^2}u^*p_\mu u+\frac{1}{M_1{}^2}u_\rho{}^*p_\mu u^\rho\right), \quad (7.19)$$

which is the sum of the 0^- and 1^+ currents

$$J_\mu{}^-=\frac{2m}{cM_0{}^2}u^*p_\mu u, \quad J_\mu{}^+=\frac{2m}{cM_1{}^2}u_\rho{}^*p_\mu u^\rho. \quad (7.20)$$

We observe that in the configuration space, the currents (7.20) are of the form

$$\begin{aligned} J_\mu{}^-&=\frac{i\hbar m}{cM_0{}^2}\left(u^*\frac{\partial u}{\partial x^\mu}-\frac{\partial u^*}{\partial x^\mu}u\right),\\ J_\mu{}^+&=\frac{i\hbar m}{cM_1{}^2}\left(u_\rho{}^*\frac{\partial u^\rho}{\partial x^\mu}-\frac{\partial u^{*\rho}}{\partial x^\mu}u_\rho\right).\end{aligned} \quad (7.21)$$

The currents $J_\mu{}^-$ and $J_\mu{}^+$ differ from the (6.11) of the [5,1] in the mass factor, since in (7.21) in place of M_0 and M_1 we have the "effective masses" $\bar{M}_0=M_0{}^2/m$ and $\bar{M}_1=M_1{}^2/m$.

VIII. SUPERMULTIPLET [5,6]

From the definitions of the $\Gamma_{\mu\nu}$, $J_{\mu\nu}$ for $N=6$, the wave equation (6.3) can be split up in the form

$$\begin{aligned}(\tau_{\mu\nu})_{ab}p^\nu\phi^\mu{}_b-mc\phi_{5a}&=0,\\ (\tau_{\mu\nu})_{ab}p^\nu\phi_{5b}-mc\phi_{\mu a}&=0.\end{aligned} \quad (8.1)$$

Eliminating $\phi_{\mu a}$ and using the relation

$$\rho^2\tau_{\mu\nu}\tau^\mu{}_\rho p^\nu p^\rho=(2\lambda^2+3+\rho^2)p^2+2\rho\Gamma_{\mu\nu}p^\mu p^\nu, \quad (8.2)$$

we obtain

$$[(2\lambda^2+3+\rho^2)p^2+2\rho\Gamma_{\mu\nu}p^\mu p^\nu-m^2c^2\rho^2]_{ab}\phi_{5b}=0. \quad (8.3)$$

Hence, using $(\Gamma_{\mu\nu}p^\mu p^\nu)^2=p^4$, we get the equation

$$[(2\lambda^2+\rho^2\pm2\rho+3)p^2-m^2c^2\rho^2]\phi_{5a}=0, \quad (8.4)$$

which describes a 1^- and a 1^+ meson with masses

$$\begin{aligned}(m\rho/M_-)^2&=(\rho+1)^2+2(1+\lambda^2),\\ (m\rho/M_+)^2&=(\rho-1)^2+2(1+\lambda^2),\end{aligned} \quad (8.5)$$

respectively. The corresponding wave functions can be written as

$$u_{1\mu}=(1/p)p^\nu\phi_{5\mu\nu}, \quad u_{2\mu}=(1/2p)g_{\mu\gamma}\epsilon^{\gamma\nu\rho\sigma}p_\nu\phi_{5\rho\sigma}, \quad (8.6)$$

where

$$\phi_{5\mu\nu}=\tfrac{1}{2}Q_{\mu\nu a}\phi_{5a}.$$

Thus the supermultiplet [5,6] consists of a pair of vector and axial-vector mesons described by $\phi_{\lambda a}$ ($\equiv\phi_{5a}$, $\phi_{\mu a}$). Using the J structure of the $U(3,1)$ representations introduced by (7.16), we may rewrite the mass formulas (8.5) in the form

$$(m\rho/M)^2=[\rho+s(s+1)-J-1]^2+s(s+1)(\lambda^2+1), \quad (8.7)$$

where $s=1$ and $J=0$, 2. In terms of the Z number, the masses (8.5) can be combined into

$$(m\rho/M)^2=(\rho+Z)^2+s(s+1)(\lambda^2+1), \quad (8.7')$$

where $s=1$, $Z=\pm1$.

The conserved current vector of the [5,6] can be written as

$$J_\mu=-i\bar{\phi}\beta^\nu\tau_{\nu\mu}\phi=(2/mc\rho^2)\phi_5{}^*\Gamma_{44} \times[(\rho^2+2\lambda^2+3)p_\mu+2\rho\Gamma_{\mu\nu}p^\nu]\phi_5, \quad (8.8)$$

where the $U(3,1)$ index b has been suppressed. Introducing the projection operators

$$S_\pm=\tfrac{1}{2}(1\pm\Gamma_{\mu\nu}p^\mu p^\nu/p^2)$$

and the definitions

$$u_\pm=S_\pm\phi_5, \quad \bar{u}=u^*\Gamma_{44},$$

we obtain

$$mc\rho^2J_\mu=\frac{2m^2\rho^2}{M_+{}^2}\bar{u}_+p_\mu u_++\frac{2m^2\rho^2}{M_-{}^2}\bar{u}_-p_\mu u_- +4(\bar{u}_-p_\mu u_--\bar{u}_+p_\mu u_++\bar{\phi}_5\Gamma_{\mu\rho}p^\rho\phi_5).$$

From

$$\Gamma_{\mu\rho}p^\rho S_\pm = p_\mu S_\pm$$

it follows that

$$J_\mu = (2m/cM_+^2)\bar{u}_+ p_\mu u_+ + (2m/cM_-^2)\bar{u}_- p_\mu u_-, \quad (8.9)$$

where $u_\pm$ satisfy the Klein-Gordon equations

$$(p^2 - c^2M_\pm^2)u_\pm = 0.$$

IX. SUPERMULTIPLET [5,10]

From the wave function $\phi_{\lambda[AB]}$ ($A, B = 1, 2, \ldots, 5$) it is clear that it describes the mesons corresponding to the states $\phi_{5[5\mu]}$ and $\phi_{5[\mu\nu]}$. The presence of the states $\phi_{5[\mu\nu]}$ implies that the mass spectrum of the [5,6] for the pair 1^- and 1^+ mesons recurs, with, of course, different values of the parameters ρ, λ, and m. The states $\phi_{5[5\mu]}$, as follows from the wave equation of the [5,10],

$$\begin{aligned}[(\rho-1)\beta^\sigma p_\sigma - imc\rho]\phi_{[AB]} \\ -(1+i\lambda)J_{\mu\rho[AB]}\beta^\mu p^\nu \eta_\nu{}^\rho - (1-i\lambda)J_{\nu\rho[AB]}\beta^\mu p^\nu \eta_\mu{}^\rho \\ +\beta^\sigma p_\sigma(g_{5A}\phi_{[5B]} - g_{5B}\phi_{[5A]}) = 0, \quad (9.1)\end{aligned}$$

satisfy the equation

$$(\beta^\rho p_\rho - imc)\phi_{[5\mu]} = 0, \quad (9.2)$$

where the $SO(3,2)$ index λ has been suppressed, and where $\eta_{\mu\nu} = \frac{1}{2}J_{\mu\nu[AB]}\phi^{[AB]}$. The $J_{\mu\nu}$ are of the same form as the $\beta_{\mu\nu}$ of $SO(3,2)$ and they satisfy the relations

$$\tfrac{1}{2}J_{\mu\nu[AB]}J_{\rho\sigma}{}^{[AB]} = g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma}, \quad (9.3)$$

$$\tfrac{1}{2}J_{\mu\nu[AB]}J^{\mu\nu}{}_{[CD]} = \bar{g}_{AD}\bar{g}_{BC} - \bar{g}_{AC}\bar{g}_{BD}, \quad (9.4)$$

where

$$\bar{g}_{AB} = g_{AB} - g_{A5}g_{B5}, \quad J_{\mu\nu} = \begin{bmatrix} M_{\mu\nu} & 0 \\ 0 & 0 \end{bmatrix} = \Lambda_+ M_{\mu\nu},$$

and the coefficients g_{AB} are the same as β_0 or $g_{\lambda\omega}$ of the $SO(3,2)$.

The wave equation (9.2) can be replaced by

$$p^\nu \phi_{\nu[5\mu]} - mc\phi_{5[5\mu]} = 0, \quad (9.5)$$

$$p_\mu \phi_{5[5\nu]} - mc\phi_{\mu[5\nu]} = 0. \quad (9.6)$$

Hence, with the obvious steps we obtain the equation

$$[p^2 - m^2c^2]\phi_{5[5\mu]} = 0, \quad (9.7)$$

which describes a 0^- and a 1^+ meson with equal mass m. The corresponding wave functions are

$$\eta = (1/p)p^\mu \phi_{5[5\mu]}, \quad \eta_\mu = \phi_{5[5\mu]} - (p_\mu/p)\eta, \quad (9.8)$$

where $p^\mu \eta_\mu = 0$.

