Nodal Structure, Nodal Flux Fields, and Flux Quantization in Stationary Quantum States*

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The nodal structure of the Schrödinger wave function is used to describe an N-particle system in a stationary quantum state. In terms of nodal lines (quantized flux lines) together with their circulation numbers, two fields are defined: the microscopic, singular nodal flux field $\mathbf{n}(\mathbf{r})$ and the macroscopic, regular nodal flux density field $\mathbf{N}(\mathbf{r})$. Their definition arises in a natural way from a quantum-mechanical vector potential defined by the gradient of the multivalued phase function associated with the pattern of quantized flux lines. The flux of the fields $\mathbf{n}(\mathbf{r})$ and $\mathbf{N}(\mathbf{r})$ is quantized and shown to be proportional to the circulation of the velocity in the system. The unit circulation for a system of bosons with a particle spin $s=0$ and mass m_0 is equal to h/m_0. In the case of charged particles, the field $(c/q)\mathbf{N}(\mathbf{r})$ resembles Jehle's formulation of the magnetic field of a lepton as a superposition of quantized flux lines. In our case, however, the presence of quantized flux lines, forming closed loops follows from the mere existence of a current density and does not have to be assumed. By a simple argument, using permutation symmetry, it is shown that a flux quantum of the size $hc/2q$ is possible.

I. INTRODUCTION

THIS paper is concerned with a system of N particles (bosons, fermions, charged or uncharged particles) in a quantum-mechanical stationary state. The nodal structure of the Schrödinger wave function is used to characterize the current and certain related fields.

Our work is based on our previous results,[1] which show that for a stationary N-particle system which can be described by a Schrödinger wave function, a current density (and hence in the case of charged particles a magnetic field created by the current) is only possible in the presence of $(3N-2)$-dimensional nodal manifolds, with nonzero circulation numbers, associated with a multivalued phase function. We here investigate a wide class of Schrödinger functions for which these $(3N-2)$-dimensional nodal manifolds are represented by one-dimensional nodal manifolds (quantized flux lines).

Using the general mathematical structure of the current, we define two fields: the nodal flux field $\mathbf{n}(\mathbf{r})$ and its macroscopic average, the nodal flux density field $\mathbf{N}(\mathbf{r})$. Their definitions arise in a natural way from a quantum-mechanical vector potential $\mathbf{A}_{qu}(\mathbf{r})$ defined, up to a factor, as the gradient of the multivalued phase function associated with the pattern of quantized flux lines. The nodal flux, i.e., the flux of the two fields $\mathbf{n}(\mathbf{r})$ and $\mathbf{N}(\mathbf{r})$, is a sum of quantized contributions each of which is associated with the flux of a single flux line.

Further, the nodal flux through a surface F is proportional to the circulation of the velocity vector of the system along the boundary of F. As a consequence, the circulation in our system is always quantized. It is shown that for bosons with a particle spin $s=0$ the circulation quantum is equal to the experimentally observed value in helium II.

In the case of charged particles the quantum-mechanical vector potential $(c/q)\mathbf{A}_{qu}(\mathbf{r})$ has properties analogous to those of a classical *magnetic* vector potential. The definition of the magnetic vector potential as the gradient of a multivalued phase function has been suggested by several authors.[2] A field with this mathematical structure—in particular, the charge-independent quantum-mechanical vector potential $\mathbf{A}_{qu}(\mathbf{r})$ defined in our article—does not have the regularity properties of a classical vector potential. We clarify in our case the relation of the singular field $\mathbf{A}_{qu}(\mathbf{r})$ to a regular potential $\mathbf{A}_N(\mathbf{r})$, defined from the nodal flux density field $\mathbf{N}(\mathbf{r})$ by the equation $\mathbf{N}(\mathbf{r})=\operatorname{curl}\mathbf{A}_N(\mathbf{r})$.

Do the fields $\mathbf{n}(\mathbf{r})$ and $\mathbf{N}(\mathbf{r})$ correspond to known classical fields? A possible answer to this question might be inspired by the behavior of $(c/q)\mathbf{A}_{qu}(\mathbf{r})$ mentioned above and by the observation that, in the case of charged particles, the field $(c/q)\mathbf{N}(\mathbf{r})$ resembles Jehle's formulation of the magnetic field of a lepton, which he defines as a superposition of quantized flux lines associated with a solution of the Dirac equation.[3,4] We emphasize that in our case, in contrast to Jehle's, the existence of a superposition pattern of quantized flux lines, forming closed loops, is already established and does not have to be assumed. In addition, we show that the flux of the field $\mathbf{N}(\mathbf{r})$ is quantized in units which can have the values of quantized magnetic flux observed in superconductors. Our results might thus give some support to Jehle's basic assumption that all magnetic flux has to be expressed in terms of quantized flux.

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[1] J. Riess, Ann. Phys. (N. Y.) **57**, 301 (1970).

[2] See especially the classical work about flux lines by Dirac [P. A. M. Dirac, Proc. Roy. Soc. (London) **A133**, 60 (1931); Phys. Rev. **74**, 817 (1948)]. An interesting discussion is given by Kaempffer [F. A. Kaempffer, in *Concepts in Quantum Mechanics* (Academic, New York, 1965), Sec. 20].
[3] H. Jehle, Int. J. Quantum Chem. **IIS**, 373 (1968).
[4] H. Jehle, Int. J. Quantum Chem. **III**, 269 (1969).

