Statistical Basis for the Many-Body Model of Matter*

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The premise that physical variables must be treated as random variables is shown to yield a purely statistical basis for the many-body model of matter in both classical and quantum forms. This derivation also yields an alternative to the conventional quantum formalism in which the usual distinction between bosons and fermions does not appear.

INTRODUCTION

TWO recent publications[1,2] have introduced a derivation of the formal structures of mechanics, thermodynamics, and quantum theory directly from the statistical nature of experimental data. Continued study of this statistical theory suggests some slight alterations in the derivation and extensions to encompass other segments of physics in a natural way.

The present paper shows that the many-body model of matter can be deduced directly from the statistical properties of data in both a classical and a quantum-mechanical format. Furthermore, we show that there is a format for the quantum model that has not been previously recognized in which there is no distinction between particles on the basis of spin.

The derivation given here is prefaced with what seems to be a more satisfactory derivation of the thermodynamic formalism. Furthermore, the entire statistical description is here expressed in the appropriate context of a duplicated experiment.

EXPERIMENTS AND STATISTICS

A cornerstone of quantitative science is the premise that any meaningful experiment can be reproduced as many times as we please; in fact, only such experiments or "systems" rightfully belong to the domain of science. However, since not every element, or "particle," of a physical system, or of the measuring instruments themselves, is subject to control by an experimenter, every physical variable must be viewed as a random variable. This was stated as a basic premise of our earlier papers and is here our justification for asserting that the proper mathematical description of an experiment is a statistical description.

Thus if we carry out any number of duplications of an experiment, in which, for example, we prepare the system and then measure certain chosen properties of the system, for instance only the variable x, we obtain a distribution of measured values. The duplication, or repetition, of the experiment is *defined* by asserting that the probability for a measured value, say

$$d\bar{P}(x|\eta)=f(x|\eta)dx, \tag{1}$$

is exactly the *same* for every duplication of the experiment. This is a conditional probability having values of parameters $\eta_1, \eta_2, \ldots, \eta_J$ fixed in the preparation of the system as contingent event. The conditioning parameters

$$\eta=(\eta_1, \eta_2, \ldots, \eta_J) \tag{2}$$

were referred to in our earlier papers as the state vector of the environment.

The embedding structure of our earlier papers takes into account the fact that once a system has been prepared, we may then measure a variety of different properties of the system. Thus we define the set of random variables

$$\xi=(\xi_1, \xi_2, \ldots, \xi_k), \quad \xi\in R \tag{3}$$

where R is the set of all possible configurations of ξ, and we require that the ξ_i have a probability distribution

$$dP(\xi|\eta)=\rho(\xi|\eta)d\tau(\xi) \tag{4}$$

such that the probability distribution of every physical variable accessible to measurement can then be defined as an integral over some subset of R. That is, every variable accessible to measurement in the experiment can be expressed as a function of the ξ_i, such as, e.g., $x(\xi)$, and the probability for a measured value of any such variable can be expressed as

$$d\bar{P}(x|\eta)=\int_{R_x}\rho(\xi|\eta)d\tau(\xi), \tag{5}$$

where

$$\xi\in R_x\subset R, \quad \text{if and only if } x\leqslant x(\xi)\leqslant x+dx, \tag{6}$$

for example. Certainly the number of ξ_i, K, must be *at least* as large as the number of functionally independent physical variables accessible to measurement, but it may be larger.

Beyond the stipulations just given, the *form* of the probability distribution $\rho(\xi|\eta)$ is essentially arbitrary, but as we pointed out in our earlier papers, every conditional probability density can be shown to be representable in the form

$$\rho(\xi|\eta)=e^{-\sigma(\xi,\eta)}/B(\eta), \tag{7}$$

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[1] R. E. Collins, Phys. Rev. **183**, 1081 (1969).
[2] R. E. Collins, in *A Critical Review of Thermodynamics*, edited by E. B. Stuart, Benjamin Gal-Or, and A. J. Brainard (Mono, Baltimore, 1970).

where $\sigma(\xi,\eta)$ is required to be a real function, while $B(\eta)$ must be represented as the integral of the exponential here over the domain $\xi\in R$. Then $\sigma(\xi,\eta)$ can be *chosen*[1] as a finite linear combination of functions $p_n(\xi)$, $n=0, 1, \ldots, M$ of the ξ_i so that

$$dP(\xi|\eta)=\frac{1}{B(q_n(\eta))}\exp[-\sum_{n=0}^{M} q_n(\eta)p_n(\xi)]d\tau(\xi), \quad \xi\in R. \quad (8)$$

It is obvious that the p_n here are additive quantities for a composite of two statistically independent systems having the same q_n, i.e., these are extensive properties of the system. It is also obvious that since the η_j enter here only in the combinations $q_n(\eta)$, we may delete notation of the η_j altogether. This implies that the environmental parameters, which may be exceedingly numerous, only condition the experiment in certain "collective" ways that are characterized by relatively few parameters, the $q_n(\eta)$, $n=0, 1, \ldots, M$; that is, if M actually is a *small* number compared to the number of η_j.

One can easily show that if the $p_n(\xi)$ are bounded, measurable functions of the ξ_j over R, and R has finite measure, then not only will the integral of this distribution over R exist, in the Lebesgue sense, but also the expectation value of each of the $p_n(\xi)$ will exist and be bounded. Thus the $p_n(\xi)$ can justifiably be selected as physical variables which may be measured in the experiment.

We now show briefly how this structure yields the formalism of thermodynamics in what seems like a natural and proper manner, and then how the many-body model of a system follows.

THERMODYNAMICS

We denote $p_0(\xi)$ by $E(\xi)$ and q_0 by $\beta=1/kT$, and then define a particular *macrostate by*

$$p_n(\xi)=p_n=\text{const}, \quad n=1, 2, \ldots, M \quad \text{if and only if } \xi\in R_s\subset R. \quad (9)$$

The probability for this state is, by the ordinary rules of probability theory,

$$\pi_s=\int_{R_s} dP(\xi|q)=\frac{1}{B}\exp(-\sum_{n=1}^{M} q_np_n)\times\int_{R_s} e^{-\beta E(\xi)}d\tau(\xi), \quad (10)$$

or

$$\pi_s=\frac{1}{B}\exp(-\sum_{n=1}^{M} q_np_n)Z, \quad (11)$$

where

$$Z=\int_{R_s} e^{-\beta E(\xi)}d\tau(\xi). \quad (12)$$

Then the most probable macrostate, which we call the "equilibrium state," is defined by

$$\partial\pi_s/\partial p_n=0, \quad n=1, 2, \ldots, M. \quad (13)$$

This yields the equilibrium values of the p_n as functions of the q_n and β through the forms

$$q_n=\partial \ln Z/\partial p_n, \quad n=1, 2, \ldots, M \quad (14)$$

where it is noted that the p_n enter Z only in the definition of R_s.