We have thus obtained a spectrum of a doublet of 1^+ and 1^-, a 0^- and a 1^+ mesons whose masses can be represented by

$$(m\rho/M)^2 = (\rho+Z)^2 + Z^2 s(s+1)(\lambda^2+1), \quad (9.9)$$

where we have $s=0$, 1 when $Z=0$, and where we have $s=1$ when $Z=\pm1$. The current vector is of the same type as (7.19) and (8.9) and is given by

$$\begin{aligned} J_\mu = \frac{2m}{c}\left(\frac{1}{M_+^2}u_+^* p_\mu u_+ + \frac{1}{M_-^2}u_-^* p_\mu u_-\right) \\ + \frac{2m}{c}\left(\frac{1}{M_0^2}\eta^* p_\mu \eta + \frac{1}{M_1^2}\eta_\rho^* p_\mu \eta^\rho\right). \quad (9.10)\end{aligned}$$

X. SUPERMULTIPLET [5,15]

As in the lower meson mass levels, the states $\phi_{5[ab]}$ are the only ones appearing in the current vector. Thus from the spin decomposition $(1,1)+(0,1)+(1,0)$ it follows that the wave function $\phi_{\lambda[ab]}$ ($\equiv \phi_{5[ab]}, \phi_{\mu[ab]}$) of the [5,15] describes a pair of 1^+ and 1^- mesons arising from the recurrence of the [5,6] structure in the [5,15], and it also contains a 0^+, a 1^-, and a 2^+ meson.

In this case the wave equation (6.3) can be split up as

$$\begin{aligned}(\tau^\mu{}_\nu)_{AB}p^\nu \phi_{\mu B} - mc\phi_{5A} = 0, \\ (\tau_{\mu\nu})_{AB}p^\nu \phi_{5B} - mc\phi_{\mu A} = 0,\end{aligned} \quad (10.1)$$

where the indices A and B represent the indices $[ab]$ and $[cd]$, respectively. Eliminating $\phi_{\mu A}$ from (10.1), we obtain

$$\begin{aligned}p^\mu p^\nu(\Gamma_\mu{}^\rho \Gamma_{\nu\rho} + 2\rho\Gamma_{\mu\nu} + \lambda^2 J^\rho{}_\mu J_{\rho\nu} + \rho^2 g_{\mu\nu})_{AB}\phi_{5B} \\ -m^2c^2\rho^2\phi_{5A} = 0, \quad (10.2)\end{aligned}$$

where

$$\Gamma^\rho{}_{\mu\{AB\}}\Gamma_{\nu\rho\{BC\}}p^\mu p^\nu = -\tfrac{3}{8}p^2 J_{\mu\nu A}J^{\mu\nu}{}_C, \quad (10.3)$$

$$J^\rho{}_{\mu[AB]}J_{\rho\nu[BC]}p^\mu p^\nu = -\tfrac{1}{4}p^2 J_{\mu\nu A}J^{\mu\nu}{}_C, \quad (10.4)$$

and where we used the definitions in Appendix A 5 of I. Hence Eq. (10.2) becomes

$$\begin{aligned}[(\tfrac{1}{4}\rho - \tfrac{1}{4}\lambda^2 - \tfrac{3}{8})p^2 J_{\rho\sigma A}J^{\rho\sigma}{}_B + \rho^2 p^2 \delta_{AB} \\ -\rho J_{\rho\gamma A}J_\sigma{}^\gamma{}_B p^\rho p^\sigma]\phi_{5B} - m^2c^2\rho^2\phi_{5A} = 0. \quad (10.5)\end{aligned}$$

Using the wave functions $\eta_{[\mu\nu]}$ and $t_{\{\mu\nu\}}$ as defined by

$$\eta_{[\mu\nu]} = \tfrac{1}{2}J_{\mu\nu[ab]}\phi_{5[ab]}, \quad t_{\{\mu\nu\}} = \tfrac{1}{2}\Gamma_{\mu\nu\{ac\}}g_{cb}\phi_{5[ab]}, \quad (10.6)$$

and using the property $\mathrm{Tr}(\Gamma_{\mu\nu}\Gamma_5) = 0$ in the wave equation (10.5), we obtain the results

$$\begin{aligned}[(\rho-1)^2 + 2(1+\lambda^2)]p^2\eta_{[\mu\nu]} - m^2c^2\rho^2\eta_{[\mu\nu]} \\ +4p^\rho(p_\mu\phi_{\rho\nu} - p_\nu\phi_{\rho\mu}) = 0, \quad (10.7)\end{aligned}$$

$$(p^2 - m^2c^2)t_{\{\mu\nu\}} = 0. \quad (10.8)$$

Equation (10.7) yields the same spectrum as the [5,6]. Equation (10.8) is satisfied by the wave functions

$$t = (1/p^2)p^\mu p^\nu t_{\{\mu\nu\}}, \quad t_\mu = (1/p)p^\nu t_{\{\mu\nu\}} - (p_\mu/p)t, \quad (10.9)$$

$$\begin{aligned}S_{\mu\nu} = t_{\{\mu\nu\}} - (1/p)(p_\mu t_\nu + p_\nu t_\mu) + \tfrac{1}{3}g_{\mu\nu}t \\ -(4/3p^2)p_\mu p_\nu t \quad (10.10)\end{aligned}$$

of the 0^+, 1^-, and 2^+ mesons, where

$$S_\mu{}^\mu=0,\quad p^\nu S_{\mu\nu}=0. \tag{10.11}$$

Hence the meson mass formula of the [5,15] can be expressed as

$$(m\rho/M)^2=(\rho+Z)^2+Z^2s(s+1)(\lambda^2+1), \tag{10.12}$$

where $Z=\pm1$, $s=1$ masses correspond to the recurrence of the [5,6] structure. The values $Z=0$ with $s=0$, 1, 2 yield the degenerate mass m for the 0^+, 1^-, 2^+ mesons in which the $Z=0$ with $s=0$ is just the recurrence of the [5,1] with opposite parity.

The corresponding conserved current vector of the [5,15] can be written down as the sum of five meson currents, each of which is separately conserved.

XI. SUPERMULTIPLET [5,20]

As in the case of the baryons, we shall discuss only one of the $N=20$ representations. Thus the [5,20]'s spin decomposition $(0,0)\otimes[(1,1)+(0,2)+(2,0)+(0,0)]$ shows that it contains 0^-, 0^+, 1^+, 2^+, 2^+, 2^- mesons described by the wave equation

$$\begin{aligned}[\Gamma_{\mu\rho}\Gamma^\rho{}_\nu+\lambda^2J_{\mu\rho}J^\rho{}_\nu+\rho^2g_{\mu\nu}+\lambda(\Gamma_\mu{}^\rho J_{\rho\nu}+J_{\rho\nu}\Gamma_\mu{}^\rho)\\+2\rho\Gamma_{\mu\nu}]_{AB}p^\mu p^\nu\phi_{5B}-m^2c^2\rho^2\phi_{5A}=0,\end{aligned} \tag{11.1}$$

where A and B represent, now, $\{ab\}$ and $\{cd\}$, respectively, and the components $\phi_{\mu A}$ of the wave function have been eliminated. As before the mesons of the [5,20] are described by the wave function $\phi_{\lambda A}$ ($\equiv\phi_{5A}$, $\phi_{\mu A}$). By using the results of Appendix A 5 of I, it can easily be shown that

$$\begin{aligned}(\Gamma_{\mu\rho}\Gamma_\nu{}^\mu)_{AB}p^\rho p^\nu=\tfrac{3}{8}p^2(4g_Ag_B+\Gamma_{\mu\nu A}\Gamma^{\mu\nu}{}_B)\\+\Gamma_{\mu\rho A}\Gamma^\rho{}_{\nu B}p^\mu p^\nu-(2/\sqrt{3})\\\times(\Gamma_{\mu\nu A}g_B+\Gamma_{\mu\nu B}g_A)p^\mu p^\nu,\end{aligned} \tag{11.2}$$

$$\begin{aligned}(J_{\mu\rho}J^\mu{}_\nu)_{AB}p^\rho p^\nu=\tfrac{1}{4}p^2\Gamma_{\mu\nu A}\Gamma^{\mu\nu}{}_B+\Gamma_{\mu\rho A}\Gamma_\nu{}^\rho{}_Bp^\mu p^\nu,\\p^\mu p^\nu(\Gamma_\mu{}^\rho J_{\rho\nu}+J_{\rho\nu}\Gamma_\mu{}^\rho)_{AB}=(2i/\sqrt{3})\\\times(g_A\Gamma_{\mu\nu B}-g_B\Gamma_{\mu\nu A})p^\mu p^\nu,\end{aligned} \tag{11.3}$$

where $g_A=g_{ab}$ and $\Gamma_{\mu\nu A}$ refer to the $N=6$ representation of the $U(3,1)$. On substituting from (11.2) and (11.3) in (11.1) we get

$$\begin{aligned}\rho^2(p^2-m^2c^2)\phi_{5A}\\+[\tfrac{3}{2}g_Ap^2+((\rho-2-2i\lambda)/\sqrt{3})p^\rho p^\sigma\Gamma_{\rho\sigma A}]\pi_1\\+g_A[((p-2+2i\lambda)/\sqrt{3})p^2]\pi_2+\tfrac{1}{4}(\lambda^2+\rho+\tfrac{3}{2})\\\times p^2\Gamma_{\rho\sigma A}\zeta^{\rho\sigma}+(1+\lambda^2-\rho)p^\rho p^\sigma\Gamma_{\rho\gamma A}\zeta_\sigma{}^\gamma=0,\end{aligned} \tag{11.4}$$

where

$$\begin{aligned}\zeta_{\{\mu\nu\}}=\Gamma_{\mu\nu A}\phi_{5A},\quad \pi_1=g_A\phi_{5A},\\\pi_2=(1/p_2)p^\mu p^\nu\zeta_{\{\mu\nu\}}.\end{aligned} \tag{11.5}$$