In the Appendix we discuss some topics pertinent to the present analysis.

II. REVIEW OF NODAL STRUCTURE OF SCHRÖDINGER WAVE FUNCTIONS

The present article is based on investigations[1] into the nodal structure of Schrödinger wave functions. We therefore briefly review some of the earlier results.

We consider a system of N identical particles with vanishing particle density at the boundaries, which can be described by a spin-free time-independent Schrödinger Hamiltonian [see Eq. (10)], whose coefficients may be singular on $(3N-3)$-dimensional subsets of the configuration space R^{3N} (e.g., Coulomb singularities are permitted). The phase $\varphi(x)$ of an eigenfunction[5] $\psi(x)=R(x)e^{i\varphi(x)}$, $x\in R^{3N}$, depends in a critical way on the $(3N-2)$-dimensional nodal manifolds[6] (denoted by M_{3N-2}) and their circulation numbers m_k,

$$m_k=\frac{1}{2\pi}\oint_{P_k}\frac{\partial}{\partial x_\alpha}\varphi(x)dx_\alpha\,, \tag{1}$$

where P_k is a closed path which encircles the kth $(3N-2)$-dimensional nodal manifold once, but no other $(3N-2)$-dimensional nodal manifold. The circulation numbers are integers, i.e., $m_k\in\{0,\pm1,\pm2,\ldots\}$.

We consider, throughout, the case where the particles are either uncharged or, if the particles are charged, where the *external* magnetic field is zero. The ($3N$-dimensional) N-current density has then the general form

$$j_\alpha(x)=\frac{\hbar}{m_0}\rho(x)\frac{\partial}{\partial x_\alpha}\varphi(x)\,, \tag{2}$$

with m_0 the particle mass and $\rho(x)$ the N-density:

$$\rho(x)=\psi^*(x)\psi(x)\,.$$

Because of relation (2) the nodal structure, which determines the phase and its $3N$-dimensional gradient $\partial\varphi(x)/\partial x_\alpha$, is reflected in the behavior of the N-current density $j_\alpha(x)$ and hence in that of the three-dimensional current density

$$\mathbf{j}(\mathbf{r})=\sum_{i=1}^{N}\mathbf{j}_i(\mathbf{r})=\sum_{i=1}^{N}\left\{\int\mathbf{j}_i(x)d\mathbf{r}_1\cdots d\mathbf{r}_{i-1}d\mathbf{r}_{i+1}\cdots d\mathbf{r}_N\right\}_{\mathbf{r}_i=\mathbf{r}}. \tag{3}$$

Here the 3-dimensional vector $\mathbf{j}_i(x)$ is defined by

$$\mathbf{j}_i(x)=(j_{3i-2}(x),\,j_{3i-1}(x),\,j_{3i}(x),)\,.$$

It has been shown in Ref. 1 that in this case of a vanishing external magnetic field there is no N-current density at all—and hence no physically observable (i.e., three-dimensional) current density $\mathbf{j}(\mathbf{r})$—unless the wave function has at least one $(3N-2)$-dimensional nodal manifold with circulation number $m\neq0$, that is, unless the phase of the wave function is multivalued. In fact, the whole N-current density can be written as

$$j_\alpha(x)=\sum_k j_\alpha{}^{(k)}(x)\,, \tag{4}$$

where $j_\alpha{}^{(k)}(x)$ is the $3N$-dimensional circulation current associated with the multivaluedness of the phase around the kth $(3N-2)$-dimensional nodal manifold $M_{3N-2}{}^{(k)}$.

Simple one-particle examples are the hydrogen states

$$\psi_{n,l,m}(\mathbf{r})=h(r,\theta)(\cos m\varphi+i\sin m\varphi)=h(r,\theta)e^{im\varphi}\,, \tag{5}$$

which have a single one-dimensional (since $N-2=1$) nodal manifold (M_1) along the z axis, around which the phase is multivalued with the circulation number m, i.e., where the circulation number is just the familiar magnetic quantum number. The current here has the form

$$\begin{aligned}\mathbf{j}(\mathbf{r})&=(\hbar/m_0)\rho(\mathbf{r})m(-y,x,0)/(x^2+y^2)\\&=(\hbar/m_0)\rho(\mathbf{r})m(\mathbf{u}_z\times\mathbf{r})/(x^2+y^2)\,,\end{aligned} \tag{6}$$

where $\mathbf{u}_z$ is a unit vector in the z direction. Formula (6) illustrates the current circulation around the nodal line.

For an arbitrary one-particle function it is possible to have simultaneously *several* one-dimensional nodal manifolds with nonzero circulation numbers. Each of the manifolds $M_1{}^{(k)}$ forms a loop which is either closed inside the system or closed by two boundary points of the system [e.g., the z axis of the hydrogen states $\psi_{n,l,m}(\mathbf{r})$ where the boundary points are at infinity].