From Eqs. (8) and (11) we then have the conditional probability for ξ, given fixed values of $p_1, p_2, \ldots, p_M$ and β as conditions:

$$dP(\xi|\beta,p)=(e^{-\beta E(\xi)}/Z)d\tau(\xi), \quad (15)$$

with $\xi\in R_s(p_1, \ldots, p_M)$. This is the canonical distribution function and, as pointed out by Mandelbrot,[3,4] the average value of E over a set of measurements is a sufficient statistic for the value of β, or, as pointed out in our earlier paper,[1] is the *observed value* of E. Our derivation here shows how the values of $p_1, p_2, \ldots, p_M$ enter into the determination of the domain R_s available to ξ and hence into the determination of the expectation value of E.

Denoting the probability density $e^{-\beta E(\xi)}/Z$ here by $\rho_s(\xi|\beta,p)$, we define

$$S=-k\int_{R_s}\rho_s\ln\rho_s d\tau, \quad k>0 \quad (16)$$

and then show that since the expectation value of $E(\xi)$ over R_s is given by

$$E_s=\int_{R_s} E(\xi)\rho_s(\xi|\beta,p)d\tau(\xi)=-\frac{\partial \ln Z}{\partial\beta}, \quad (17)$$

we have

$$dS=k\beta dE_s+k\sum_{n=1}^{M}\frac{\partial \ln Z}{\partial p_n}dp_n, \quad (18)$$

where Z is the function of $p_1, p_2, \ldots, p_n$ and β defined in Eq. (12). In particular, if we demand that these p_n have their most probable values for the fixed q_m, $m=0, 1, 2, \ldots, M$, then according to Eq. (14)

$$dS=k\beta dE_s+k\sum_{n=1}^{M} q_n dp_n, \quad (19)$$

which is the fundamental identity of thermodynamics.

The only significant change here from the formal structure of our earlier papers is in the way the set R_s is defined and the inclusion of $q_0=\beta$ from the outset as one of the fixed parameters q_n, with $E(\xi)$ in the role of the conjugate $p_n(\xi)$. The significance of the requirement that the p_n, $n=1, 2, \ldots, M$, have their most probable values to define thermodynamic equilibrium, and the generalization to the other ensembles of thermo-

[3] Benoit B. Mandelbrot, Ann. Math. Stat. **33**, 1021 (1962).
[4] Benoit B. Mandelbrot, J. Math. Phys. **5**, 164 (1964).

dynamics by fixing only some of these p_n, remains essentially the same as given in our earlier paper.[2] Now we show how the many-body model follows naturally from this statistical theory.

MANY-BODY MODEL

We now consider the variables ξ_i, $i=1, 2, \ldots, K$ of the embedding space and their distribution function

$$\rho_s(\xi_1, \xi_2, \ldots, \xi_K|\beta,p)=e^{-\beta E(\xi_1,\xi_2,\ldots,K)}/Z(\beta,p). \quad (20)$$

This is a probability distribution defined over the space $R_s \subset R$ having volume element $d\tau(\xi)$.

We assume that by a suitable *choice* of the embedding variables this K-dimensional space can be represented as a direct-product space, or Cartesian product, of N identical spaces. In particular, we will require that

$$K/N=g, \quad (21)$$

where g, as well as K and N, is an integer. While not necessary for the formal structure, this requirement on the embedding will be seen to bring it into the form familiar to us in statistical mechanics.

With this requirement, we have

$$R_s=R_1\otimes R_2\otimes R_3\otimes\cdots\otimes R_N, \quad (22)$$

with $\xi_1, \xi_2, \ldots, \xi_g$ belonging to R_1, $\xi_{g+1}, \ldots, \xi_{2g}$ belonging to R_2, etc. These N subsets of the ξ_i are now denoted as the vectors $\hat{\xi}_1, \hat{\xi}_2, \ldots, \hat{\xi}_N$ and the volume element of R_s is written as

$$d\tau(\xi)=d\tau_1(\hat{\xi}_1)d\tau_2(\hat{\xi}_2)\cdots d\tau_N(\hat{\xi}_N). \quad (23)$$

Then we see ρ_s as the joint probability distribution for configurations of these N vectors, that is, $\rho_s(\hat{\xi}_1,\hat{\xi}_2,\ldots,\hat{\xi}_N)$, where we delete notation of the conditional parameters for simplicity in notation.

Now we observe that by the ordinary rules of probability theory, this joint distribution can be factored in the form

$$\begin{aligned}\rho_s(\hat{\xi}_1,\hat{\xi}_2,\ldots,\hat{\xi}_N)=&\{\rho_1(\hat{\xi}_1)\rho_2(\hat{\xi}_2)\cdots\rho_N(\hat{\xi}_N)\}\\ &\times\{C_{12}(\hat{\xi}_1,\hat{\xi}_2)\cdots C_{ij}(\hat{\xi}_i,\hat{\xi}_j)\cdots\}\\ &\times\{C_{123}(\hat{\xi}_1,\hat{\xi}_2,\hat{\xi}_3)\cdots C_{ijk}(\hat{\xi}_i,\hat{\xi}_j,\hat{\xi}_k)\cdots\}\cdots\\ &\quad C_{123\ldots N}(\hat{\xi}_1,\hat{\xi}_2,\ldots,\hat{\xi}_N). \quad (24)\end{aligned}$$

Here $\rho_i(\hat{\xi}_i)$ is the probability distribution for the vector $\hat{\xi}_i$ alone, $C_{ij}(\hat{\xi}_i,\hat{\xi}_j)$ is the two-vector correlation function for $\hat{\xi}_i$, and $\hat{\xi}_j$, $i\neq j$, $C_{ijk}(\hat{\xi}_i,\hat{\xi}_j,\hat{\xi}_k)$ is the three-vector correlation of $\hat{\xi}_i$, $\hat{\xi}_j$, $\hat{\xi}_k$, $i\neq j\neq k$, and similarly for all the higher-order correlations up to the N-vector correlation $C_{12\cdots N}(\hat{\xi}_1,\hat{\xi}_2,\ldots,\hat{\xi}_N)$.