By multiplying (11.4) by g_A and $\Gamma_{\mu\nu A}$, we obtain the equations

$$[(\rho^2+9)p^2-m^2c^2\rho^2]\pi_1+2\sqrt{3}(\rho-2+2i\lambda)p^2\pi_2=0, \tag{11.6}$$

$$\begin{aligned}[((\rho+1)^2+2(1+\lambda^2))p^2-m^2c^2\rho^2]\zeta_{\{\mu\nu\}}\\+4(1+\lambda^2-\rho)p^\rho(p_\mu\zeta_{\{\rho\nu\}}+p_\nu\zeta_{\{\rho\mu\}}-\tfrac{1}{2}p_\rho g_{\mu\nu}\pi_2)\\+(8/\sqrt{3})(\rho-2-2i\lambda)(p_\mu p_\nu-\tfrac{1}{4}g_{\mu\nu}p^2)\pi_1=0.\end{aligned} \tag{11.7}$$

Hence, using the definition (11.5) of the π_2, we obtain the equation

$$\begin{aligned}\{[(\rho-2)^2+5+8\lambda^2]p^2-m^2c^2\rho^2\}\pi_2\\+2\sqrt{3}(\rho-2-2i\lambda)p^2\pi_1=0,\end{aligned} \tag{11.8}$$

which together with (11.6) yields the spectrum

$$\begin{aligned}(m\rho/M)^2=[(\rho-1)^2+4+4(1+\lambda^2)]\\+4Z[(1+\lambda^2-(\rho-1))^2+(1+\lambda^2)(\rho-1)]^{1/2}\end{aligned} \tag{11.9}$$

for the 0^- (π_2) and 0^+ (π_1) mesons corresponding to $Z=1$ and $Z=-1$, respectively.

The wave functions for the 1^+ and 2^- mesons are given by

$$\pi_{2\mu}=(1/p)p^\nu\zeta_{\{\mu\nu\}}-(p_\mu/p)\pi_2, \tag{11.10}$$

$$\begin{aligned}\pi_{\{\mu\nu\}}=\zeta_{\{\mu\nu\}}-(1/p)(p_\mu\pi_{2\nu}+p_\nu\pi_{2\mu})+\tfrac{1}{3}g_{\mu\nu}\pi_2\\-(4/3p^2)p_\mu p_\nu\pi_2,\end{aligned} \tag{11.11}$$

where, as can easily be seen, the restrictions

$$p^\mu\pi_{2\mu}=0,\quad g^{\mu\nu}\pi_{\{\mu\nu\}}=0,\quad p^\nu\pi_{\{\mu\nu\}}=0$$

are satisfied. From definitions (11.10), (11.11) and Eqs. (11.7), (11.6), and (11.8), we obtain

$$(p^2-M_1{}^2c^2)\pi_{2\mu}=0,\quad (p^2-M_2{}^2c^2)\pi_{\{\mu\nu\}}=0, \tag{11.12}$$

where

$$(m\rho/M_1)^2=(\rho-1)^2+6(1+\lambda^2) \tag{11.13}$$

and

$$(m\rho/M_2)^2=(\rho+1)^2+2(1+\lambda^2) \tag{11.14}$$

represent the mass spectrum of the 1^+ and 2^- mesons, respectively.

The remaining pair of 2^+ mesons are described by a wave function whose spin content is of the form $(0,2)+(2,0)$. If we multiply (11.4) by $\tfrac{1}{2}[(\Gamma_{\mu\nu}+iJ_{\mu\nu})\times\Gamma_5]_{ab}$ and contract with respect to $U(3,1)$ indices, we obtain the equation

$$(p^2-m^2c^2)\kappa_{\{\mu\nu\}}=0, \tag{11.15}$$

where we used the relations (4.26), and where

$$\kappa_{\mu\nu}=\tfrac{1}{2}\,\mathrm{Tr}[(\Gamma_{\mu\nu}\Gamma_5+iJ_{\mu\nu}\Gamma_5)\phi_5]. \tag{11.16}$$

Equation (11.15), with the restrictions $g^{\mu\nu}\kappa_{\mu\nu}=0$ and $p^\nu\kappa_{\mu\nu}=0$, describes the pair of 2^+ with the equal mass m.

The mass spectrum corresponding to the states $\pi_{1\mu}$, $\pi_{\{\mu\nu\}}$, and $\kappa_{\mu\nu}$ can be expressed in a single mass relation of the form

$$(m\rho/M)^2=(\rho+Z)^2+[s(s+1)-4Z]Z^2(1+\lambda^2), \tag{11.17}$$

where ($Z=-1$, $s=1$), ($Z=1$, $s=2$), ($Z=0$, $s=2$).

XII. SUPERMULTIPLETS [10,1] AND [10,4]

So far we have only discussed the $N_0=4_B$ (baryons) and $N_0=5$ (mesons) representations of the group $SO(3,2)$. The $N_0=10$ representation of the $SO(3,2)$ in the wave equation (6.3) together with the $N=1$, 4, 6, 10, 15, ..., representations of the $U(3,1)$ also describe mesons. As pointed out before, the two-sequence structure of the mesons as described by $N_0=5$ and $N_0=10$ is similar to the two-sequence structure of the fermions (baryons and leptons) described by the $N_0=4_B$ and $N_0=4_l$ representations of the $SO(3,2)$. One basic difference between $N_0=5$ and $N_0=10$ arises from the fact that the latter supermultiplets are more crowded than the former ones.

The $N_0=10$ representation of the $SO(3,2)$ can be obtained from the analogy with the $N_0=5$. First we observe that the ten matrices for the five-dimensional representation of the $SO(3,2)$ can be expressed in the form

$$(\beta_{\xi\eta})_\lambda{}^\omega=i(\delta_\xi{}^\omega g_{\eta\lambda}'-\delta_\eta{}^\omega g_{\xi\lambda}'), \qquad (12.1)$$

where the indices ξ, η, λ, $\omega=1, 2, \ldots, 5$, and where $g_{\eta\lambda}'$ represents the matrix $\beta_5\beta_0$, β_0 being the metric in the space of $SO(3,2)$ [see (3.8) of I]. The definitions (12.1) do, of course satisfy commutation rules of the $SO(3,2)$. From (12.1) it is easily seen that

$$\beta_{5\mu}=\beta_\mu \quad (\mu, \nu=1, 2, 3, 4), \quad g_{\mu\nu}'=g_{\mu\nu}, \quad g_{55}'=-1,$$

and the remaining six components are just $\beta_{\mu\nu}$, the spin matrices of the Kemmer algebra.

The ten-dimensional representation of the $SO(3,2)$ is given by

$$(\beta_{\xi\eta})_{[[\lambda\omega],[\zeta\epsilon]]}=-\tfrac{1}{2}i[(\beta_{\xi\phi})_{\lambda\omega}(\beta_\eta{}^\phi)_{\zeta\epsilon} -(\beta_{\eta\phi})_{\lambda\omega}(\beta_\xi{}^\phi)_{\zeta\epsilon}], \qquad (12.2)$$

where all the indices run from 1 to 5, and where

$$\tfrac{1}{2}(\beta_{\xi\eta})_{\lambda\omega}(\beta^{\xi\eta})_{\zeta\epsilon}=g_{\xi\epsilon}'g_{\eta\zeta}'-g_{\xi\zeta}'g_{\eta\epsilon}'. \qquad (12.3)$$

The ten β's as defined by (12.2) obey the commutation rules of the $SO(3,2)$. By using definition (12.1), Eq. (12.2) can be written in the form

$$\begin{aligned}(\beta_{\xi\eta})_{[[\lambda\omega],[\zeta\epsilon]]}=-\tfrac{1}{2}i[&g_{\lambda\zeta}'(g_{\omega\eta}'g_{\zeta\epsilon}'-g_{\omega\xi}'g_{\eta\epsilon}')\\ &+g_{\omega\zeta}'(g_{\xi\lambda}'g_{\eta\epsilon}'-g_{\xi\epsilon}'g_{\lambda\eta}')\\ &+g_{\lambda\epsilon}'(g_{\omega\xi}'g_{\eta\zeta}'-g_{\omega\eta}'g_{\xi\zeta}')\\ &+g_{\omega\epsilon}'(g_{\eta\lambda}'g_{\xi\zeta}'-g_{\xi\lambda}'g_{\eta\zeta}')], \qquad (12.4)\end{aligned}$$

where

$$\beta_\mu=\beta_{5\mu},$$

and the remaining six are the spin matrices of the $N_0=10$ representation of the $SO(3,2)$.