In the case of an N-particle system, the phase can[7,8] have the general form

$$\varphi(x)=\sum_{i=1}^{N}f_i(\mathbf{r}_i)+\chi(x),\quad x\in R^{3N}. \tag{7}$$

Each of the functions $f_i(\mathbf{r}_i)$ is either a constant or is multivalued around (*in general, several*) one-dimensional nodal manifolds $M_1{}^{(k)}$ (quantized flux lines) of the density

$$\rho_i(\mathbf{r}_i)=\int\rho(x)d\mathbf{r}_1\cdots d\mathbf{r}_{i-1}d\mathbf{r}_{i+1}\cdots d\mathbf{r}_N \tag{8}$$

in a way analogous to the general behavior of a one-particle phase discussed above. Clearly the manifolds $M_1{}^{(k)}$ correspond to $(3N-2)$-dimensional nodal manifolds of the N-density $\rho(x)$, i.e., to the earlier mentioned manifolds $M_{3N-2}{}^{(k)}$. In Eq. (7), $\chi(x)$ is a step function

[5] It is sufficient for the following to consider only space functions (see Ref. 1, Sec. I).

[6] A nodal point x_0 is defined by $\Psi(x_0)=0$.

[7] Investigations in Sec. 5.7 of Ref. 8 suggest that the structure (7) might in fact be the most general form of a phase leading to a nonzero current $\mathbf{j}(\mathbf{r})$. We emphasize that the case where the functions $f_i(\mathbf{r})$ are not all identical is possible (Sec. V).

[8] J. Riess, *Swiss Federal Institute of Technology Thesis No. 4048* (Verlag, Zürich, 1968).

which is constant in each connected nodeless region. Such a function always belongs to any phase $\varphi(x)$,[1] but is not of importance for the following.

In this article we consider wave functions which have the phase structure (7). In this case each of the manifolds $M_{3N-2}^{(k)}$ with nonzero circulation number m_k gives rise to a circulation contribution

$$\mathbf{j}^{(k)}(\mathbf{r})=\left\{\int \mathbf{j}_i^{(k)}(x)d\mathbf{r}_1 d\mathbf{r}_2\cdots d\mathbf{r}_{i-1}d\mathbf{r}_{i+1}\cdots d\mathbf{r}_N\right\}_{\mathbf{r}_i=\mathbf{r}} \quad (9)$$

to the three-dimensional current density $\mathbf{j}(\mathbf{r})$, where in formula (9) the manifold $M_{3N-2}^{(k)}$ associated with the $3N$-dimensional vector $j_\alpha^{(k)}(x)$ is manifest in the subspace of the ith particle, i.e., $M_1^{(k)}$ is a nodal one-dimensional manifold of the density $\rho_i(\mathbf{r}_i)$ as defined by (8).

Summarizing, we can say: To each system described by a wave function of our class there belongs a superposition of definite patterns of fixed quantized flux lines $M_1^{(k)}$. The physical current $\mathbf{j}(\mathbf{r})$ is composed of elementary circulation currents $\mathbf{j}^{(k)}(\mathbf{r})$ which basically have the form of the hydrogen current (6).

III. DEFINITION OF QUANTUM-MECHANICAL VECTOR POTENTIAL

In this section we define a quantum-mechanical vector potential for systems of both charged and uncharged particles. In order to motivate this definition, we present a chain of arguments which show that in the case of charged particles there is a certain equivalence between this quantum-mechanical vector potential and a *magnetic* vector potential of an internal magnetic field.

While a unified theory of quantum mechanics and electrodynamics does not exist at present, a first step towards an integrated theory is reflected in the well-known rule defining how the vector potential $\mathbf{A}_{\text{ext}}(\mathbf{r})$ of an *external* field appears in the Schrödinger Hamiltonian:

$$H[\mathbf{A}_{\text{ext}}(\mathbf{r})]=\frac{1}{2m_0}\sum_{\alpha=1}^{3N}\left(\frac{\hbar}{i}\frac{\partial}{\partial x_\alpha}-\frac{q}{c}\mathcal{A}_\alpha(x)\right)^2+V(x). \quad (10)$$

Here q denotes the charge of the particles and $\mathcal{A}_\alpha(x)$ is the αth component of the $3N$-dimensional vector

$$\begin{aligned}\mathcal{A}(x)&=(\mathcal{A}_1(x),\mathcal{A}_2(x),\ldots,\mathcal{A}_{3N}(x))\\&=(\mathbf{A}_{\text{ext}}(\mathbf{r}_1),\mathbf{A}_{\text{ext}}(\mathbf{r}_2),\ldots,\mathbf{A}_{\text{ext}}(\mathbf{r}_N)). \quad (11)\end{aligned}$$

As a consequence of (10), the $3N$-dimensional current density has the form

$$j_\alpha(x)=\frac{q}{m_0 c}\rho(x)\left(\frac{\hbar c}{q}\frac{\partial}{\partial x_\alpha}\varphi(x)-\mathcal{A}_\alpha(x)\right). \quad (12)$$

This suggests that the $3N$-dimensional gradient of the phase, times $\hbar c/q$, is equivalent in a certain sense to a $3N$-dimensional vector potential of the *internal* magnetic field, generated by the $3N$-dimensional current

$$j_\alpha'(x)=\frac{q}{m_0 c}\rho(x)\frac{\hbar c}{q}\frac{\partial}{\partial x_\alpha}\varphi(x).$$