There are, of course, N of the ρ_i functions, $\frac{1}{2}N(N-1)$ of the C_{ij} functions, etc., up to just one N-vector function. All of these functions must be non-negative and furthermore the integral of each $\rho_i(\hat{\xi}_i)$ over the corresponding R_i domain must be unity. Thus, by the arguments of our earlier paper,[1] each of these $\rho_i(\hat{\xi}_i)$ functions must have the functional structure appropriate to *any* conditional probability density:

$$\rho_i(\hat{\xi}_i|\beta,p)=\exp[-\sigma_i(\hat{\xi}_i,\beta,p)]/z_i(\beta,p). \quad (25)$$

Here, and in all following equations, we indicate dependence on the p_n by the single symbol p. By the same sort of argument, each of the C functions has a similar structure. Thus, for example,

$$C_{ij}(\hat{\xi}_i,\hat{\xi}_j,p)=\exp[\beta-G_{ij}(\hat{\xi}_i,\hat{\xi}_j,\beta,p)]/L_{ij}(\beta,p). \quad (26)$$

Therefore Z must be the product of all those factors indicated which do not involve the $\hat{\xi}_i$, and we must then have from Eq. (24)

$$\begin{aligned}\beta E(\hat{\xi}_1,\hat{\xi}_2,\ldots,\hat{\xi}_N)=&\sum_{i=1}^{N}\sigma_i(\hat{\xi}_i,\beta,p)+\sum_{\substack{i=1\\(i\neq j)}}^{N}\sum_{j=1}^{N}G_{ij}(\hat{\xi}_i,\hat{\xi}_j,\beta,p)\\ &+\cdots+G_{12\ldots N}(\hat{\xi}_1,\hat{\xi}_2,\ldots,\hat{\xi}_N,\beta,p). \quad (27)\end{aligned}$$

But the left-hand member is independent of $p_1, p_2, \ldots, p_N$ and has β as a factor; consequently, the right-hand member must be independent of the p_n also, and each term must contain β only as a factor. Then, dividing out β, this assumes the form

$$\begin{aligned}E(\hat{\xi}_1,\hat{\xi}_2,\ldots,\hat{\xi}_N)=&\sum_{i=1}^{N}\epsilon_i(\hat{\xi}_i)+\sum_{\substack{i=1\\(i\neq j)}}^{N}\sum_{j=1}^{N}v_{ij}(\hat{\xi}_i,\hat{\xi}_j)+\cdots\\ &+v_{12\ldots N}(\hat{\xi}_1,\hat{\xi}_2,\ldots,\hat{\xi}_N), \quad (28)\end{aligned}$$

where each v-function is a corresponding G divided by β, while each ϵ_i is a σ_i divided by β. We then interpret this in terms of "particle interactions."

Thus, $\epsilon_i(\hat{\xi}_i)$ is the contribution to the "energy" E due to particle i alone; $v_{ij}(\hat{\xi}_i,\hat{\xi}_j)$ is the contribution to the energy due to interaction of particle i with particle j, and all other v-functions are similarly read as interaction energies arising from more than two particles. This structure of the energy function E must *always* exist simply because of the functional structure of a joint probability distribution. Of course, we could also have arrived at this structure in terms of the K individual variables ξ_i, instead of the N vectors $\hat{\xi}_j$ appearing here. However, the form exhibited is now interpretable as the energy of a collection of N particles each having g degrees of freedom. In terms of an embedding, we can always construct a many-to-one map which produces this form.

The reduction of this form to that corresponding to a collection of N *identical* particles is made by demanding that all functions have the same form, i.e., all ϵ_i functions have the same form, and similarly for all the v-functions.

The observed value of E, defined as the average value of E over a set of duplications of an experiment, is the single sufficient statistic for estimating the value of β. Any such estimate is an approximation to the expectation value of E. This expectation value is, by

the law of large numbers, the value toward which all such averages tend as the number of duplications becomes large; thus we now examine the expectation value of E.

For given values of $p_1, p_2, \ldots, p_M$, as well as fixed β, we have the expectation value

$$E_s(\beta,p)=\int_{R_1}\int_{R_2}\cdots\int_{R_N}E(\hat{\xi}_1,\hat{\xi}_2,\ldots,\hat{\xi}_N)\times\rho_s(\hat{\xi}_1,\ldots,\hat{\xi}_N|\beta,p)d\tau_1d\tau_2\cdots d\tau_N, \quad (29)$$

which is a *conditional* expectation value. Into this we insert our expression for $E(\xi)$ deduced above, retaining now only those interactions, or correlations, up to second order, to obtain

$$E_s(\beta,p)=\sum_{i=1}^{N}\int_{R_1}\int_{R_2}\cdots\int_{R_N}\epsilon_i(\hat{\xi}_i)\rho_s d\tau_1d\tau_2\cdots d\tau_N+\sum_{\substack{i=1\\(i\neq j)}}^{N}\sum_{j=1}^{N}\int_{R_1}\int_{R_2}\cdots\int_{R_N}v_{ij}(\hat{\xi}_i,\hat{\xi}_j)\rho_s d\tau_1d\tau_2\cdots d\tau_N. \quad (30)$$

In the integrals of the single sum here, we may execute all integrations except that over R_i, and use the fact that

$$\rho_i(\hat{\xi}_i|\beta,p)=\int_{R_1}\int_{R_2}\cdots\int_{\substack{R_j\\(j\neq i)}}\cdots\int_{R_N}\rho_s d\tau_1d\tau_2\cdots d\tau_j\cdots d\tau_N \quad (31)$$

to write this sum as

$$E_{s(1)}(\beta,p)=\sum_{i=0}^{N}\int_{R_i}\epsilon_i(\hat{\xi}_i)\rho_x(\hat{\xi}_i|\beta,p)d\tau_i. \quad (32)$$

But for identical particles every one of these integrals is exactly the same and hence

$$E_{s(1)}(\beta,p)=N\int_{R_1}\epsilon(\hat{\xi}_1)\rho_1(\hat{\xi}_1|\beta,p)d\tau_1 \quad (33)$$

represents all the single-particle contributions to E_s. Here we use ϵ to represent any one of the ϵ_i.