The action of the new β's on the wave function $\phi_{[\zeta\epsilon]}$ of the [10,1] is given by

$$(\beta_{\xi\eta})_{[[\lambda\omega],[\zeta\epsilon]]}\phi^{[\xi\epsilon]}=i(g_{\omega\eta}'\phi_{[\xi\omega]}+g_{\omega\xi}'\phi_{[\lambda\eta]} +g_{\xi\lambda}'\phi_{[\eta\omega]}+g_{\eta\lambda}'\phi_{[\omega\xi]}). \qquad (12.5)$$

Hence putting $\xi=5$, $\eta=\mu$, we obtain the action of the ten-dimensional β's on the wave function as

$$(\beta_\mu)_{[[\lambda\omega],[\zeta\epsilon]]}\phi^{[\zeta\epsilon]}=i(g_{\omega\mu}'\phi_{[5\lambda]}+g_{\omega5}'\phi_{[\lambda\mu]} +g_{5\lambda}'\phi_{[\mu\omega]}+g_{\mu\lambda}'\phi_{[\omega5]}).$$

Thus the wave equation

$$(\beta^\mu p_\mu-imc)\phi=0 \qquad (12.6)$$

of the [10,1] can be split up as

$$p^\nu\phi_{[\mu\nu]}-mc\phi_{[5\mu]}=0, \qquad (12.7)$$

$$p_\mu\phi_{[5\nu]}-p_\nu\phi_{[5\mu]}+mc\phi_{[\mu\nu]}=0. \qquad (12.8)$$

From (12.7) we get the restriction $p^\mu\phi_{[5\mu]}=0$ and from (12.8) the equation

$$(p^2-m^2c^2)\phi_{[5\mu]}=0. \qquad (12.9)$$

Hence the wave function $\phi_{[\xi\eta]}$ $(\equiv\phi_{[5\mu]}, \phi_{[\mu\nu]})$ describes a 1^- meson of mass m.

For the supermultiplet [10,4], the wave equation (6.3) can be written as

$$(\Gamma^\mu{}_\gamma+\lambda J^\mu{}_\gamma)_\rho{}^\sigma p^\gamma\phi_{[\mu\nu]\sigma}+\rho p^\mu\phi_{[\mu\nu]\rho} +mc\rho\phi_{[5\nu]\rho}=0, \qquad (12.10)$$

$$(\Gamma_{\nu\gamma}+\lambda J_{\nu\gamma})_\rho{}^\sigma p^\gamma\phi_{[5\mu]\sigma}-(\Gamma_{\mu\gamma}+\lambda J_{\mu\gamma})_\rho{}^\sigma p^\gamma\phi_{[5\nu]\sigma} +\rho(p_\nu\phi_{[5\mu]\rho}-p_\mu\phi_{[5\nu]\rho})-mc\rho\phi_{[\mu\nu]\rho}=0. \qquad (12.11)$$

Now the wave function $\phi_{[\xi\eta]\mu}$ $(\equiv\phi_{[5\nu]\mu}, \phi_{[\rho\sigma]\mu})$ of the [10,4], as follows from the spin decomposition $(\frac{1}{2},\frac{1}{2})\otimes(\frac{1}{2},\frac{1}{2})$, describes two 0^+ mesons, a pair of 1^+, 1^- mesons, a 2^- meson, and a 1^- meson. Upon using the definitions of the $\Gamma_{\mu\nu}$, $J_{\mu\nu}$ of $N=4$ in (12.10) and (12.11), we obtain the equations

$$(\rho+\tfrac{1}{2})p^\rho\phi_{[\rho\nu]\mu}-(1+i\lambda)p_\mu\phi_{[\rho\nu]}{}^\rho -(1-i\lambda)p^\rho\phi_{[\mu\nu]\rho}+mc\rho\phi_{[5\nu]\mu}=0, \qquad (12.12)$$

$$\begin{aligned}(\rho+\tfrac{1}{2})&(p_\mu\phi_{[5\nu]\rho}-p_\nu\phi_{[5\mu]\rho})\\ &-(1+i\lambda)p_\rho(\phi_{[5\nu]\mu}-\phi_{[5\mu]\nu})\\ &-(1-i\lambda)p^\sigma(g_{\mu\rho}\phi_{[5\nu]\sigma}-g_{\nu\rho}\phi_{[5\mu]\sigma})\\ &+mc\rho\phi_{[\mu\nu]\rho}=0. \qquad (12.13)\end{aligned}$$

On eliminating $\phi_{[\mu\nu]\rho}$ between (12.12) and (12.13), we get the equation for the components $\phi_{[5\mu]\nu}$ in the form

$$\begin{aligned}\{[(\rho+\tfrac{1}{2})^2&+(1+\lambda^2)]p^2-m^2c^2\rho^2\}\phi_{[5\nu]\mu}\\ &-(1+\lambda^2)p^2\phi_{[5\mu]\nu}+(\rho-\tfrac{1}{2}-i\lambda)(1+i\lambda)pp_\mu\eta_{3\nu}\\ &-(\rho+\tfrac{1}{2})^2pp_\nu\eta_{3\mu}+(\lambda^2+3-4\rho)pp_\mu\eta_{1\nu}+(1-i\lambda)\\ &\times(\rho-\tfrac{1}{2}+i\lambda)pp_\nu\eta_{1\mu}+[3\lambda^2+4-(\rho+\tfrac{3}{2})^2p_\mu p_\nu\eta_2\\ &+g_{\mu\nu}(\rho+\tfrac{1}{2})(1-i\lambda)p^2\eta_2\\ &+(\rho+\tfrac{1}{2})(1+i\lambda)p_\mu p_\nu\eta_1=0, \qquad (12.14)\end{aligned}$$

where the wave functions of the two 0^+, two 1^-, and one 1^+ mesons are given by

$$\eta_1=\phi_{[5\mu]}{}^\mu, \quad \eta_2=(1/p^2)p^\mu p^\nu\phi_{[5\nu]\mu}, \qquad (12.15)$$

$$\begin{aligned}\eta_{1\mu}&=(1/p)p^\rho\phi_{[5\mu]\rho}-(p_\mu/p)\eta_2,\\ \eta_{3\mu}&=(1/p)p^\rho\phi_{[5\rho]\mu}-(p_\mu/p)\eta_2,\end{aligned} \qquad (12.16)$$

$$\eta_{2\mu}=(1/2p)g_{\mu\gamma}\epsilon^{\gamma\nu\rho\sigma}p_\nu\phi_{[5\rho]\sigma}, \qquad (12.17)$$

respectively.

The above definitions of the wave functions together with equation (12.14) yield the mass spectra

$$(m\rho/M)^2=(\rho+\tfrac{1}{2})^2+3(1+\lambda^2) \tag{12.18}$$

for the 0^+ pair and

$$(m\rho/M)^2=(\rho+\tfrac{1}{2})^2+2(1+\lambda^2) \tag{12.19}$$

for the 1^+ meson. The wave functions $\eta_{1\mu}$ and $\eta_{3\mu}$ lead to the spectrum

$$(m\rho/M)^2=\tfrac{1}{2}A+\tfrac{1}{2}Z[A^2-8(1+\lambda^2)^2]^{1/2} \tag{12.20}$$

for the two 1^- mesons, where

$$A=(\rho-\tfrac{3}{2})^2+3(1+\lambda^2) \tag{12.21}$$

and $Z=\pm 1$.

The wave function of the 2^- meson is given by

$$\begin{aligned}\eta_{\{\mu\nu\}}=(1/2p^2)\epsilon^{\rho\sigma\alpha\beta}p_\sigma[g_{\mu\rho}p_\nu\phi_{[5\alpha]\beta}+g_{\nu\rho}p_\mu\phi_{[5\alpha]\beta}]\\ -(1/p)(p_\mu\eta_{2\nu}+p_\nu\eta_{2\mu}),\end{aligned} \tag{12.22}$$

which satisfies the five restrictions

$$g^{\mu\nu}\eta_{\{\mu\nu\}}=0,\quad p^\nu\eta_{\{\mu\nu\}}=0.$$

The wave function $\eta_{\{\mu\nu\}}$, like $\eta_{2\mu}$, satisfies the equation

$$[((\rho+\tfrac{1}{2})^2+2(1+\lambda^2))p^2-m^2c^2\rho^2]\eta_{\{\mu\nu\}}=0. \tag{12.23}$$

Hence we obtain the interesting result that the mass of the 2^- meson is equal to the mass of the 1^+ meson given by (12.19).