In fact, consider a system of charged particles in a vanishing external magnetic field (as we do in the remainder of this article), which carries a nonvanishing current

$$\mathbf{j}(\mathbf{r})=\sum_{i=1}^{N}\mathbf{j}_i(\mathbf{r}).$$

For the special case where

$$\mathbf{j}_i(\mathbf{r})=(\hbar/m_0 c)\rho_i(\mathbf{r})\,\text{grad}f(\mathbf{r}),\quad i=1,2,\ldots,N \quad (13)$$

i.e., where $f_i(\mathbf{r})=f(\mathbf{r})$ for all i, this system *formally* carries a magnetic field $\text{curl}\mathbf{A}_{qu}{}^q(\mathbf{r})$ with

$$\mathbf{A}_{qu}{}^q(\mathbf{r})=(\hbar c/q)\,\text{grad}f(\mathbf{r}), \quad (14)$$

as is easily shown by a formal gauge transformation of the vector potential of the external zero magnetic field. Of course, $\text{curl}A_{qu}{}^q(\mathbf{r})=(\hbar c/q)$ curl grad$f(\mathbf{r})$ vanishes where grad$f(\mathbf{r})$ is continuous; however, it does not vanish on a nodal line with nonzero circulation number. We repeat (see Sec. II) that such a nodal line with nonzero circulation number must exist if the current does not vanish.

Kaempffer[2] discusses various aspects of a definition analogous to (14) for the electromagnetic potentials. We would like to point out that the field (14) does not have the regularity properties of a classical vector potential since it is infinite at the nodal lines $M_1^{(k)}$, and Stokes's theorem cannot be applied in the usual sense.

If we write $\mathbf{A}_{qu}{}^q(\mathbf{r})$ in the form

$$A_{qu}{}^q(\mathbf{r})=(c/q)A_{qu}(\mathbf{r}), \quad (14')$$

then the field $\mathbf{A}_{qu}(\mathbf{r})$ is independent of the charge of the particles. We call $\mathbf{A}_{qu}(\mathbf{r})$ defined by (14) and (14′) the quantum-mechanical vector potential associated with the current (13). We shall extend this definition to the general case (7) of the phase. In this general case the *quantum-mechanical vector potential of the particle system* is defined as

$$\mathbf{A}_{qu}(\mathbf{r})=(\hbar/N)\sum_{i=1}^{N}\text{grad}f_i(\mathbf{r}). \quad (15)$$

The special case (14) where all the functions $f_i(\mathbf{r})$ are equal is included in definition (15). The quantum-mechanical vector potential $\mathbf{A}_{qu}(\mathbf{r})$ has a meaning for both charged and uncharged particles since it depends only on the phase of the wave function.

Let us now consider the line integral of the field $\mathbf{A}_{qu}(\mathbf{r})$ along the boundary ∂F of a surface F. We call it the *nodal flux* ϕ_F. Using (1), (7), and (15), we obtain

$$\phi_F=\oint_{\partial F}\mathbf{A}_{qu}(\mathbf{r})\cdot d\mathbf{r}=\frac{\hbar}{N}\sum_{i=1}^{N}\oint_{\partial F}\frac{\partial}{\partial\mathbf{r}}f_i(\mathbf{r})\cdot d\mathbf{r}$$

$$=\frac{\hbar}{N}\sum_{k'}2\pi m_{k'}, \quad (16)$$

where k' labels the intersections of the manifolds $M_1{}^{(k)}$ with the surface F.[9] (According to the definition of the manifolds $M_1{}^{(k)}$, many of the intersections with different labels k' may be at the same point in three-dimensional space.) The circulation number which belongs to the k'th intersection is denoted by $m_{k'}$, where the sign is defined relative to the (right-handed) orientation on the surface F.

Since the numbers $m_{k'}$ are integers, we can see from (16) that the flux ϕ_F is composed of smallest *flux units* of absolute value h/N. Such a unit is generated for instance by a single manifold $M_1{}^{(k)}$ (associated with $M_{3N-2}{}^{(k)}$) with circulation number $m_k=1$. The factor $1/N$ can be interpreted as the probability of observing among N indistinguishable particles the one to which a definite manifold $M_1{}^{(k)}$ belongs. The flux unit is independent of the area of F, as a consequence of relation (1), and depends only on whether the line $M_1{}^{(k)}$ intersects F or not.

IV. NODAL FLUX DENSITY FIELD

We now consider the nodal flux ϕ_F to be the flux of a certain field. In order to define this field, we express ϕ_F as a surface integral

$$\phi_F=(h/N)\int_F\sum_k m_k\delta(x_{t_k}(\mathbf{r}))\delta(y_{t_k}(\mathbf{r}))\mathbf{u}_{t_k}\cdot d\mathbf{S}. \quad (17)$$

Here t_k is the family of parameters describing the nodal lines $M_1{}^{(k)}$ (together with the position vectors $\mathbf{R}_{t_k}$) and $x_{t_k}(\mathbf{r})$, $y_{t_k}(\mathbf{r})$, $z_{t_k}(\mathbf{r})$ are the coordinates of a space vector $\mathbf{r}$ described in a local, right-handed, Cartesian coordinate system with origin $\mathbf{R}_{t_k}$. The z_{t_k} axis is taken tangent to the manifold $M_1{}^{(k)}$ at the point $\mathbf{R}_{t_k}$ and the orientation of the (x_{t_k},y_{t_k}) plane is the same as the given orientation of the surface F. The vector $\mathbf{u}_{t_k}$ is a unit vector in the z_{t_k} direction and $\delta(\cdots)$ denotes Dirac's one-dimensional δ function.