In the same way we can execute all integrations in the double sum except those over R_i and R_j, and use the fact that

$$\rho_{ij}(\hat{\xi}_i,\hat{\xi}_j|\beta,p)=\int_{R_1}\cdots\int_{R_k}\cdots\int_{R_N}\rho_s d\tau_1\cdots d\tau_k\cdots d\tau_N \quad (k\neq j\neq i) \quad (34)$$

to obtain the double sum as

$$E_{s(2)}(\beta,p)=\sum_{i=1}^{N}\sum_{j=1}^{N}\int_{R_i}\int_{R_j}v_{ij}(\hat{\xi}_i,\hat{\xi}_j)\times\rho_{ij}(\hat{\xi}_i,\hat{\xi}_j|\beta,p)d\tau_id\tau_j, \quad (35)$$

where ρ_{ij} is the joint probability distribution for $\hat{\xi}_i$ and $\hat{\xi}_j$ and can always be expressed as

$$\rho_{ij}(\hat{\xi}_i,\hat{\xi}_j|\beta,p)=\rho_i(\hat{\xi}_i|\beta,p)\rho_j(\hat{\xi}_j|\beta,p)C_{ij}(\hat{\xi}_i,\hat{\xi}_j,\beta,p), \quad (36)$$

as indicated previously. Again we can make a reduction for identical particles, namely,

$$E_{s(2)}(\beta,p)=\tfrac{1}{2}N(N-1)\int_{R_1}\int_{R_2}v_{12}(\hat{\xi}_1,\hat{\xi}_2)\times\rho_{12}(\hat{\xi}_1,\hat{\xi}_2|\beta,p)d\tau_1d\tau_2, \quad (37)$$

or

$$E_{s(2)}(\beta,p)=\tfrac{1}{2}N(N-1)\int_{R_1}\int_{R_2}V(\hat{\xi}_1,\hat{\xi}_2,\beta,p)\times\rho_1(\hat{\xi}_1|\beta,p)\rho_1(\hat{\xi}_2|\beta,p)d\tau_1d\tau_2, \quad (38)$$

with the quantity V defined as

$$V(\hat{\xi}_1,\hat{\xi}_2,\beta,p)=v_{12}(\hat{\xi}_1,\hat{\xi}_2)C_{12}(\hat{\xi}_1,\hat{\xi}_2,\beta,p). \quad (39)$$

The latter form for $E_{s(2)}$ is obtained by using Eq. (36).

We use a common subscript on $\rho_1(\hat{\xi}_1|\beta,p)$ and $\rho_1(\hat{\xi}_2|\beta,p)$ because for identical particles these are the *same* function but of different arguments. We also note the very important fact that for identical particles, the two-particle distribution function $\rho_{12}(\hat{\xi}_1,\hat{\xi}_2|\beta,p)$ must be invariant under exchange of $\hat{\xi}_1$ and $\hat{\xi}_2$ and hence $C_{12}(\hat{\xi}_1,\hat{\xi}_2|\beta,p)$, and therefore $v_{12}(\hat{\xi}_1,\hat{\xi}_2)$ must be invariant under exchange of $\hat{\xi}_1$ and $\hat{\xi}_2$. With these properties in mind, we now turn to the construction of the quantum formalism for this model.

QUANTIZATION FOR NO INTERACTIONS

In our earlier paper[1] we made the transition from ordinary probability theory to a quantum formalism by exploiting the non-negative property of probabilities. We do this now by observing that any probability density, such as $\rho_1(\hat{\xi}_1|\beta,p)$, for example, has the two properties

$$\rho_1(\hat{\xi}_1|\beta,p)\geqslant 0 \quad (40)$$

and

$$\int_{R_1}\rho_1(\hat{\xi}_1|\beta,p)d\tau_1=1. \quad (41)$$

These are exploited by noting that Eq. (40) admits the structure

$$\rho_1(\hat{\xi}_1|\beta,p)=\psi^*(\hat{\xi}_1,\beta,p)\psi(\hat{\xi}_1,\beta,p), \quad (42)$$

which determines a complex function ψ, having complex conjugate ψ^*, in terms of ρ_1 to within an arbitrary phase function. Furthermore, since Eq. (41) becomes

$$\int_{R_1}|\psi|^2d\tau_1=1, \quad (43)$$

it follows that $\psi(\hat{\xi}_1,\beta,p)$ is Lebesgue square integrable

over R_1, and therefore possesses an expansion

$$\psi(\hat{\xi}_1,\beta,p)=\sum_{n=1}^{\infty} b_n^*(\beta,p)U_n(\hat{\xi}_1,p), \tag{44}$$

where the $U_n(\hat{\xi}_1,p)$ are a complete set of orthonormal functions spanning the function space of all such Lebesgue square-integrable functions over R_1. That is, we have a set $U_n(\hat{\xi}_1,p)$, $n=1, 2, \cdots$, such that[5]

$$\int_{R_1} U_n(\hat{\xi}_1,p)U_m^*(\hat{\xi}_1,p)d\tau_1=\delta_{nm}, \tag{45}$$

where δ_{nm} is the Kronecker δ. Here the fact that these $U_n(\hat{\xi}_1,p)$ must depend on the p_j is evident, because the domain R_1 is dependent on these quantities. Of course, if the domain R_1 were unbounded in any one of its coordinates, then an integral representation could replace the corresponding sum here.

The expansion for ψ in the U_n can be inserted into Eq. (43), to obtain, by virtue of Eq. (45),

$$\sum_{n=1}^{\infty} b_n^*(\beta,p)b_n(\beta,p)=b^\dagger b=1, \tag{46}$$

where b is a column matrix with elements $b_n(\beta,p)$, and $b^\dagger$ is its adjoint. This is an *identity* in β and the p_j that is valid for all values of these arguments.

Now we employ this representation for $\rho_1(\hat{\xi}_1|\beta,p)$ in the expression for $E_{s(1)}$ as defined in Eq. (33) above, that is, Eq. (42) for ρ_1 and Eq. (44) for ψ. This yields

$$E_{s(1)}=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} Nb_n^*b_m\bar{\epsilon}_{bm}'=Nb^\dagger\bar{\epsilon}'b, \tag{47}$$

where $\bar{\epsilon}'$ is the Hermitian matrix with elements defined by

$$\bar{\epsilon}_{nm}'(p)=\int_{R_1}\epsilon(\hat{\xi}_1)U_n(\hat{\xi}_1,p)U_m^*(\hat{\xi}_1,p)d\tau_1, \tag{48}$$

which are explicitly functions of the p_j defining the prepared state of the system.