XIII. SUPERMULTIPLET [10,6]

The supermultiplet is described by the wave function $\phi_{[\lambda\omega]a}$ ($\equiv\phi_{[5\mu]a}$, $\phi_{[\mu\nu]a}$) which satisfies the equations

$$(\tau^\rho{}_\nu)_{ab}p^\nu\phi_{[\rho\mu]b}+mc\phi_{[5\mu]a}=0, \tag{13.1}$$

$$(\tau_{\nu\rho})_{ab}p^\rho\phi_{[5\mu]b}-(\tau_{\mu\rho})_{ab}p^\rho\phi_{[5\nu]b}-mc\phi_{[\mu\nu]a}=0. \tag{13.2}$$

Hence, eliminating $\phi_{[\mu\nu[a}$, we obtain the equation

$$\begin{aligned}(\tau^\rho{}_\nu\tau_{\mu\sigma})_{ab}p^\nu p^\sigma\phi_{[5\rho]b}-(\tau^\rho{}_\nu\tau_{\rho\sigma})_{ab}p^\nu p^\sigma\phi_{[5\mu]b}\\ +m^2c^2\phi_{[5\mu]a}=0,\end{aligned} \tag{13.3}$$

where

$$\rho^2\tau^\rho{}_\nu\tau_{\rho\sigma}p^\nu p^\sigma=(2\lambda^2+3+\rho^2)p^2+2\rho\Gamma_{\mu\nu}p^\mu p^\nu.$$

By using the definitions of the $\Gamma_{\mu\nu}$, $J_{\mu\nu}$ as given in Appendix A of I, Eq. (13.3) can be expressed in the form

$$\begin{aligned}&\{[(\rho-1)^2+2(1+\lambda^2)]p^2-m^2c^2\rho^2\}\phi_{[5\mu],[\nu\rho]}-(\rho-1)^2p_\mu p^\gamma\phi_{[5\gamma],[\nu\rho]}+(1+\lambda^2-4\rho)p^\gamma[p_\rho\phi_{[5\mu],[\gamma\nu]}-p_\nu\phi_{[5\mu],[\gamma\rho]}]\\ &+(1+\lambda^2)[(g_{\mu\rho}p_\nu-g_{\mu\nu}p_\rho)p\Gamma_1+p^\gamma(p_\rho\phi_{[5\nu],[\mu\gamma]}-p_\nu\phi_{[5\rho],[\mu\gamma]})+p^2(\phi_{[5\rho],[\mu\nu]}-\phi_{[5\nu],[\mu\rho]})]+(1+i\lambda)\\ &\times(\rho+i\lambda)p_\mu p^\gamma(\phi_{[5\rho],[\gamma\nu]}-\phi_{[5\nu],[\gamma\rho]})+(1-i\lambda)(\rho-i\lambda)p^\gamma(p_\rho\phi_{[5\gamma],[\mu\nu]}-p_\nu\phi_{[5\gamma],[\mu\rho]})+(\rho-1)(1+i\lambda)\\ &\times p^2(g_{\mu\nu}\Gamma_{3\rho}-g_{\mu\rho}\Gamma_{3\nu})+(\rho-1)(1-i\lambda)p_\mu(p_\nu\Gamma_{1\rho}-p_\rho\Gamma_{1\nu})=0,\end{aligned} \tag{13.4}$$

where

$$\phi_{[5\mu],[\nu\rho]}=Q_{\nu\rho a}\phi_{[5\mu]a},$$

and the definitions

$$\begin{aligned}\Gamma_1&=(1/p)p^\rho\phi_{[5\sigma],[\rho\sigma]},\\ \Gamma_{1\mu}&=\phi_{[5\sigma],[\mu\sigma]}-(1/p)p_\mu\Gamma_1,\\ \Gamma_{3\mu}&=(1/p^2)p^\rho p^\sigma\phi_{[5\beta],[\mu\sigma]}\end{aligned} \tag{13.5}$$

represent the wave functions of the 0^+, 1^-, 1^- mesons, respectively. The wave functions of the remaining 0^-, 1^+, 1^+, 2^-, 2^- mesons are given by

$$\Gamma_2=(1/2p)\epsilon^{\mu\nu\rho\sigma}p_\mu\phi_{[5\nu],[\rho\sigma]}, \tag{13.6}$$

$$\Gamma_{2\mu}=(1/2p^2)g_{\mu\gamma}\epsilon^{\gamma\alpha\rho\sigma}p_\alpha p^\nu\phi_{[5\nu],[\rho\sigma]}, \tag{13.7}$$

$$\Gamma_{4\mu}=(1/2p^2)g_{\mu\gamma}\epsilon^{\gamma\alpha\rho\sigma}p_\alpha p^\nu\phi_{[5\rho],[\nu\sigma]}, \tag{13.8}$$

$$\begin{aligned}\Gamma_{1\{\mu\nu\}}=(1/2p)\epsilon^{\gamma\alpha\rho\sigma}p_\alpha(g_{\mu\gamma}\phi_{[5\rho],[\nu\sigma]}+g_{\nu\gamma}\phi_{[5\rho],[\mu\sigma]})\\ -(1/p)(p_\mu\Gamma_{4\nu}+p_\nu\Gamma_{4\mu}),\end{aligned} \tag{13.9}$$

$$\begin{aligned}\Gamma_{2\{\mu\nu\}}=(1/2p)\epsilon^{\gamma\alpha\rho\sigma}p_\alpha(g_{\mu\gamma}\phi_{[5\nu],[\rho\sigma]}+g_{\nu\gamma}\phi_{[5\mu],[\rho\sigma]})\\ -(1/p)(p_\mu\Gamma_{2\nu}+p_\nu\Gamma_{2\mu}),\end{aligned} \tag{13.10}$$

respectively. Thus, as seen from the spin decomposition $(\frac{1}{2},\frac{1}{2})\otimes[(0,1)+(1,0)]$, the supermultiplet [10,6] contains eight mesons. As in the previous computations, once the wave functions are defined, the calculation of the mass spectra follows from performing the implied operations on Eq. (13.4).

From the definitions Γ_1 and Γ_2 and Eq. (13.4) we obtain the mass spectrum

$$(m\rho/M)^2=(\rho+Z)^2+4(1+\lambda^2), \tag{13.11}$$

where $Z=+1$ and -1 for the Γ_1 and Γ_2, respectively. The spectrum corresponding to the pairs $(\Gamma_{1\mu},\Gamma_{3\mu})$ and $(\Gamma_{2\mu},\Gamma_{4\mu})$ is given by

$$(m\rho/M)^2=\tfrac{1}{2}C+\tfrac{1}{2}Z[C^2-8(1+\lambda^2)^2]^{1/2}, \tag{13.12}$$

where

$$C=(\rho+Z')^2+3(\lambda^2+1) \tag{13.13}$$

and $Z'=\mp 1$ corresponds to the pairs of 1^- and 1^+, respectively, and where in both cases $Z=\pm 1$.

For the pair of 2^- mesons we get the results

$$\begin{aligned}(m\rho/M)^2&=(\rho-1)^2+4(1+\lambda^2),\\ (m\rho/M)^2&=(\rho-1)^2+1+\lambda^2,\end{aligned} \tag{13.14}$$

respectively. Hence we see that the masses of the 0^- and one of the 2^- mesons are equal.

XIV. DISCUSSION OF MESON SPECTRUM

The branching of the meson spectrum into two classes arising from the five- and ten-dimensional representations of $SO(3,2)$ does not necessarily imply a classification of the strange and nonstrange mesons into different groups. Like the half-integral-spin particles (fermions), the integral-spin particles (bosons) belong to two different reducible representations of the G symmetry $[=SO(3,2)\otimes U(3,1)]$. The fundamental differences (if any) between $N_0=5$ and $N_0=10$ mesons could manifest themselves in their strong, weak, and electromagnetic interactions.

However, like baryons, mesons also can be regarded as identical objects in the space of spin, Z, charge, parity, etc.: No two mesons can be put into the same state. Thus the wave function of two mesons must be antisymmetric with respect to an interchange of their spin, Z, charge, parity, etc., quantum numbers as a group. In this way the "super-exclusion principle" can form the basis for the proliferation of mesons. Figure 2 is a diagrammatic illustration of the meson spectrum where, as in the case of baryons, an "internal supermultiplet" of mesons lies in the plane containing the axis of the coaxial cylinders and the mesons of equal spin and parity. Therefore different internal supermultiplets of mesons are obtained by fixed rotations of the plane around the common axis of the cylinders.

The predicted periodicity, observed for the baryons, as based on the recurrence of the low-lying level structures, like [5,1], [5,6], at the higher levels prevails also in the meson spectrum. Although the spectra [10,10], [10,15], etc., have not been calculated in this paper, we expect that the low-lying levels [10,1], [10,6] will recur in the higher mass levels. Figure 2, which represents the low-lying levels from [5,1] to [5,20] and [10,1] to [10,6] and also the uncalculated levels [10,10] to [10,20], contains only ten $J^P=0^-$ mesons, where the parity assignment is based on the wave functions ϕ_λ of [5,1] and $\phi_{[\lambda\omega]}$ of [10,1] which describe $J^P=0^-$ and $J^P=1^-$ [since $1^-=(0,\frac{1}{2})\otimes(\frac{1}{2},0)$] mesons, respectively.

