Factorization of Satellite Terms in the N-Particle Dual Amplitude. II. The Single Planar Dual Loop*

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(Received 16 March 1970)

The amplitude for a single planar dual loop is written for a loop constructed from a factorized N-particle dual amplitude with satellite terms included. The endpoint essential singularity, already present in the nonsatellite case, is strengthened by the increased degeneracy of the states, and there is no possibility of canceling the singularity by manipulating the satellite coefficients.

I. INTRODUCTION

PERHAPS the most remarkable property of the N-particle generalization of the beta-function model for four particles is its factorization at each internal pole.[1] This property is made self-evident in the operator coherent-state formalism of Nambu[2] and of Fubini, Gordon and Veneziano.[3]

With the factorized N-particle dual amplitude as a starting point, and local field theory as a guide, it is an attractive program[4] to attempt unitarization of the model by constructing dual loop diagrams in the same way that Cutkosky's rules allow construction of loop diagrams from the Born term in quantum electrodynamics. In this way one introduces an imaginary part to the scattering amplitude and builds up a finite width to the resonances. By consideration of nonplanar loop diagrams, one hopes to find the diffractive scattering contribution, or Pomeranchukon singularity, generated by ordinary Regge trajectories.[5]

Unfortunately, the simplest single planar dual loop constructed in this way gives rise to an integral with an end point essential singularity. The singular behavior arises because of the large degeneracy of the states which circulate in the loop diagram.

A formalism has recently been developed by one of us[6] in which factorization of satellite terms[7] in the N-particle dual amplitude is exhibited explicitly in an operator coherent-state formalism. Using this formalism, one can investigate the end point essential singularity of the single planar dual loop with satellite contribution included. The degeneracy of states is increased[8] but, on the other hand, a number (possibly infinite) of arbitrary satellite coefficients is introduced. In this paper we show that the end point essential singularity is strengthened by the increased level degeneracy and that there is no possibility of cancellation by manipulation of the satellite coefficients.

In the nonsatellite case, a means of renormalizing the endpoint singularity which preserves the desirable features of the amplitude has been proposed by Neveu and Scherk[9] and extended by them in collaboration with other members of the Princeton group.[10] This technique can presumably be extended to deal with the end point singularity found in the present paper when satellite contributions are included.

The plan of the paper is as follows. In Sec. II we find the satellite contributions to the single loop. In Sec. III we discuss how unequal intercepts and satellite terms may be combined in the present formalism. Finally, in Sec. IV there is some discussion.

II. EQUAL INTERCEPTS WITH SATELLITES

Let us write the nonsatellite N-particle dual amplitude corresponding to the configuration of Fig. 1 as[11]

$$A_N=\int_0^1 \prod_{i=1}^{N-3} dx_i \tilde{A}_N. \tag{2.1}$$

We now modify this amplitude as follows (see I):

$$A_N'=\int_0^1 \prod_{i=1}^{N-3} dx_i \tilde{A}_N \exp\Big[\sum_{\text{all } j,k} u_{jk}(1-u_{jk})g(u_{jk})\Big], \tag{2.2}$$

where the u_{jk} correspond to all possible diagonals of the dual diagram in momentum space and may be defined in terms of the integration variables of Eq. (2.1) as

* Supported in part by the U. S. Atomic Energy Commission.
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[2] Y. Nambu [University of Chicago Report No. EFI 69-64 (1969)], talk presented at the International Conference on Symmetries and Quark Models, Wayne State University, 1969 (unpublished).
[3] S. Fubini, D. Gordon, and G. Veneziano, Phys. Letters **29B**, 679 (1969).
[4] K. Kikkawa, B. Sakita, and M. A. Virasoro, Phys. Rev. **184**, 1701 (1969); S. Fubini and G. Veneziano (Ref. 1), note added; D. Amati, C. Bouchiat, and J. L. Gervais, Nuovo Cimento Letters **2**, 399 (1969).
[5] P. G. O. Freund and R. J. Rivers, Phys. Letters **29B**, 510 (1969); G. Frye and L. Susskind, *ibid.* **31B**, 589 (1970).
[6] P. H. Frampton, Phys. Letters **32B**, 195 (1970), hereafter referred to as I.
[7] D. J. Gross, Nucl. Phys. **B13**, 467 (1969).
[8] D. K. Sinclair, SUNY at Stony Brook report, 1970 (unpublished).
[9] A. Neveu and J. Scherk, Phys. Rev. D **1**, 2355 (1970).
[10] T. H. Burnett, D. J. Gross, A. Neveu, J. Scherk, and J. H. Schwartz, Phys. Letters **32B**, 115 (1970).
[11] In the notation of Eq. (1) of Ref. 6,

$$\tilde{A}_N=\langle 0|V(p_{N-1})D(s_{N-3},x_{N-3})V(p_{N-2})\cdots V(p_2)|0\rangle.$$

The notation for satellites of the present Sec. II follows that of Ref. 6.