Since $\bar{\epsilon}'$ is Hermitian, it can be diagonalized by a unitary transformation; thus we introduce a unitary matrix T;

$$E_{s(1)}=Nb^\dagger T^\dagger\bar{\epsilon}'T^\dagger Tb=Nc^\dagger\bar{\epsilon}c, \tag{49}$$

where the elements of the column matrix c are

$$c_n(\beta,p)=\sum_{m=1}^{\infty} T_{nm}(p)b_m(\beta,p) \tag{50}$$

and $\bar{\epsilon}$ is the diagonal matrix with elements

$$\bar{\epsilon}_{nm}=\delta_{nm}\bar{\epsilon}_n=\delta_{nm}\int_{R_1}\epsilon(\hat{\xi}_1)\psi_n(\hat{\xi}_1,p)\psi_m^*(\hat{\xi}_1,p)d\tau_1. \tag{51}$$

[5] The single index n is equivalent to a set of independent indices $n_1, n_2, \ldots, n_g$, equal in number to the dimensionality of R_1.

Here the functions $\psi_n(\hat{\xi}_1,p)$ are given by

$$\psi_n(\hat{\xi}_1,p)=\sum_{m=1}^{\infty} T_{nm}(p)U_m(\hat{\xi}_1,p) \tag{52}$$

and also form an orthonormal basis for the function space, as can be readily verified.

That the unitary matrix T must have elements $T_{nm}(p)$ which are functions of the p_j follows from the fact that the matrix $\bar{\epsilon}'$ to be diagonalized by T has elements which are functions of these quantities.

The unit-norm property of the expansion coefficients expressed in Eq. (46) is preserved by this unitary transformation:

$$b^\dagger b=b^\dagger T^\dagger Tb=c^\dagger c=\sum_{n=1}^{\infty} c_n^*(\beta,p)c_n(\beta,p)=1, \tag{53}$$

and since Eq. (49) has the explicit form

$$E_{s(1)}(\beta,p)=N\sum_{n=1}^{\infty}\bar{\epsilon}_n(p)c_n^*(\beta,p)c_n(\beta,p), \tag{54}$$

we see that a particular interpretation of the $c_n^*c_n$ is admitted, namely, we see $c_n^*c_n$ as the conditional probability for a particle to have a state corresponding to the discrete value $\bar{\epsilon}_n(p)$ of ϵ.

Thus we see the entire embedding structure recast in terms of identical particles, each of which has the discrete energy spectrum, $\bar{\epsilon}_n(p)$, $n=1, 2, \ldots$, with the spectrum explicitly determined by those parameters p_j, $j=1, 2, \ldots, M$ defining the preparation of the system.

Furthermore, we see that we can interpret the numbers

$$\bar{N}_n=Nc_n^*c_n \tag{55}$$

appearing in Eq. (54) as the expectation values for the number of particles having states corresponding to the energy value $\bar{\epsilon}_n$. Thus, in the *absence of interactions*, we see a different embedding of the physical variables in a space of hidden variables suggested, i.e., a substitution. We define the new set of variables

$$\lambda_l=\lambda_l(\xi_1,\xi_2,\ldots,\xi_K,p), \quad l=1, 2, \ldots, K \tag{56}$$

and then have the quantity E as a function of the λ_l, having the p_j as parameters, defined over $\lambda\in\mathfrak{R}(p)$. The probability distribution for the λ_l is obtained from Eq. (15) with the inverse of this substitution,

$$\xi_i=\xi_i(\lambda_1,\lambda_2,\ldots,\lambda_K,p), \quad i=1, 2, \ldots, K \tag{57}$$

in the form

$$dP(\lambda|\beta,p)=[e^{-\beta E(\lambda,p)}/z(\beta,p)]d\tau(\lambda), \quad \lambda\in\mathfrak{R}(p). \tag{58}$$

We then use a Hammel basis representation[6] of $E(\lambda,p)$,

$$E(\lambda,p)=\sum_{n=1}^{\infty}\bar{\epsilon}_n(p)N_n(\lambda), \tag{59}$$

[6] Jacob Korevaar, *Mathematical Methods* (Academic, New York, 1968), Vol. 1, pp. 373 ff.

where the $N_n(\lambda)$ are linearly independent functions and are therefore members of the Hamel basis of L^2 on $\mathfrak{R}(p)$. Only a *finite* number of the $N_n(\lambda)$ are nonzero for any configuration of λ and p.

This retains the quantity E as the quantity whose average value is the single sufficient statistic for β; in particular, we have the expectation value

$$\int_{\mathfrak{R}} E(\lambda,p)dP(\lambda|\beta,p)=\sum_{n=1}^{\infty}\bar{\epsilon}_n(p)\bar{N}_n(\beta,p), \quad (60)$$

where the $\bar{N}_n$ are the expectation values of the $N_n(\lambda)$ and must be exactly the same as those defined in Eq. (55).

An interesting feature of this embedding in terms of noninteracting particles is that it yields directly the familiar method of most probable distribution. Thus from Eq. (57) we have the probability for particular values of the N_n:

$$P(N_1,N_2,\ldots,|\beta,p)=\frac{1}{Z}\exp(-\beta\sum_{1}^{\infty}\bar{\epsilon}_nN_n)\Gamma, \quad (61)$$

where

$$\Gamma=\int_{\mathfrak{R}_N}d\tau(\lambda) \quad (62)$$

with

$$\lambda\in\mathfrak{R}_N\subset\mathfrak{R} \quad \text{if and only if} \quad N_n(\lambda)=N_n,\ n=1,2,\ldots,$$

$$\text{and} \quad \sum_{n=1}^{\infty}N_n=N, \quad (63)$$

and Γ is explicitly the number of ways that the particular set of numbers $N_1, N_2, \ldots$, can occur subject to the constraint of fixed N.

Then the *most probable* set of N_n values is determined by maximizing $P(N_1,N_2,\ldots,|\beta,p)$, subject to the constraint of fixed N by the method of undertermined multipliers; this yields

$$\frac{\partial\ln\Gamma}{\partial N_n}-\beta\bar{\epsilon}_n+\gamma=0, \quad n=1,2,\ldots \quad (64)$$

where γ is the undetermined multiplier of N. This is *exactly* the structure ordinarily derived from a much less well-defined basis.