We call the singular field defined by

$$\mathbf{n}(\mathbf{r})=(h/N)\sum_k m_k\delta(x_{t_k}(\mathbf{r}))\delta(y_{t_k}(\mathbf{r}))\mathbf{u}_{t_k} \quad (18)$$

the *nodal flux field*. It is infinite on the nodal lines and zero otherwise.

Let us now consider a system of *macroscopic* dimensions. By this we understand that the region where the particle density is considerably different from zero is of nonatomic dimension. We remark that the region where the particle density is very small or zero must not be neglected, since the flux lines $M_1{}^{(k)}$ are not restricted to a region occupied by matter.

If the manifolds $M_1{}^{(k)}$ all form closed loops within a microscopic range, the nodal flux ϕ_F through any macroscopic surface F is zero. If the flux lines $M_1{}^{(k)}$ are not closed within a microscopic range, then the nodal flux ϕ_F through a macroscopic surface F may not be zero.

It is therefore possible, in certain cases, to replace the microscopic (singular) field $\mathbf{n}(\mathbf{r})$ by a macroscopic mean value $\mathbf{N}(\mathbf{r})$ where the singularities are averaged out. We call this macroscopic field $\mathbf{N}(\mathbf{r})$ the *nodal flux density field*. For the description of macroscopic properties it is in general sufficient to know the mean value for a large number of manifolds $M_1{}^{(k)}$. To obtain the mean values at the point $\mathbf{r}$, we isolate a plane surface F_0 with center $\mathbf{r}$, intersected by a large number of manifolds $M_1{}^{(k)}$, determine its maximum total flux $\phi_{F_0{}^{\max}}$ by varying the angular position of F_0 to a final position $F_0{}^{\max}$ such that $\phi_{F_0{}^{\max}}$ is a maximum, and finally divide by the area of F_0. We do this for each space point $\mathbf{r}$. For this mean to have significance the surface F_0 must be sufficiently large that the macroscopic value is constant for microscopic changes of $\mathbf{r}$. In summary, the nodal flux density field $\mathbf{N}(\mathbf{r})$ is defined by

$$\mathbf{N}(\mathbf{r})=\langle\mathbf{n}(\mathbf{r})\rangle_{\text{av}}=\{\phi_{F_0{}^{\max}(\mathbf{r})}/\text{area}(F_0)\}\mathbf{u}(\mathbf{r}), \quad (19)$$

where $\mathbf{u}(\mathbf{r})$ is a unit vector perpendicular to $F_0{}^{\max}(\mathbf{r})$. We remark that the definitions are the same inside and outside the material medium.

Since the lines of type $M_1{}^{(k)}$ are closed or go to infinity (see Sec. II), the flux of the field $\mathbf{N}(\mathbf{r})$ through any closed surface is zero. This implies

$$\operatorname{div}\mathbf{N}(\mathbf{r})=0, \quad (20)$$

since $\mathbf{N}(\mathbf{r})$ is sufficiently regular to apply Gauss's theorem. Consequently, we can write

$$\mathbf{N}(\mathbf{r})=\operatorname{curl}\mathbf{A}_N(\mathbf{r}), \quad (21)$$

where the new field $\mathbf{A}_N(\mathbf{r})$ may be called the nodal density vector potential. Equations (20) and (21) are a consequence of the connectivity properties of the nodal manifolds $M_1{}^{(k)}$. The vector potential $\mathbf{A}_N(\mathbf{r})$ is a regular field in contrast to the singular quantum-mechanical vector potential $\mathbf{A}_{qu}(\mathbf{r})$ defined earlier.

Once we know $\mathbf{N}(\mathbf{r})$, the field $\mathbf{A}_N(\mathbf{r})$ can be evaluated by solving Eq. (21). Nevertheless, it is instructive to see directly, i.e., without solving Eq. (21), how the vector potential $\mathbf{A}_N(\mathbf{r})$ is related to the vector potential $\mathbf{A}_{qu}(\mathbf{r})$, i.e., to the phase of the wave function. We shall demonstrate this for the case of uniformly distributed nodal manifolds M_1 parallel to the z axis, a situation corresponding to a homogeneous field $\mathbf{N}(\mathbf{r})=\mathbf{N}$ in the z direction.

Let us formally replace each nodal manifold $M_1{}^{(k)}$ of

[9] A curved line $M_1{}^{(k)}$ may intersect F several times.

the distribution by a hydrogen-type manifold M_1 [compare (5) and (6)] which has the same spatial position and circulation number. This substitution leaves the flux of each nodal manifold $M_1{}^{(k)}$ invariant since the flux is determined solely by the circulation number, and spatial position, of the manifolds $M_1{}^{(k)}$. Since the field $\mathbf{N}(\mathbf{r})$ has been defined by the flux only, the same field $\mathbf{N}(\mathbf{r})$ is obtained whether, in definition (15), the proper phase of the wave function is used, or the corresponding formal superposition of hydrogen phases. The latter case leads to the vector potential [compare (5) and (6)]

$$\mathbf{A}_{qu}'(\mathbf{r})=\sum_k \frac{\hbar m_k}{N}\frac{\mathbf{u}_z\times[\mathbf{r}-\mathbf{R}_k(z)]}{(x-X_k)^2+(y-Y_k)^2}, \tag{22}$$

where $\mathbf{u}_z$ is a unit vector in the z direction and $\mathbf{R}_k(z)=(X_k,Y_k,z)$, with X_k and Y_k fixed, is the position of the manifold $M_1{}^{(k)}$ with circulation number m_k.