As in the case of the baryon spectra, here also we can derive "sum rules." Thus we may write

$$\left(\frac{\sqrt{3}}{M_1}\right)^2-\left(\frac{1}{M_0}\right)^2=\frac{1}{m^2}\left(2+\frac{6}{\rho}-\frac{3}{2\rho^2}\right) \qquad (14.1)$$

for the [5,4], and

$$\left(\frac{1}{M_-}\right)^2-\left(\frac{1}{M_+}\right)^2=\frac{4}{\rho m^2} \qquad (14.2)$$

for the [5,6], [5,10], and [5,15], where in each case m has a different interpretation. Because of the insufficient number of relations, ρ and m are not eliminated. The inverse-mass-squared relations, as contrasted to the inverse-mass relations for the baryons, are the characteristic feature of the meson sum rules. For the [5,20], putting $\rho^2=\frac{1}{2}(\rho+1)^2+\frac{1}{2}(\rho-1)^2-1$ and solving

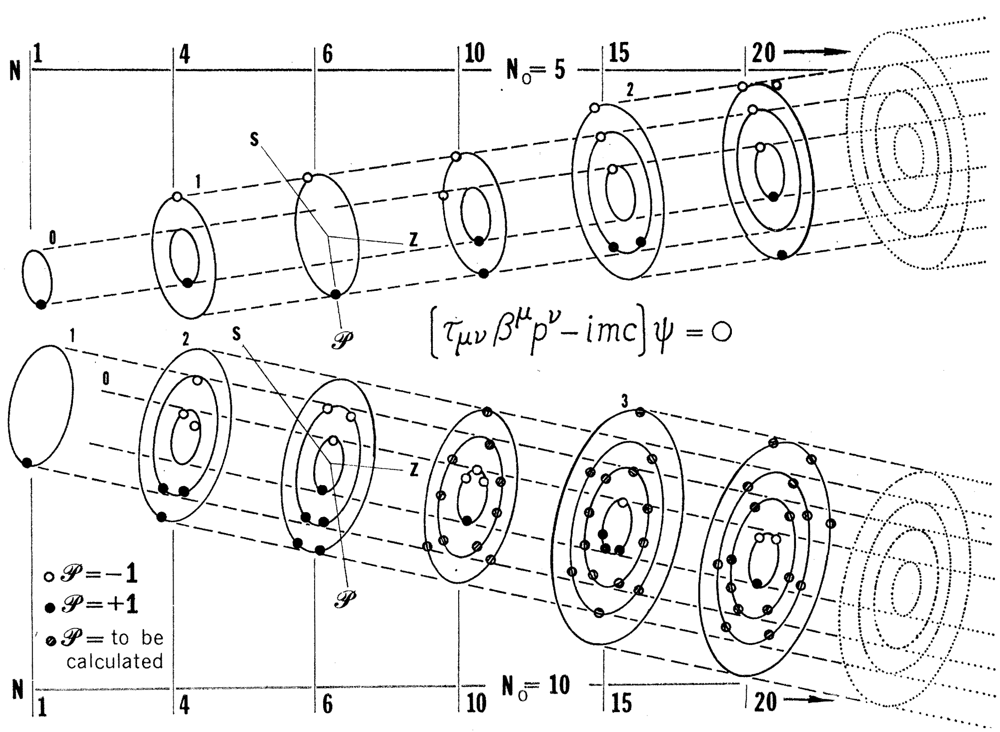

FIG. 2. Meson supermultiplets. The planes perpendicular to the axes contain mesons of different spins and parities, forming the space-time supermultiplets. For each branch, a hyperplane through the common axis may be constructed in such a way that it contains mesons of equal spin and parity, hence defining an internal supermultiplet.

(11.9), (11.13), and (11.14) for $(\rho+1)^2$, $(\rho-1)^2$, and $1+\lambda^2$, and substituting in the identity

$$\{\tfrac{1}{4}[(\rho+1)^2-(\rho-1)^2]\}^2=\tfrac{1}{2}[(\rho+1)^2+(\rho-1)^2]-1,$$

we obtain the sum rule

$$[(a_1+a_2-2a_0)(a_1-a_2)+4(7a_1-a_2-10a_0)]^2 =4(4+a_1-a_2)(4a_0+a_2-3a_1), \quad (14.3)$$

where

$$a_1=\frac{m^2}{M_1{}^2},\quad a_2=\frac{m^2}{M_2{}^2},\quad a_0=\tfrac{1}{2}m^2\left(\frac{1}{M_{\pi 1}{}^2}+\frac{1}{M_{\pi 2}{}^2}\right).$$

For the [10,4], from (12.18)–(12.20), we obtain the sum rule

$$\left(\frac{1}{M_0{}^2}-\frac{1}{M_1{}^2}\right)^2=\frac{1}{2}\left(\frac{1}{M_+M_-}\right)^2, \quad (14.4)$$

where M_0, M_1, and $M_\pm$ are defined by (12.18), (12.19), and (12.20), respectively.

The sum rules for the [10,6] follow from the four mass relations (13.12) in the form

$$M_+{}^+M_-{}^+=M_+{}^-M_-{}^-, \quad (14.5)$$

where lower and upper $(+,-)$ signs refer to $Z=\pm 1$ and $Z'=\pm 1$, respectively. From (13.11) and (13.14) we further obtain the sum rule

$$\frac{1}{M_b{}^2}-\frac{1}{M_2{}^2}=\frac{3}{\sqrt{2}}\frac{1}{M_+{}^+M_-{}^+}, \quad (14.6)$$

where M_b is the mass value corresponding to $Z=-1$ in (13.11) and M_2 is given by the second relation of (13.14). The remaining sum rule involving M_a [$Z=1$ in (13.11)] is

$$\left(\frac{1}{M_a}\right)^2-\left(\frac{1}{M_+{}^+}\right)^2-\left(\frac{1}{M_-{}^+}\right)^2=\frac{1}{\sqrt{2}}\frac{1}{M_+{}^+M_-{}^+}. \quad (14.7)$$

Others are given by

$$\left(\frac{1}{M_b}\right)^2-\left(\frac{1}{M_+{}^-}\right)^2-\left(\frac{1}{M_-{}^-}\right)^2=\frac{1}{\sqrt{2}}\frac{1}{M_+{}^-M_-{}^-} \quad (14.8)$$

and

$$\left(\frac{1}{M_+{}^-}\right)^2+\left(\frac{1}{M_-{}^-}\right)^2-\left(\frac{1}{M_2}\right)^2=\frac{\sqrt{2}}{M_+{}^-M_-{}^-}. \quad (14.9)$$

In the absence of the internal quantum numbers, the best way to compare the above results with the existing (and in some cases changing) experimental data is by a direct computer analysis.[4]

[4] The computer scanning of the baryon and meson spectra of this theory has been carried out by A. Perlmutter and the results, which compare favorably with experiment, appear in *Proceedings of the Coral Gables Conference on Fundamental Interactions at High Energy, University of Maimi, 1970*, edited by J. Iverson, R. Williams, and A. Perlmutter (Gordon and Breach, New York, 1970).

XV. MASSLESS SPIN-1 PARTICLE AND PHOTON

It has been pointed out that in the limit $\rho=\infty$ the wave equation reduces to Dirac or Kemmer type of equation describing equal-mass particles of different spins and that the resulting equation is invariant under $U(3,1)$ transformations. An entirely different class of equations refer to those arising from the limiting case of $\rho=0$, $\lambda=0$, viz.,

$$\Gamma_{\mu\nu}\beta^\mu p^\nu\phi=0. \quad (15.1)$$

For Eq. (15.1) the G symmetry [$\equiv SO(3,2)\otimes U(3,1)$] is *exactly* broken. We have thus obtained a new formulation of the massless particles. Consider now the $\rho=0$, $\lambda=0$ limit of the supermultiplet [5,4] where now the components $\phi_{5\mu}$ and $\phi_{\mu\nu}$ of the wave function $\phi_{\xi\mu}$ ($\xi=1,\ldots,5$, $\mu=1,\ldots,4$) decouple and (15.1) yields the equations

$$\tfrac{1}{2}p_\mu\phi_{5\nu}-p_\nu\phi_{5\mu}-g_{\mu\nu}p^\rho\phi_{5\rho}=0, \quad (15.2)$$

$$\tfrac{1}{2}p^\nu\phi_{\nu\mu}-p^\nu\phi_{\mu\nu}-p_\mu\phi_\nu{}^\nu=0. \quad (15.3)$$

Multiplying (15.2) by $g^{\mu\nu}$, we get

$$p^\mu\phi_{5\mu}=0, \quad (15.4)$$

and using this result in (7.10) and setting $\rho=\lambda=0$ there, we get the wave equation

$$p^2\phi_{5\mu}=0 \quad (15.5)$$

for a massless spin-1 particle, the photon. Equation (15.3) together with (7.1) and (7.2) leads to

$$p^2\mathfrak{A}_\mu=0, \quad (15.6)$$

where

$$\mathfrak{A}_\mu=(1/p)p^\rho\phi_{\rho\mu}-(2p^\rho/p)\phi_{\mu\rho}+p_\mu(1/p^3)p^\rho p^\sigma\phi_{\rho\sigma}, \quad (15.7)$$
$$p^\mu\mathfrak{A}_\mu=0.$$

Thus the remaining components $\phi_{\mu\nu}$ of the wave function $\phi_{\xi\mu}$ provide the same result as do the components $\phi_{5\mu}$. From (15.4) and (15.5) or (15.6) and (15.7) it is clear that for the supermultiplet [5,4] the limiting case $\rho=\lambda=0$ yields the description of the spin-1 massless particle in terms of the vector potential rather than the electromagnetic field itself.