Later we will see that our statistical basis for physical theory leads quite naturally to two unique forms for the quantity Γ defined above and hence to the two fundamental forms of quantum statistics, but now we show how the familiar operator structure of the quantum theory appears here.

Since the $c_n(\beta,p)$ are, according to Eq. (53), members of a square-summable sequence, these have a representation on an abstract vector space which is a separable Hilbert space. Then we recognize Eq. (49) for $E_{s(1)}$ as the matrix representation of a particular abstract vector structure. Thus, with $\mathbf{H}_0(p)$ an abstract operator having the complete set of eigenvectors $|n\rangle$, $n=1, 2, \ldots$ spanning the vector space, with corresponding eigenvalues $\bar{\epsilon}_n(p)$, we have

$$\mathbf{H}_0|n\rangle=\bar{\epsilon}_n(p)|n\rangle, \quad n=1, 2, \ldots. \quad (65)$$

Also, we have a vector $|\beta,p\rangle$, or, for brevity, just $|s\rangle$, defined on this space as

$$|s\rangle=\sum_{n=1}^{\infty}\langle n|s\rangle|n\rangle, \quad (66)$$

such that the complex numbers $c_n(\beta,p)$ of Eq. (49) are given by

$$c_n(\beta,p)=\langle n|s\rangle \quad (67)$$

and also

$$\langle s|s\rangle=1. \quad (68)$$

This last equation is the vector equivalent of Eq. (53) and we see Eq. (54) as

$$N\langle s|\mathbf{H}_0|s\rangle=N\sum_{n=1}^{\infty}\bar{\epsilon}_n(p)\langle s|n\rangle\langle n|s\rangle, \quad (69)$$

which is exactly the form familiar to us in quantum theory. But now we turn to the pair interaction contributions to E_s, that is, $E_{s(2)}$.

QUANTIZATION OF PAIR INTERACTION TERMS

We consider the form of $E_{s(2)}(\beta,p)$ given in Eq. (37). Here the joint distribution ρ_{12} has the properties

$$\rho_{12}(\hat{\xi}_1,\hat{\xi}_2|\beta,p)=\rho_{12}(\hat{\xi}_2,\hat{\xi}_1|\beta,p)\geqslant 0 \quad (70)$$

and

$$\int_{R_1}\int_{R_2}\rho_{12}(\hat{\xi}_1,\hat{\xi}_2|\beta,p)d\tau_1d\tau_2=1. \quad (71)$$

Thus, just as in the case of the single-particle distribution function, we can introduce a complex function and its conjugate as

$$\rho_{12}={}_p\psi(\hat{\xi}_1,\hat{\xi}_2,\beta,p)\ {}_p\psi^*(\hat{\xi}_1,\hat{\xi}_2,\beta,p) \quad (72)$$

and this "pair wave function" ${}_p\psi$ must be Lebesgue square integrable over the product space $R_1\otimes R_2$. Thus, ${}_p\psi(\hat{\xi}_1,\hat{\xi}_2,\beta,p)$ must have an expansion in a complete set of orthonormal functions spanning the space of all such square-integrable functions.

The single-particle wave functions $\psi_n(\hat{\xi}_1,p)$ and $\psi_m(\hat{\xi}_2,p)$, $n, m=1, 2, \ldots$ introduced above are defined on R_1 and R_2, respectively, and can be combined in product form to constitute a basis set on the product space $R_1\otimes R_2$. But the manner in which these must be combined to form a basis set is dictated by the fact that ρ_{12} in Eq. (72) above must be invariant under exchange of $\hat{\xi}_1$ and $\hat{\xi}_2$, as dictated in Eq. (70).

There are only *two* distinct ways of combining the one-particle functions to form two-particle orthonormal-basis functions such that ${}_p\psi\ {}_p\psi^*$ will be invariant under this exchange, namely,

$${}_B\psi_{nm}(\hat{\xi}_1,\hat{\xi}_2,p)$$
$$=\tfrac{1}{2}[\psi_n(\hat{\xi}_1,p)\psi_m(\hat{\xi}_2,p)+\psi_m(\hat{\xi}_1,p)\psi_n(\hat{\xi}_2,p)] \quad (73)$$

or

$$ {}_F\psi_{nm}(\hat{\xi}_1,\hat{\xi}_2,p) = \tfrac{1}{2}[\psi_n(\hat{\xi}_1,p)\psi_m(\hat{\xi}_2,p)-\psi_m(\hat{\xi}_1,p)\psi_n(\hat{\xi}_2,p)]. \quad (74) $$

These are the Bose and Fermi wave functions, either set of which forms an orthonormal basis on $R_1\otimes R_2$.

Here we see that for the case of Fermi-type functions, the joint distribution ρ_{12} will vanish identically for $\hat{\xi}_1=\hat{\xi}_2$, i.e., if we write

$$ {}_p\psi(\hat{\xi}_1,\hat{\xi}_2,\beta,p)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_{nm}^*(\beta,p)\ {}_F\psi_{nm}(\hat{\xi}_1,\hat{\xi}_2,p) \quad (75) $$

and then substitute this into Eq. (72). Thus we conclude that we can define two kinds of particles, namely, bosons or fermions, such that for the former, any number of particles may occupy the same state, but for the latter, no more than one may occupy a given state. Therefore, the two basic ways of counting are introduced into our formalism, and we see that there are two *unique* forms for Γ as defined in Eq. (62) above, i.e., the usual Bose and Fermi expressions.

Our point here is that all of this simply follows from the statistical nature of measurements and the functional structure of probabilities as exploited in our embedding procedure. Furthermore, all of the conventional operator structure of the quantum theory of interactions also follows. Thus we will show first how the conventional operator formalism follows and then how a revised operator formalism is deduced.