We now consider an arbitrary *noninfinitesimal* surface F parallel to the xy plane. The surface F has in this case the direction of a surface $F_0{}^{\max}$ which we used earlier to define the field $\mathbf{N}(\mathbf{r})$. Hence the line integral of $\mathbf{A}_{qu}'(\mathbf{r})$ along the boundary ∂F is by definition equal to

$$\oint_{\partial F}\mathbf{A}_{qu}'(\mathbf{r})\cdot d\mathbf{r}=|\mathbf{N}|\,(\text{area of } F). \tag{23}$$

On the other hand, if we replace in (23) the field $\mathbf{A}_{qu}'(\mathbf{r})$ by the numerator of $\mathbf{A}_{qu}'(\mathbf{r})$ times $\pi/(\text{area of } F)$, we obtain the same value:

$$\frac{\pi}{(\text{area of } F)}\left\{\sum_k \frac{\hbar m_k}{N}\oint_{\partial F}\{\mathbf{u}_z\times[\mathbf{r}-\mathbf{R}_k(z)]\}\cdot d\mathbf{r}\right\}$$

$$=\frac{\pi}{(\text{area of } F)}\left\{\sum_{k\in F}\frac{\hbar m_k}{N}\right\}\oint_{\partial F}(\mathbf{u}_z\times\mathbf{r})\cdot d\mathbf{r} \tag{24a}$$

$$=\tfrac{1}{2}|\mathbf{N}|\oint_{\partial F}(\mathbf{u}_z\times\mathbf{r})\cdot d\mathbf{r} \tag{24b}$$

$$=|\mathbf{N}|\,(\text{area of } F). \tag{24c}$$

Here $k\in F$ denotes the manifolds $M_1{}^{(k)}$ intersecting F. In deriving (24b), relation (16) together with the definition of the nodal flux density field $\mathbf{N}(\mathbf{r})$ is used. As a consequence of (24b), and since in our case $|\mathbf{N}|\mathbf{u}_z=\mathbf{N}$, we can for the calculation of the macroscopic flux replace $\mathbf{A}_{qu}'(\mathbf{r})$ by the new vector potential

$$\mathbf{A}''(\mathbf{r})=\tfrac{1}{2}(\mathbf{N}\times\mathbf{r}). \tag{25}$$

This $\mathbf{A}''(\mathbf{r})$ is clearly a macroscopic vector potential. It satisfies Eq. (21) and is therefore a vector potential $\mathbf{A}_N(\mathbf{r})$.

This whole process illustrates how a singular quantum-mechanical vector potential $\mathbf{A}_{qu}(\mathbf{r})$ can be transformed into a regular (classical) potential $\mathbf{A}_N(\mathbf{r})$. The singularities of order $1/r$ which describe the area independence of the flux associated with a single manifold $M_1{}^{(k)}$ are removed by the average process over the macroscopic surface F, in the step from (23) to (25). The steps involved to obtain (22) and (24a) are both in classical language called gauge transformations. However, they differ in the fact that the former transformation changes neither the microscopic flux ϕ_F and the (microscopic) field $\mathbf{n}(\mathbf{r})$, nor the corresponding macroscopic quantities, whereas the latter transformation leaves only the macroscopic flux ϕ_F and the (macroscopic) field $\mathbf{N}(\mathbf{r})$ invariant.

V. APPLICATIONS AND DISCUSSION

We have seen that the nodal structure is useful in describing some characteristic properties of stationary quantum systems which are represented by wave functions with a phase structure (7). In this section we discuss, in particular, some properties related to the quantization of the nodal flux ϕ_F. We have shown that the nodal flux unit cannot be smaller than h/N. This is the flux difference of two states whose sums of numbers $m_{k'}$ in (16) differ by one. In many cases this situation is forbidden by the permutation symmetry of the space functions, which depends on the total spin S and the spin s of the individual particles.[10]

We consider two examples.

(a) The space functions are totally symmetric with respect to particle permutation. (For $N>2s+1$ this situation is only possible for bosons and it is the only possible case for bosons with spin $s=0$.) In this case the phase functions can only have the structure

$$\varphi(x)=\sum_{i=1}^{N} f(\mathbf{r}_i)+\chi(x),$$

where all the $f_i(\mathbf{r})$ are now identical, and, in particular, all have the same circulation numbers. Therefore the smallest nodal flux difference is equal to

$$\Delta\phi_F=h. \tag{26}$$

We now consider the velocity

$$\mathbf{v}(\mathbf{r})\equiv(1/N)\sum_{i=1}^{N}\mathbf{j}_i(\mathbf{r})/\rho_i(\mathbf{r})$$

$$=(1/N)\sum_{i=1}^{N}(\hbar/m_0)\,\mathrm{grad}f_i(\mathbf{r})=(1/m_0)\mathbf{A}_{qu}(\mathbf{r}). \tag{27}$$