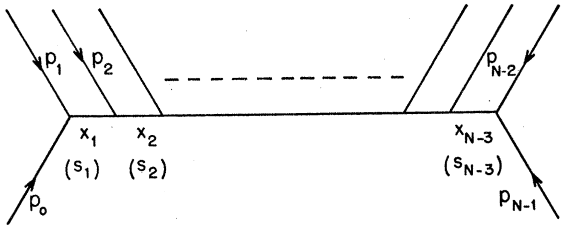

FIG. 1. Diagram indicating notation for the multiperipheral tree diagram.

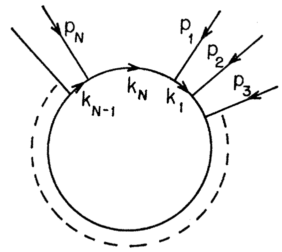

FIG. 2. Diagram indicating notation for the single planar loop diagram.

enumerated below:

$$u_{1,m+1}=x_m\,,\quad 1\leqslant m\leqslant (N-3) \tag{2.3}$$

$$u_{l+1,m+1}=\frac{(1-x_lx_{l+1}\cdots x_{m-1})(1-x_{l-1}x\cdots x_m)}{(1-x_{l-1}x_l\cdots x_{m-1})(1-x_lx_{l+1}\cdots x_m)}\,,\quad 1\leqslant l<m\leqslant (N-2)\,,\quad \tfrac{1}{2}(N-2)(N-3)\text{ variables}. \tag{2.4}$$

$$\text{Total}=\tfrac{1}{2}N(N-3)\text{ variables}.$$

For $g(u)$ an arbitrary analytic function, the amplitude may be rewritten, using the operator formalism of I, as

$$A_N'=\langle 0|\,V'(p_{N-2})P(s_{N-3})V'(p_{N-3})\cdots P(s_1)V'(p_1)\,|0\rangle\,, \tag{2.5}$$

in which

$$P(s)=\int_0^1 dx\,x^{-1-\alpha(s)+H'}M^\dagger(x)x^LM(x)\,\exp\{x(1-x)[g(x)+g(1-x)]\}\,, \tag{2.6}$$

$$M(x)=\exp[\sum_{r=1}^{\infty}A^{r,s}(W_r{}^0{}_{,s}x^r+\sum_{p=1}^{r}W_r{}^p{}_{,s}x^p)]\,,$$

$$L=\sum_{r=1}^{\infty}A^{r,s,\dagger}A^r{}_{,s}\,, \tag{2.7}$$

$$u_{jk}(1-u_{jk})g(u_{jk})=\sum_{r=1}^{\infty}\sum_{s=0}^{r}\sum_{p,q=0}^{r}W_r{}^p{}_{,s}W_r{}^q{}_{,s}G_r{}^{ss}x_{j-2}{}^px_{k-1}{}^q(x_{j-1}x_j\cdots x_{k-2})^r\,, \tag{2.8}$$

$$G_r{}^{ss'}=\pm\delta_{ss'}\,, \tag{2.9}$$

and the operators $A^{r,s}$, $A^{r,s\dagger}$ satisfy

$$[A^{r,s},A^{r',s'\dagger}]=\delta_{rr'}G_r{}^{ss'}\,, \tag{2.10}$$

with $s=0,\,1,\,2,\,\ldots,\,r$.

Because of the indefinite metric in Eq. (2.9), we see that in general the ghost problem is heightened in the amplitude with satellites. Already we know[12] that the Ward-like identities arising from reflection symmetry are not useful to remove ghost states when satellites are present.