The pair wave function ${}_p\psi$ above can be written in the form

$$ {}_p\psi(\hat{\xi}_1,\hat{\xi}_2,\beta,p)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\alpha_{nm}^*(\beta,p)\psi_n(\hat{\xi}_1,p)\psi_m(\hat{\xi}_2,p), \quad (76) $$

where one of the two conditions

$$ \alpha_{nm}{}^*=\pm\alpha_{mn}{}^* \quad (77) $$

must hold, with the plus sign applying to bosons and the minus to fermions, i.e.,

$$ \alpha_{nm}{}^*=\tfrac{1}{2}(c_{nm}{}^*\pm c_{mn}{}^*). \quad (78) $$

Furthermore, from the norm condition, Eq. (71), we see that

$$ \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\alpha_{kl}\alpha_{mn}{}^*=1; \quad (79) $$

we then obtain $E_{s(2)}$ in the form

$$ E_{s(2)}(\beta,p)=\tfrac{1}{2}N(N-1)\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\alpha_{kl}{}^*\alpha_{mn} \times\langle k,l|v|m,n\rangle, \quad (80) $$

where

$$ \langle k,l|v|m,n\rangle=\int_{R_1}\int_{R_2} v_{12}(\hat{\xi}_1,\hat{\xi}_2)\psi_k(\hat{\xi}_1,p)\psi_l(\hat{\xi}_2,p) \times\psi_m{}^*(\hat{\xi}_1,p)\psi_n{}^*(\hat{\xi}_2,p)d\tau_1 d\tau_2 \quad (81) $$

simply by using Eq. (76) for ${}_p\psi$ and ${}_p\psi^*\ {}_p\psi$ for ρ_{12} in Eq. (37).

This expression for $E_{s(2)}$ is also equivalent to an expression in an abstract operator formalism, but now the abstract vector space must be defined as a direct product space of two identical vector spaces. Thus we require two abstract operators like $\mathbf{H}_0$ above, say, $\mathbf{H}_0{}^1$ acting only in the first space and $\mathbf{H}_0{}^2$ acting only in the second, and a set of basis vectors

$$ |m,n\rangle=|m\rangle|n\rangle, \quad m,n=1,2,\ldots \quad (82) $$

such that,[7]

$$ \begin{aligned} \mathbf{H}_0{}^1|m,n\rangle&=\bar{\epsilon}_m|m,n\rangle,\\ \mathbf{H}_0{}^2|m,n\rangle&=\bar{\epsilon}_n|m,n\rangle. \end{aligned} \quad (83) $$

We also require an operator $\mathbf{v}$ which acts in both spaces such that

$$ \langle k,l|\mathbf{v}|m,n\rangle=\langle k,l|v|m,n\rangle \quad (84) $$

is the quantity defined above, and a vector defined on the space as

$$ |s\rangle=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\langle m,n|s\rangle|m,n\rangle \quad (85) $$

such that

$$ \langle s|s\rangle=1 \quad (86) $$

and

$$ \langle m,n|s\rangle=\alpha_{mn}(\beta,p). \quad (87) $$

Then we have the complete expression for E_s as the sum of $E_{s(1)}$ and $E_{s(2)}$ in the abstract format as

$$ E_s=\tfrac{1}{2}N^2\langle s|\mathbf{H}_0{}^1+\mathbf{H}_0{}^2+\frac{N-1}{N}\mathbf{V}|s\rangle, \quad (88) $$

which is in the form familiar to us in conventional treatments of the many-body problem in quantum theory. Here, however, this formal structure has been deduced directly from a purely statistical basis. But now we show how an alternative to this conventional structure is deduced from this same statistical basis.

MODIFIED QUANTIZED FORM OF INTERACTION TERMS

Instead of beginning with the form for $E_{s(2)}$ in Eq. (37) which involves the function ρ_{12}, we now use the form in Eq. (38) which involves the single-particle function ρ_1 of the two arguments $\hat{\xi}_1$ and $\hat{\xi}_2$. We have already seen, in our discussion of the no-interaction case above, that ρ_1 is expressed as $\psi^*\psi$ and the single-particle wave function has the expansion given in Eq. (44). However, by virtue of Eqs. (50) and (52), we see that

$$ \psi=b^\dagger U=b^\dagger T^\dagger TU=c^\dagger\Psi, \quad (89) $$

or,

$$ \psi(\hat{\xi}_1,\beta,p)=\sum_{n=1}^{\infty}c_n{}^*(\beta,p)\psi_n(\hat{\xi}_1,p), \quad (90) $$

[7] Notation of dependence on β and the p_j is suppressed.

and the $\psi_n(\hat{\xi}_1,p)$ are also an orthonormal basis for L^2 on R_1, i.e., the space of Lebesgue square-integrable functions in R_1.

Thus we use this representation for $\rho_1(\hat{\xi}_1|\beta,p)$ and $\rho_1(\hat{\xi}_2|\beta,p)$ in Eq. (38) to obtain

$$E_{s(2)}=\tfrac{1}{2}N(N-1)\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}c_k^*c_l^*c_mc_n \times\langle k,l|\,V\,|m,n\rangle, \quad (91)$$

where we have introduced the notation

$$\langle k,l|\,V\,|m,n\rangle=\int_{R1}\int_{R2}\psi_k(\hat{\xi}_1,p)\psi_l(\hat{\xi}_2,p)\psi_m^*(\hat{\xi}_1,p) \times\psi_n^*(\hat{\xi}_2,p)V(\hat{\xi}_1,\hat{\xi}_2,\beta,p)d\tau_1d\tau_2. \quad (92)$$

The fundamental difference between this representation of $E_{s(2)}$ and that given in Eq. (80) is that in Eq. (80) the correlation between particles is partially expressed in the interaction potential $v_{12}(\hat{\xi}_1,\hat{\xi}_2)$ and partially in the expansion coefficients, the α_{kl}^* and their complex conjugates, but here *all* correlation between particles is expressed in the function $V(\hat{\xi}_1,\hat{\xi}_2,\beta,p)$.

When carried over to the abstract vector representation, we now require an operator $\mathbf{V}$ acting in both spaces of the direct product space as defined above by the basis vectors $|m\rangle|n\rangle$, but now the vector $|s\rangle$, as defined on the direct product space above, must also be the direct product of vectors defined on each of the two spaces. That is, we require

$$|s\rangle=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\langle m,n|s\rangle|m,n\rangle \quad (93)$$

to be simply

$$|s\rangle=\sum_{m=1}^{\infty}c_m(\beta,p)|m\rangle\sum_{n=1}^{\infty}c_n(\beta,p)|n\rangle, \quad (94)$$

or

$$|s\rangle=|s\rangle_{(1)}|s\rangle_{(2)}. \quad (95)$$

Then we see that with $\mathbf{V}$ defined such that

$$\langle k,l|\mathbf{V}|m,n\rangle=\langle k,l|\,V\,|m,n\rangle \quad (96)$$

as given in Eq. (92), the complete expression for E_s is

$$E_s=\tfrac{1}{2}N^2\langle s|\mathbf{H}_0{}^1+\mathbf{H}_0{}^2+\frac{N-1}{N}\mathbf{V}|s\rangle, \quad (97)$$

or

$$E_s=\tfrac{1}{2}N^2\,{}_{(1)}\langle s|\,{}_{(2)}\langle s|\mathbf{H}_0{}^1+\mathbf{H}_0{}^2+\frac{N-1}{N}\mathbf{V}|s\rangle_{(1)}|s\rangle_{(2)}. \quad (98)$$

The fundamental differences in the structure of E_s given here and that in Eq. (88) above are in the product structure of the vector $|s\rangle$ and the fact that the operator $\mathbf{V}$ here must contain β as a parameter, whereas the operator $\mathbf{v}$ does not.