Its circulation along a closed path P is equal to

$$\oint_P \mathbf{v}(\mathbf{r})\cdot d\mathbf{r}=(1/m_0)\oint_P \mathbf{A}_{qu}(\mathbf{r})\cdot d\mathbf{r}=(1/m_0)\phi_F, \tag{28}$$

[10] For the mathematical theory of representations of permutation groups (as applied to quantum mechanics) see, e.g., M. Hamermesh, in *Group Theory and its Application to Physical Problems* (Addison-Wesley, Reading, Mass., 1962), Chap. 7.

where the surface F is bounded by P; i.e., the circulation along a closed line is proportional to the nodal flux through a surface which is bounded by that line. Hence *the circulation is quantized*, together with the nodal flux. In particular, the nodal flux unit (26) corresponds to the circulation unit h/m_0. This is the value which has been experimentally observed in helium II by Vinen[11] and by Rayfield and Reif.[12] We emphasize that, as a consequence of the nodal structure of the wave function, the circulation is *always* quantized in our particle system. Moreover, for bosons with spin $s=0$ the circulation unit is fully determined by the permutation symmetry of the particles.

(b) The space functions belong to a permutation symmetry which corresponds to a Young diagram with two rows (or two columns) of length $N/2$. (If N is large, we can assume N even without loss of generality.) For a space function of a fermion system with particle spin $s=\frac{1}{2}$ and a total spin $S=0$ this is the only possible symmetry.

In this case the number of identical phase functions is no longer necessarily N. Consider, for example, space functions belonging to this symmetry which are antisymmetric in the even-numbered particles and again antisymmetric in the odd-numbered particles. Then the $N/2$ functions $f_i(\mathbf{r})$, i even, are equal and the $N/2$ functions $f_i(\mathbf{r})$, i odd, are equal. The smallest possible nodal flux difference arises then if two space functions differ by a unit change in the sum of circulation numbers associated with each $f_i(\mathbf{r})$ of one only of the two sets (e.g., i even). This implies $\Delta\phi_F=h/2$, which corresponds to a circulation difference $h/2m_0$.

We remark that, in case (b), symmetry alone does not prevent the occurrence of even smaller flux units. The theoretical deduction of the occurrence of certain definite flux differences requires in this case also a consideration of the energy, to determine what total numbers of equal phase functions $f_i(\mathbf{r})$ correspond to relatively stable particle states. (Conversely, these numbers could be determined from circulation measurements.)

In this context it is interesting to mention the *magnetic* flux differences of the size $hc/2q$, which have been observed in superconductors between states of trapped flux.[13] It is therefore tempting, in the case of charged particles, to identify the flux $(c/q)\phi_F$ with the magnetic flux through the surface F generated by the particles. If we could, in fact, make this identification, then, as a natural consequence, we would have to consider the field $(c/q)\mathbf{A}_{qu}(\mathbf{r})$ as the quantum-mechanical or microscopic *magnetic* vector potential, the field $(c/q)\mathbf{n}(\mathbf{r})$ as the microscopic magnetic field, and the field $(c/q)\mathbf{N}(\mathbf{r})$ as the macroscopic magnetic field together with its vector potential $(c/q)\mathbf{A}_N(\mathbf{r})$. As to the field $(c/q)\mathbf{A}_{qu}(\mathbf{r})$, we have already seen in Sec. III that its behavior is analogous to that of an internal magnetic field.

The field $(c/q)\mathbf{N}(\mathbf{r})$ resembles Jehle's formulation of the magnetic field of a lepton,[3,4] which he defines as a superposition of assumed quantized flux lines associated with a solution of the Dirac equation. In our case a superposition pattern of quantized flux lines always exists if there is a current flowing in the system and hence, in the case of charged particles, if there is a magnetic field created by this current.

APPENDIX

Aharonov and Bohm[14] have discussed experiments[15] which show the effects of electromagnetic potentials on charged particles in a field-free region. These effects, which in further discussions by Furry and Ramsey[16] and by Weisskopf[17] have been shown to fulfill basic quantum-mechanical consistency requirements, might appear as an indication of a nonlocal feature acquired by wave functions in an electromagnetic field (compare Kaempffer[2]).

In this connection, we would like to recall the identification of the flux $(c/q)\phi_F$ with the magnetic flux together with its consequences mentioned at the end of Sec. V. A magnetic theory based on this identification would bear nonlocal features due to the role of the phase whose gradient (which would be comparable to the magnetic vector potential) is in general not equal to zero in regions where the density of flux lines (which would be proportional to the magnetic field) vanishes.

In fact, the situation of Aharonov and Bohm's magnetic experiment applied to a stationary quantum system would represent an example par excellence for one of the characteristic features of such a theory: the fact that the presence of a magnetic field would lead to a multivalued phase function.