A more interesting result is obtained from the limiting state of the supermultiplet [5,6]. In this case also the components ϕ_{5a} and $\phi_{\mu a}$ ($a=1,2,\ldots,6$) for $\rho=\lambda=0$ are decoupled and Eq. (15.1) yields the field equations

$$g_{\mu\rho}p^\sigma\phi_{5[\nu\sigma]}-g_{\mu\nu}p^\sigma\phi_{5[\rho\sigma]}-(p_\mu\phi_{5[\nu\rho]}+p_\nu\phi_{5[\rho\mu]}+p_\rho\phi_{5[\mu\nu]})=0, \quad (15.8)$$

$$p_\mu\phi^\rho{}_{[\nu\rho]}-p_\nu\phi^\rho{}_{[\mu\rho]}+p^\rho(\phi_{\mu[\nu\rho]}+\phi_{\nu[\rho\mu]}+\phi_{\rho[\mu\nu]})=0, \quad (15.9)$$

where

$$\phi_{5[\mu\nu]}=\tfrac{1}{2}Q_{\mu\nu a}\phi_{5a},\quad \phi_{\rho[\mu\nu]}=\tfrac{1}{2}Q_{\mu\nu a}\phi_{\rho a}. \quad (15.10)$$

From (15.8), multiplying by $g^{\mu\nu}$, we obtain

$$p^{\rho}\phi_{5[\mu\rho]}=0\,. \tag{15.11}$$

Hence Eq. (15.8) becomes

$$p_{\mu}\phi_{5[\nu\rho]}+p_{\nu}\phi_{5[\rho\mu]}+p_{\rho}\phi_{5[\mu\nu]}=0\,. \tag{15.12}$$

Equations (15.11) and (15.12) are just Maxwell's equations, and in this case the spin-1 field (photon) is described by fields rather than by potentials. Equation (15.9), putting $\phi^{\rho}{}_{[\mu\rho]}=\mathcal{A}_{\mu}$, yields

$$f_{\mu\nu}+p^{\rho}(f_{\mu[\nu\rho]}+f_{\nu[\rho\mu]}+f_{\rho[\mu\nu]})=0\,, \tag{15.13}$$

where

$$f_{\mu\nu}=p_{\mu}\mathcal{A}_{\nu}-p_{\nu}\mathcal{A}_{\mu}\,. \tag{15.14}$$

Operating by p^{ν} on (15.13), we get

$$p^{\nu}f_{\mu\nu}=0\,, \tag{15.15}$$

which together with (15.14) leads to Maxwell's equations, where we used the fact that the tensor $\phi_{\mu[\nu\rho]}+\phi_{\nu[\rho\mu]}+\phi_{\rho[\mu\nu]}$ is fully antisymmetric.

In the presence of a source, Eq. (15.1) can be replaced by

$$(\Gamma_{\mu\nu}\beta^{\mu}p^{\nu})_{ab}\phi_{b}=Q_{\mu\nu a}\beta^{\mu}J^{\nu}\,, \tag{15.16}$$

where the $SO(3,2)$ index ξ $(=1, 2, \ldots, 5)$ has been suppressed, and where the source J^{ν} represents five currents of the form $J^{\nu}{}_{\xi}$. The explicit forms of Eq. (15.16) are given by

$$\begin{aligned}g_{\mu\rho}p^{\sigma}\phi_{5[\nu\sigma]}-g_{\mu\nu}p^{\sigma}\phi_{5[\rho\sigma]}&\\-(p_{\mu}\phi_{5[\nu\rho]}+p_{\nu}\phi_{5[\rho\mu]}+p_{\rho}\phi_{5[\mu\nu]})&\\=g_{\mu\rho}J_{\nu}-g_{\mu\nu}J_{\rho}&\end{aligned} \tag{15.17}$$

and

$$\begin{aligned}p_{\mu}\phi^{\rho}{}_{[\nu\rho]}-p_{\nu}\phi^{\rho}{}_{[\mu\rho]}&\\+p^{\rho}(\phi_{\rho[\mu\nu]}+\phi_{\mu[\nu\rho]}+\phi_{\nu[\rho\mu]})=J_{[\mu\nu]}\,,&\end{aligned} \tag{15.18}$$

where

$$J_{[\mu\nu]}=J_{\mu\nu}-J_{\nu\mu}\,.$$

From (15.17), multiplying by $g^{\mu\nu}$, we obtain

$$\begin{aligned}p^{\rho}\phi_{5[\mu\rho]}&=J_{\mu}\,,\\p_{\mu}\phi_{5[\nu\rho]}+p_{\nu}\phi_{5[\rho\mu]}+p_{\rho}\phi_{5[\mu\nu]}&=0\,.\end{aligned} \tag{15.19}$$

Equations (15.18) also lead to Maxwell's equations where the current vector is defined as

$$J_{\mu}=(1/p)p^{\nu}J_{[\mu\nu]}\,, \tag{15.20}$$

which, of course, is conserved since $p^{\mu}J_{\mu}=0$.

According to this theory the photon is described by the wave function $\phi_{\xi a}$ $(\equiv\phi_{5a},\phi_{\mu a})$ obeying Eq. (15.1). However, the components ϕ_{5a} and $\phi_{\mu a}$ are decoupled and each, both in the absence and also in the presence of external sources, describes the same field. Although Eq. (15.1) is a special case of the fundamental wave equation (1.6), the former could form a basis to propose the latter equation for all the particles.

From the periodicity or the recurrence of the [5,6] in the [5,10] and [5,15], it follows that the special case $\rho=\lambda=0$ recurs as Maxwell's equations in the latter supermultiplets.

XVI. MASSLESS 2+ PARTICLE AND GRAVITON

The special state $\rho=0$, $\lambda=0$ for the supermultiplet [5,20] leads to an interesting result. For $N=20$, the wave equation (15.1) for the massless integral-spin particles becomes

$$\begin{Bmatrix}\rho\\\mu\ \ \nu\end{Bmatrix}+\tfrac{1}{2}g^{\alpha\beta}\begin{Bmatrix}\sigma\\\alpha\ \ \beta\end{Bmatrix}(\delta_{\mu}{}^{\rho}g_{\sigma\nu}+\delta_{\nu}{}^{\rho}g_{\sigma\mu}-\delta_{\sigma}{}^{\rho}g_{\mu\nu})$$
$$-(1/\sqrt{3})(\delta_{\nu}{}^{\rho}p_{\mu}+\delta_{\mu}{}^{\rho}p_{\nu}-\tfrac{1}{2}g_{\mu\nu}p^{\rho})\gamma_{1}=0\,, \tag{16.1}$$

where Christoffel symbols

$$\begin{Bmatrix}\rho\\\mu\ \ \nu\end{Bmatrix}$$

are defined by

$$\begin{Bmatrix}\rho\\\mu\ \ \nu\end{Bmatrix}=\tfrac{1}{2}g^{\rho\sigma}(p_{\nu}\gamma_{\mu\sigma}+p_{\mu}\gamma_{\nu\sigma}-p_{\sigma}\gamma_{\mu\nu})\,, \tag{16.2}$$

and where the wave functions $\gamma_{\mu\nu}$ and γ_{1} are

$$\gamma_{\mu\nu}=(\Gamma_{\mu\nu})_{\{ab\}}\phi_{5\{ab\}}\,,\quad \gamma_{1}=g_{ab}\phi_{5\{ab\}}\,, \tag{16.3}$$

and

$$g^{\mu\nu}\gamma_{\mu\nu}=0\,. \tag{16.4}$$

Furthermore, Eqs. (11.6)–(11.8) yield, for $\rho=0$, $\lambda=0$, the results

$$p^{2}\gamma_{1}=0\,,\quad p^{2}\gamma_{2}=0\,, \tag{16.5}$$

$$p^{2}\gamma_{\mu\nu}+\tfrac{4}{3}p^{\rho}(p_{\mu}\gamma_{\nu\rho}+p_{\nu}\gamma_{\mu\rho})-(16/3\sqrt{3})p_{\mu}p_{\nu}\gamma_{1}=0\,, \tag{16.6}$$

where

$$p^{2}\gamma_{2}=p^{\mu}p^{\nu}\gamma_{\mu\nu}\ (=0)\,.$$

However, the wave function $\phi_{5\{ab\}}$ for $\rho=0$, $\lambda=0$ is real and therefore $\gamma_{1}=0$. Hence, contracting Eq. (16.1) with $g^{\mu\nu}$, we obtain the result

$$p^{\nu}\gamma_{\mu\nu}=0\,, \tag{16.7}$$

and (16.6) yields the equation

$$p^{2}\gamma_{\mu\nu}=0\,. \tag{16.8}$$

Equation (16.8) together with the restrictions (16.4) and (16.7) describes a massless 2^{+} particle. Now using (16.2) in the form

$$\begin{Bmatrix}\rho\\\mu\ \ \rho\end{Bmatrix}=0\,, \tag{16.9}$$

we can rewrite Eqs. (16.7) and (16.8) in the form

$$R_{\mu\nu}=0\,, \tag{16.10}$$

where

$$R_{\mu\nu}=p_\rho\left\{\begin{matrix}\rho\\ \mu\quad\nu\end{matrix}\right\}-p_\nu\left\{\begin{matrix}\rho\\ \mu\quad\rho\end{matrix}\right\} \tag{16.11}$$

is the linear and flat space form of the curvature tensor in general relativity.