The operator structure given here is of a form that evidently has not been considered before in the study of the many-body problem. It appears to offer a means for solving some problems that are not amenable to solution by the conventional formalism, but this remains to be demonstrated.

SOME GENERAL FEATURES OF EMBEDDING AND QUANTIZATION FORMALISM

In the discussion above we have seen the many-body model of matter emerge in a natural way from our embedding formalism simply as a consequence of an admissable choice of the embedding, namely, as an embedding of the physical variables in a space of hidden variables that is chosen to be a Cartesian product space of N identical spaces. Furthermore, the distribution function for these variables was required to be invariant under all permutations of these spaces.

However, the *quantum* features of the theory that emerged from this embedding were not a consequence of this particular choice of the embedding space. Every probability distribution for any set of variables can always be constructed as a summation, or integral, over a distribution function for a larger collection of random variables. That is, every space of random variables can always be embedded in a larger space of random variables. Furthermore, since every probability is non-negative and has sum, or integral, unity over the whole space available to its arguments, a wave function can always be introduced and a representation of this function on a complete set of orthonormal functions always exists; consequently, every probability formalism automatically yields a corresponding "quantum" formalism.

The exact meaning of the "quantum states" corresponding to these eigenvalues is not completely clear. In our earlier paper,[1] we introduced a partition of the space R available to ξ on the basis of a countable collection of values of the fixed variables p_j. Here, however, this is not the case. On the contrary, we have here exhibited a collection of quantum states corresponding to the same fixed values of the p_j, while in our earlier paper the quantum states were put into one-to-one correspondence with p configurations. Thus, considerable study of this point is in order and hopefully will be accomplished shortly. (In an earlier paper,[8] we presented one aspect of this study.)

CONCLUSION

We have clearly demonstrated that the statistical description of a physical system *always* admits a many-body model by an embedding procedure, both in a classical and a quantum format. Furthermore, we have shown that in the quantum formalism not only do we obtain the usual formalism in terms of the mutually exclusive boson and fermion wave functions, but also in a new format which is neither of these.

[8] R. E. Collins, Phys. Rev. D 1, 379 (1970).

This new many-body quantum formalism retains all correlations between particles in a correlation function, whereas the conventional formalism puts part of the correlation in the choice of symmetries or antisymmetries of the wave functions and part in the form of the interaction potential. The possibilities opened by this new format for the treatment of physical problems are now being explored.

In summary, the developments presented here give further support to our contention that much of the formal structure of physical theory can be deduced directly from the statistical properties of experimental data. Perhaps one of the most exciting possibilities opened up by this is that it indicates the way for applying the formal structures of physical theory to nonphysical problems as well.

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Statistical Basis for Quantum Theory: The Embedding Formalism*

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The formulation of quantum theory on the basis of the statistical definition of a duplicated experiment, as initiated in a recent paper, is used to reexamine the embedding formalism introduced in earlier papers. It is shown that the familiar relations between momentum and coordinate representations are given a statistical basis by the embedding theory. It is also shown that a complete quantum theory includes three fundamental sets of operators and that one of these seems to represent an operator formalism for thermodynamics.

INTRODUCTION

IN the development of any new way of looking at old ideas, one progresses in an irregular fashion and must continuously reexamine each step. A recent paper[1] presented a derivation of a generalized quantum formalism directly from the statistical definition of a duplicated experiment, but two earlier papers[2,3] presented derivations of some parts of the formalism on the same basis by a very different argument; these employed an embedding of one space in another.

These embedding arguments were rather crude and, in the case of the first paper[2] particularly, they now seem rather contrived and definitely improperly presented. After finding the more direct path from the statistical description of an experiment to a quantum formalism, as presented in the most recent paper,[1] the embedding formalism has been reexamined. This has yielded what seems to be a more coherent picture of the statistical basis of quantum theory. In particular, it provides a purely statistical formulation of representation theory as used in conventional quantum mechanics. However, this reexamination has also yielded a better understanding of the operator structure of the recent paper[1] as well, and points to an enlarged operator formalism for thermodynamics. This is presented here beginning with the statistical definition of a duplicated experiment.

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[1] R. E. Collins, Phys. Rev. D 1, 379 (1970).
[2] R. E. Collins, Phys. Rev. 183, 1081 (1969).
[3] R. E. Collins, preceding paper, Phys. Rev. D 2, 658 (1970).

DUPLICATED EXPERIMENTS, STATISTICS AND TIME

The results of a duplicated experiment can in principle be summarized as a spectrum of configurations of the variables determined by measurement in the experiment (the observables) together with the probability for each such configuration (frequency of occurrence). The spectrum of values for any observable is in general determined by values of parameters which must be fixed in order to duplicate the conditions of the experiment; also the probability for the occurrence of a configuration is conditioned by these same parameters.

Thus let

$$r=(x,y,z,\dots) \tag{1}$$

be the set of all observables whose values are determined in each duplication and suppose r contains K independent variables. Then each observable can be viewed as a function of the elements of a vector of K independent components,

$$\lambda=(\lambda_1,\lambda_2,\dots,\lambda_K), \tag{2}$$

and a distinct configuration is labeled by a configuration of λ. Thus we represent any observable as

$$x(\lambda,\eta,t)=x_\lambda(\eta,t), \tag{3}$$

where $\eta_1, \eta_2, \dots, \eta_M$ and t label values of conditioning parameters that must be the same in every duplication of the experiment. Here $\eta_1, \eta_2, \dots, \eta_M$, denoted by η, are those parameters that are individually identified and fixed in each duplication, while t is introduced as a single parameter label for all other parameters which