The theory would also be automatically gauge and path independent, since the magnetic field would be determined by the charge density and the circulation numbers of the phase which both are gauge and path independent. (For a gauge- and path-independent theory in the framework of the present formulation of quantum electrodynamics, see Rohrlich and Strocchi.[18])

These promising properties of such a (nonrelativistic) theory, including the ability to describe quantized magnetic flux (see Sec. V), would, however, be balanced by an inconsistency with Maxwell's theory applied to a quantum-mechanical current: The magnetic field

[11] W. F. Vinen, Proc. Roy. Soc. (London) **A260**, 218 (1961).
[12] G. W. Rayfield and F. Reif, Phys. Rev. **136**, A1194 (1964).
[13] B. S. Deaver, Jr., and W. M. Fairbank, Phys. Rev. Letters **7**, 43 (1961); R. Doll and M. Nabauer, *ibid.* **7**, 51 (1961); cf. also W. A. Little and R. D. Parks, *ibid.* **9**, 9 (1962).
[14] Y. Aharonov and D. Bohm, Phys. Rev. **115**, 485 (1959).
[15] The experiment with magnetic flux enclosed by a split electron beam was performed by Chambers [R. G. Chambers, Phys. Rev. Letters **5**, 3 (1960)].
[16] W. H. Furry and N. F. Ramsey, Phys. Rev. **118**, 623 (1960).
[17] V. F. Weisskopf, in *Boulder Lectures* (Interscience, New York, 1960), Vol. III, p. 54.
[18] F. Rohrlich and F. Strocchi, Phys. Rev. **139**, B476 (1965).

which, on the basis of the identifications discussed, could be derived from a multivalued phase function does not in general correspond to the magnetic field obtained from the appropriate quantum-mechanical current $\mathbf{j}(\mathbf{r})$ by applying Maxwell's inhomogeneous equations. [Take, e.g., the state $\psi_{n,l,m}(\mathbf{r})$ of an isolated hydrogen atom whose phase is multivalued around the z axis.] It is the aim of further studies to gain more insight into the nature of this inconsistency.

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B Meson and Its Current

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We examine the assumption that the B meson (1220 MeV, $J^{PC}=1^{+-}$) and its chiral partner are gauge fields. This leads to the introduction of an internal $SU(6)$ group which contains $SU(3)\times SU(3)$ as a subgroup. We discuss the appropriate formalism and mass degeneracy breaking by means of the use of nonlinear realizations of the group.

INTRODUCTION

The assumption that hadrons display a chiral $SU(3)\times SU(3)$ symmetry[1] broken by the partially conserved axial-vector current (PCAC) condition has produced many successful results, especially when these results refer to massless pions.[2] However, for physical pions the situation is not that satisfactory. The application of techniques like the use of Ward identities or equivalent phenomenological Lagrangians[3] makes clear that from chiral symmetry and PCAC one can obtain at most correlations between a certain number of low-energy processes and masses relations which are determined only when supplemented by extra assumptions such as, for example, vector-meson dominance[4] and high-energy behavior.[5,6] The possibility of constructing nonlinear representations besides the usual linear ones adds an extra degree of arbitrariness to the task of assigning the experimentally known particles to chiral multiplets. A large number of papers have been devoted to the study of the multiplets characterized by $J^{PC}=(1^{--},1^{++})$, $(0^{-+},0^{++})$, and $(\frac{1}{2}^{+})$, and their physics is well understood both in its successes and in its limitations.

In the present article we discuss the inclusion within the chiral scheme of the B meson (1220 MeV, $J^{PC}=1^{+-}$). In doing so one immediately faces the problems of assigning the B to a particular representation, linear or nonlinear, and of finding (or predicting) its partners. Assuming a linear representation, the chiral companion of the B octet (or nonet) should be a 1^{-} $SU(3)$ multiplet with positive or negative charge conjugation according to whether we have a $(1,8)+(8,1)$ or a $(3,3^*)+(3^*,3)$ representation. The B decay modes and the experimental possibility for a 1^{--} object at a mass around 1650 MeV seem to favor the assignment to a $(3,3^*)+(3^*,3)$. As is shown in this article, this choice offers the possibility of enlarging the symmetry by assuming that all spin-1 particles are gauge-field quanta, the corresponding (internal) symmetry group being $SU(6)$. General considerations about $SU(3)\times SU(3)$ as a subgroup of some larger one have been made earlier,[7,8] and there are many possibilities. The group we choose here appears to be the minimal one that would include the known spin-1 mesons and contain all the results of chiral symmetry.

As a framework for the discussion of the consequences of our assumptions, we shall use the phenomenological Lagrangian approach together with vector-meson dominance in the theoretical form developed by Lee, Weinberg and Zumino.[9] In Sec. I is presented a very compact formalism for the treatment of representations of the chiral symmetry group. Besides its simplicity, it has the value of suggesting that the enlarged group is

[1] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962); Physics **1**, 63 (1964).
[2] See, e.g., S. L. Adler and R. F. Dashen, *Current Algebras and Their Applications to Particle Physics* (Benjamin, New York, 1968).
[3] The review article by S. Gasiorowicz and D. A. Geffen, Rev. Mod. Phys. **41**, 531 (1969), contains an extensive list of references on the use of these techniques.
[4] J. J. Sakurai, Ann. Phys. (N. Y.) **11**, 1 (1960).
[5] S. Weinberg, Phys. Rev. Letters **18**, 507 (1967).
[6] T. Das, V. S. Mathur, and S. Okubo, Phys. Rev. Letters **18**, 761 (1967).
[7] S. Coleman and S. Glashow, Ann. Phys. (N. Y.) **17**, 41 (1962); S. Coleman, *ibid.* **24**, 37 (1963).
[8] M. Gell-Mann and Y. Ne'eman, Ann. Phys. (N. Y.) **30**, 360 (1964).
[9] T. D. Lee, S. Weinberg, and B. Zumino, Phys. Rev. Letters **18**, 1029 (1967).