If the reality of the $\phi_{5\{ab\}}$ is not assumed, then the contraction of (16.1) with respect to the indices ρ and ν yields

$$p_\mu\gamma_1=(4/3\sqrt{3})p^\rho\gamma_{\mu\rho}. \tag{16.12}$$

Using this in (16.1) and the equation obtained from (11.7) by setting $\rho=\lambda=0$, we obtain the results

$$p_\mu\gamma_{\nu\rho}-p_\nu\gamma_{\mu\rho}-p_\rho\gamma_{\mu\nu}+\tfrac{1}{9}p^\sigma(5g_{\nu\rho}\gamma_{\sigma\mu}-g_{\mu\rho}\gamma_{\sigma\nu}-g_{\mu\nu}\gamma_{\rho\sigma})=0, \tag{16.13}$$

$$p^2\gamma_{\mu\nu}+\tfrac{4}{3}(p_\nu p^\sigma\gamma_{\mu\sigma}-(7/9)p_\mu p^\sigma\gamma_{\sigma\nu}-(1/18)g_{\mu\nu}p^2\gamma_2)=0. \tag{16.14}$$

By operating on (16.13) with p^ρ and then eliminating the resulting expression for γ_2 from (16.14) we obtain

$$p^2\gamma_{\mu\nu}-(44/45)(p_\mu p^\rho\gamma_{\nu\rho}-p_\nu p^\rho\gamma_{\mu\rho})=0, \tag{16.15}$$

which implies

$$p^2\gamma_{\mu\nu}=0,\quad p_\mu p^\rho\gamma_{\nu\rho}-p_\nu p^\rho\gamma_{\mu\rho}=0. \tag{16.16}$$

Equations (16.16) are satisfied by (16.12). Thus, together with (16.4) and (16.12), Eqs. (16.16) provide eight conditions on the wave function or the "gravitational potentials" $\gamma_{\mu\nu}$. Therefore, like any other massless particle of any spin, the graviton has two independent states of polarization.

XVII. SUMMARY AND CONCLUSIONS

The detailed discussion of the wave equation for all the particles (hadrons, leptons, photon, graviton, ...) first presented in I has been continued here. From the G-symmetry $[=SO(3,2)\otimes U(3,1)]$ point of view, all free particles can be put into three classes.

(i) For massless particles of all spins, the special case of $\rho=0$, $\lambda=0$, the G symmetry is exactly broken.

(ii) For massive leptons ($\rho_l<1$), the G symmetry is badly broken.

(iii) For hadrons ($\rho_H>1$), the G symmetry is approximately broken.

All free leptons belong to the $[4_l,N]$ reducible representations of G. All free baryons belong to the $[4_B,N]$ reducible representations of G and all the free mesons as well as photons, gravitons, ..., belong to the $[5,N]$ and $[10,N]$ reducible representations of G. Furthermore, we have seen that in the space of s, $\mathcal{P}$, Z, charge, etc., all baryons and also all mesons are identical particles obeying a "super-exclusion principle." The wave function of any pair of baryons or pair of mesons is antisymmetrical with respect to an interchange of their respective quantum numbers s, Z, $\mathcal{P}$, ... Thus the super-exclusion principle can form the basis for the meson and baryon proliferation. The limiting case $\rho=\infty$ as pointed out in I results in the Dirac or Kemmer type of equation depending on the representation of the G symmetry. However, the limiting case $\rho=0$, $\lambda=0$ leads for the $[5,4]$, $[5,6]$, $[5,10]$, and $[5,15]$ to Maxwell's equations and for the $[5,20]$ to the wave equation for a 2^+ massless particle, the graviton. In fact, the result for the $[5,20]$ has been expressed in terms of the linearized form of Einstein's field equations in general relativity.

The limiting case $\rho=0$, $\lambda=0$, in the free massive lepton and the baryon wave equations (1.3) and (1.4), yields

$$\Gamma_{\mu\nu}\gamma_5\gamma^\mu p^\nu\phi_l=0 \tag{17.1}$$

and

$$\Gamma_{\mu\nu}\gamma^\mu p^\nu\phi_b=0, \tag{17.2}$$

which, by applying the projection operator $\frac{1}{2}(1+i\gamma_5)$, can be transformed into the same type of equations. In fact, both (1.3) and (1.4) and also (17.1) and (17.2) have the same principal quantum numbers $[N_0,N]$, which is not the case for the $[5,N]$ and $[10,N]$, where N_0 assumes different values. Therefore a real physical situation (i.e., the presence of interaction) does require a coupling between Eqs. (1.3) and (1.4) to describe weak interactions. The coupling in question, because of the requirement of Lorentz invariance, is between $[4_l,N]$ and $[4_b,N]$, where N is the same for both leptonic and baryonic supermultiplets. It must be observed that the requirement of identical N for the l and b supermultiplets does not rule out the possibility of weak interaction of a higher-level b multiplet with a lower-level l multiplet. The periodicity of the level structure does allow intersupermultiplet interactions. Furthermore, the lowest supermultiplets, because of (17.1) and (17.2), in the weak interaction of leptons and baryons are $[4_l,4]$ and $[4_b,4]$. The levels $[4_l,1]$ and $[4_b,1]$, being of $U(1)$ symmetry, do not recur. Hence we see that this theory, in the lowest level $[4,4]$ for $\rho=0$, predicts two different $s=\frac{1}{2}$ two-component neutrinos. All of the above points and application of the same ideas to the strong interactions will be amplified in the next paper.

For the baryons, as seen in Fig. 1 and in the corresponding mass formulas for the various supermultiplets, the mass region in the neighborhood of m is, for both parities, densely populated. All of these, with the right parities, are found to be fittable with m, assuming the mass value in the range 1650–1700 MeV. The five $J^P=\frac{1}{2}^-$, $\frac{3}{2}^-$ baryons of equal mass in the $[4,15]$ and also the two $J^P=\frac{1}{2}$ baryons in $[4,10]$ fall into line with the experimental range of mass measurements. The situation with the mesons presents the same kind of picture. At this point we do not anticipate a strenuous effort in establishing further accord between this theory and the experiment. The presence of normal spin-parity series ($J^P=0^+, 1^-, 2^+$) in the [5,15] has a good candidate for it in the 1270–1300 MeV region, e.g., the A_2. Of the 50 mesons analyzed so far, only ten $J^P=0^-$ have been predicted. The theory also predicts $J^P=2^-$ mesons for which there is no strong experimental evidence at present.

In both Figs. 1 and 2, different supermultiplets may be related by means of the internal symmetries for which, we believe, the theory does contain sufficient provisions. Therefore the various members of, for example, pions and kaons should lie in different supermultiplets. Thus a plane through the equal-spin and equal-parity mesons (or baryons) of a given period like, for example, the [5,6] and its recurrence in the [5,10], [5,15] will contain *internal supermultiplets* or two isospin triplets of 1^- and 1^+ mesons in contrast to the space-time supermultiplets which lie in a plane perpendicular to the axis of the concentric cylinders. The latter supermultiplets contain particles of different spins and parities but belong to the same representation of the G symmetry. No one supermultiplet (internal or space-time) can be defined (i.e., s, Z, $\mathcal{Q}$, $\mathcal{P}$, internal quantum numbers) completely without the other.

The algebraic structure of the mass levels for the baryons and mesons are shown in their sum rules. The sum rules for the baryons are linear in the inverse masses, while for the mesons the sum rules are obtained in terms of the squares of the inverse masses. This property prevails throughout all the supermultiplets. Furthermore, the recurrence of the lower-level mass relations in the higher levels, like the recurrence of the [4,6] in [4,10], [4,15] for $Z=\pm1$ and of [4,10] in [4,15] for $Z=0$ and similar recurrences in the meson spectra, should aid in the identification of the members with similar properties but in different supermultiplets. This periodic nature of the hadron spectroscopy which extends through the entire spectrum of the G symmetry is a fundamental result of the theory. The periodicity discussed for the hadrons is also valid for the leptons and will provide a desirable flexibility for the concept of interaction in this theory.

Finally, we must point out that the representation of $SO(3,2)$ as well as of the Kemmer-Duffin algebra (1.7) by the set of ten matrices $\beta_5\beta_\mu$ and $\beta_{\mu\nu}$ implies the existence of integral-spin lepton supermultiplets $[5_l,N]$ and $[10_l,N]$ with similar properties as the $[4_l,N]$. It is quite reasonable to assume that the $[5_l,N]$ and $[10_l,N]$ are coupled to matter very weakly. For example, the so-far-unobserved W boson (or the integral-spin leptons) may be classified within the supermultiplets $[5_l,N]$ or $[10_l,N]$. Furthermore, the presence of $[5_l,N]$ and $[10_l,N]$ type of particles could imply the existence of a new kind of weak interaction.

The next paper of this series, besides leptons, will discuss the weak, strong, and electromagnetic interactions as inferred from this theory. In particular, we shall derive the sum rules for the magnetic moments of the baryon supermultiplets.

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