The N-particle single loop of Fig. 2 is described by the amplitude

$$B_N'=\int d^4k\,M_N'\,, \tag{2.11}$$

where

$$M_N'=\mathrm{Tr}[P(k_N{}^2)V'(p_1)P(k_1{}^2)V'(p_2)\cdots P(k_{N-1}{}^2)V'(p_N)]\,, \tag{2.12}$$

which is constructed as in Amati *et al.*[4]

We may perform the trace in the higher dimensions separately. For a given mode, the trace is

$$\frac{1}{\pi}\int d\,\mathrm{Re}\Lambda d\,\mathrm{Im}\Lambda\,\exp(-|\Lambda|^2)\langle\Lambda|\,M^\dagger(x_N)x_N{}^LM(x_N)M^\dagger(x_{N-1})\cdots M^\dagger(x_1)x_1{}^LM(x_1)\,|\Lambda\rangle$$

$$=\frac{1}{(1-w^r)^{r+1}}\exp\Bigg\{\sum_{s=0}^{r}\Bigg[\frac{w^r}{1-w^r}\sum_{i,j=1}^{N}[(\sum_{p=1}^{r}W_r{}^{sp}x_i{}^p+W_r{}^{s0}x_i{}^r)\rho_{i-1}{}^r\rho_j{}^{-r}(\sum_{q=1}^{r}W_r{}^q{}_{,s}x_j{}^q+W_r{}^0{}_{,s}x_j{}^r)]$$

$$+\sum_{1\leqslant i<j\leqslant N}[(\sum_{p=1}^{r}W_r{}^{sp}x_i{}^p+W_r{}^{0s}x_i{}^r)\rho_{i-1}{}^r\rho_j{}^{-r}(\sum_{q=1}^{r}W_r{}^q{}_{,s}x_j{}^q+W_r{}^0{}_{,s}x_j{}^r)]\Bigg]\Bigg\}\,. \tag{2.13}$$

[12] R. J. Rivers, Phys. Rev. (to be published).

Here

$$\rho_i=\prod_{j=1}^{i} x_j \quad \text{and} \quad w=\rho_N. \tag{2.14}$$

Summing over the modes and vector components, we arrive at

$$M_N'=\int \prod_{i=1}^{N} (dx_i x_i^{-\alpha(k_i^2)-1}(1-x_i)^{-c}\exp\{x_i(1-x_i)[g(x_i)+g(1-x_i)]\})$$

$$\times\prod_{r=1}^{\infty}\left\{\frac{1}{(1-w^r)^{r+5}}\exp\left[-\frac{1}{r(1-w^r)}\sum_{i,j=1}^{N} C_{ij}{}^r p_i\cdot p_j\right]\exp\left[\sum_{s=0}^{r}\left(\frac{w^r}{1-w^r}\sum_{i,j=1}^{N}[(\sum_{p=1}^{r} W_r{}^{sp}x_i{}^p+W_r{}^{s0}x_i{}^r)\right.\right.\right.$$

$$\times\rho_{i-1}{}^r\rho_j{}^{-r}(\sum_{q=1}^{r} W_r{}^q{}_{,s}x_j{}^q+W_r{}^0{}_{,s}x_j{}^r)]+\sum_{1\leqslant i<j\leqslant N}[(\sum_{p=1}^{r} W_r{}^{sp}x_i{}^p+W_r{}^{0,s}x_i{}^r)\rho_{i-1}{}^r\rho_j{}^{-r}$$

$$\left.\left.\left.\times(\sum_{q=1}^{r} W_r{}^q{}_{,s}x_j{}^q+W_r{}^0{}_{,s}x_j{}^r)]\right)\right]\right\}, \tag{2.15}$$

where $C_{ij}{}^n$ is defined by Amati *et al.*[4]

We must now consider the behavior of the integrand in Eq. (2.15) for $w\to 1$. The most obvious singularity is from the factor

$$\prod_{r=1}^{\infty}\frac{1}{(1-w)^{r+5}}\overset{w\to 1}{\sim}\exp[\zeta(3)(1-w)^{-2}], \tag{2.16}$$

where $\zeta(z)$ is the Rieman ζ function. It is necessary to investigate whether this strong singularity can be canceled by adjusting the satellite coefficients

$$C_r{}^{pq}=\sum_{s=0}^{r} W_r{}^{ps}W_r{}^{qs}G_r{}^{ss} \tag{2.17}$$

of Eq. (2.8). Note that the factor containing

$$\exp\left[\frac{1}{r(1-w^r)}\sum_{i,j=1}^{N} C_{ij}{}^r p_i\cdot p_j\right] \tag{2.18}$$

is well behaved at $w=1$ because of momentum conservation, as in Amati *et al.*[4] We need consider only the exponent

$$\sum_{r=1}^{\infty}\sum_{s=0}^{r}\frac{w^r}{1-w^r}\sum_{i,j=1}^{N}[(\sum_{p=1}^{r} W_r{}^{ps}x_i{}^p+W_r{}^0{}_{,s}x_i{}^r)\rho_{i-1}{}^r\rho_j{}^{-r}(\sum_{q=1}^{r} W_r{}^q{}_{,s}x_j{}^q+W_r{}^0{}_{,s}x_j{}^r)]$$

$$=\sum_{r=1}^{\infty}\frac{w^r}{1-w^r}\sum_{i,j=1}^{N}[\sum_{p,q=0}^{r} C_r{}^{pq}x_i{}^p\rho_{i-1}{}^r\rho_j{}^{-r}x_j{}^q-\sum_{q=1}^{r} C_r{}^{0q}(1-x_i{}^r)\rho_{i-1}{}^r\rho_j{}^{-r}x_j{}^q-\sum_{p=1}^{r} C_r{}^{p0}x_i{}^p\rho_{i-1}{}^r\rho_j{}^{-r}(1-x_j{}^r)$$

$$-C_r{}^{00}(1-x_i{}^r x_j{}^r)\rho_{i-1}{}^r\rho_j{}^{-r}]. \tag{2.19}$$

It is now straightforward to show that if we define expansion coefficients by $x(1-x)g(1-x)=C_nx^n$, then the first of the three terms in Eq. (2.19) goes to the limit $\sum C_n\zeta(2n)$ when we approach the point $w=1$ along any edge of the unit hypercube volume of integration. From the fact that $x(1-x)g(x)$ must be analytic almost everywhere in $0\leqslant x\leqslant 1$, we can derive that the limit is always finite. The other terms in Eq. (2.19) also have finite values in the limit $w\to 1$.

Returning to Eq. (2.15) for M_N', we can see that the end point essential singularity for equal intercepts with satellites is given by the term

$$\prod_{r=1}^{\infty}(1-w^r)^{-d(r)-4}, \tag{2.20}$$

where $d(r)$ is the dimension of the operators A^{rs} for mode r. In the present case, $d(r)=(r+1)$.

The essential singularity is stronger than that without satellites because of the greater density of states. The asymptotic degeneracy of the level $\alpha(s)=n$ for $n\to\infty$ is given by $D(n)\sim\exp(\text{const } n^{2/3})$ in the case with satellites[6,8] compared to $D(n) \sim \exp(\text{const } n^{1/2})$ without satellites.

Further, as the main conclusion of this section, we notice that it is not possible to arrange the satellite coefficients C_r^{pq} such that the amplitude is finite for the single planar dual loop.

III. UNEQUAL INTERCEPTS

In Sec. II we took the Regge intercepts $\alpha(0)$ to be the same in all channels.

It is straightforward to combine satellite terms with unequal intercepts. As an example we consider the case of most physical interest, namely, the case where all external scalar lines are considered to have negative G parity. In a multiperipheral configuration, the intermediate-state trajectories then alternate between even and odd G parity. Isospin complications will be neglected.

The operators $a_\mu^{(r)} a_\mu^{(r)+}$ of Refs. 2 and 3 are extended to a fifth dimension.[13] The metric of the fifth dimension is the same as the three space components. The 5-momentum is defined by

$$p_\alpha = (p_\mu, \sqrt{(\tfrac{1}{2}d)}),$$

where $d = 2[\alpha_{\text{odd}}(0) - \alpha_{\text{even}}(0)]$. The scalar mode can be quantized as in Amati *et al.*[4] since only the even-G-parity two-particle channels contribute to this mode. For the nonsatellite case, the trace in the dual loop goes through exactly as in Amati *et al.*[4] and the result is an extra partition function in the integrand arising from the fifth dimension. In the limit $w \to 1$ the other factors are finite.

If we include G-parity intercepts and satellite terms, the essential singularity is from a factor

$$\prod_{r=1}^{\infty} (1-w^r)^{-d(r)-5}.$$

IV. SUMMARY

We have shown that introduction of satellite terms into the N-particle dual amplitude leads to a strengthening of the end point essential singularity of the single dual loop due to the increased density of states. In Sec. II we showed that there is no possibility of a cancellation by manipulation of the satellite coefficients. In Sec. III we indicated how differing intercepts may straightforwardly be introduced with satellite terms present.

For the nonsatellite case the end point singularity has been treated by Neveu and Scherk,[9,10] who give a prescription for separating a finite part from the divergent integral. It will be interesting to find whether such a prescription can be generalized to the case of satellites.

[13] S. Fubini and G. Veneziano, Ann. Phys. (N. Y.) (to be published); see also Y. Nambu (Ref. 2).