Corrections to the Soft-Pion Theorem in K_{l3} Decay*

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Using appropriate continuity arguments in the $SU^{(+)}(3)\otimes SU^{(-)}(3)$ subgroup space, a new sum rule has been derived for K_{l3} decays. This sum rule incorporates corrections to the soft-pion theorem and contains K_{l3} form factors defined at a point much closer to the physical decay region.

IT is well known[1] that the soft-pion theorem[2] involving form factors in K_{l3} decay becomes exact in the limit of the chiral $SU^{(+)}(2)\otimes SU^{(-)}(2)$ symmetry. In the usual notation, one may write

$$f_+(-m_K^2)+f_-(-m_K^2)-f_K/f_\pi=O(\epsilon_\pi)\,, \qquad (1)$$

where the parameter ϵ_π denotes the departure from $SU^{(+)}(2)\otimes SU^{(-)}(2)$-symmetry limit. It has been argued by Dashen and Weinstein[3] that if one changes the argument of the form factors from $-m_K^2$ to $m_\pi^2-m_K^2$, one obtains a relation which, in principle, is expected to be more accurate than (1):

$$f_+(m_\pi^2-m_K^2)+f_-(m_\pi^2-m_K^2)-f_K/f_\pi=O(\epsilon_\pi\epsilon_8)\,, \qquad (2)$$

where ϵ_8 denotes departures from the $SU(3)$-symmetry limit. This relation follows if one notes that the left-hand side of (2) vanishes not only in the $SU^{(+)}(2)\otimes SU^{(-)}(2)$ limit but also in the $SU(3)$ limit. Since one expects ϵ_8 to be also a small parameter in suitable dimensionless units, the sum rule (2) is expected to be better than (1).

The purpose of this paper is to point out that one can obtain a sum rule which, in the spirit of Ref. 3, holds to a degree of accuracy even higher than (1) or (2), and which furthermore involves the K_{l3} form factors at a point closer to the physical region.

We would first like to show from the arguments of Ref. 3 alone that there exists a sum rule given by

$$(m_K^2+m_\pi^2)f_+(\Delta)+(m_K^2-m_\pi^2)f_-(\Delta) = m_K^2 f_K/f_\pi+m_\pi^2 f_\pi/f_K+O(\epsilon_\pi\epsilon_K\epsilon_8^2)\,, \qquad (3)$$

where ϵ_K denotes the departure from the chimeral[4] $SU(3)$ limit, and Δ is a function of m_π^2 and m_K^2 satisfying the conditions

$$\begin{aligned}\Delta &= -m_K^2 \quad [SU^{(+)}(2)\otimes SU^{(-)}(2)\text{ limit}]\\ &= -m_\pi^2 \quad [\text{chimeral } SU(3)\text{ limit}]\\ &= O(\epsilon_8^2) \quad [\text{usual } SU(3)\text{ limit}].\end{aligned} \qquad (4)$$

The proof of the relation (3) is quite straightforward. Consider the expression

$$T=m_K^2[f_+(\Delta)+f_-(\Delta)-f_K/f_\pi] +m_\pi^2[f_+(\Delta)-f_-(\Delta)-f_\pi/f_K]. \qquad (5)$$

It is simple to check that T vanishes for each of the following subgroups of the $SU^{(+)}(3)\otimes SU^{(-)}(3)$ symmetry: $SU^{(+)}(2)\otimes SU^{(-)}(2)$, $SU(3)$, and chimeral $SU(3)$. To see this, note that in the $SU^{(+)}(2)\otimes SU^{(-)}(2)$ limit, $m_\pi^2\to 0$, so that from (4) and the soft-pion theorem (1), T must vanish. In the chimeral $SU(3)$ limit[4] where $m_K^2\to 0$, using the soft-kaon analog of (1)

$$f_+(-m_\pi^2)-f_-(-m_\pi^2)-f_\pi/f_K=O(\epsilon_K)\,, \qquad (6)$$

one obtains the result that T must vanish once again in the chimeral $SU(3)$ limit. Finally, in the $SU(3)$-symmetric limit, note that from (4) and the Ademollo-Gatto theorem one has $f_+(\Delta)=1+O(\epsilon_8^2)$. Also, since $f_-(\Delta)=O(\epsilon_8)$ and $f_K/f_\pi=1+O(\epsilon_8)$, it is simple to verify that not only does T vanish in the $SU(3)$ limit, but also $T=0$ even if the first-order $SU(3)$-violating terms are included. Note that the relative sign between the two terms in Eq. (5) is crucial for this purpose. The sum rule (3) now follows immediately.

It is worth mentioning that if, for instance, the multiplicative factors m_K^2 and m_π^2 in front of the brackets in Eq. (5) are changed to m_K^4 and m_π^4, respectively, the resulting sum rule would also be accurate to the same order as (3). There is, however, no ambiguity here, since the new sum rule would differ from (3) by terms also of the same order $\epsilon_\pi\epsilon_K\epsilon_8^2$. Also note that higher powers of m_K^2 and m_π^2 would weaken the sum rule from a practical point of view, since one would be neglecting $O(\epsilon_\pi\epsilon_K\epsilon_8^2)$ compared with terms that go to zero, increasingly faster,[1] in the full $SU^{(+)}(3)\otimes SU^{(-)}(3)$ limit ($m_K^2\to 0$, $m_\pi^2\to 0$).

Note that the functional dependence of Δ on m_π^2 and m_K^2 is unknown except for the conditions (4), and any expression for Δ that satisfies these conditions would also satisfy the sum rule (3). However, consistent with the conditions (4), we can in principle vary the functional dependence of Δ to generate a set of values of Δ corresponding to the physical values of the π and K masses. All these values of Δ would then satisfy the sum rule (3), differing only by terms of order $\epsilon_\pi\epsilon_K\epsilon_8^2$. Clearly, however, there exists a "best" value Δ_0 which satisfies $T=0$ where there are no terms proportional to $\epsilon_\pi\epsilon_K\epsilon_8^2$ in (3).

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[1] See, e.g., R. F. Dashen, Phys. Rev. **183**, 1245 (1969).
[2] C. Callan and S. Treiman, Phys. Rev. Letters **16**, 153 (1966); V. S. Mathur, S. Okubo, and L. K. Pandit, *ibid.* **16**, 371 (1966); M. Suzuki, *ibid.* **16**, 212 (1966). The form factors $f_\pm(q^2)$ are the usual ones encountered in the K_{l3} decay. Note that in the $SU(3)$ limit, $f_+(0)=1$, $f_-(q^2)=0$. For the physical decay region, $-m_l^2 < q^2 < -(m_K-m_\pi)^2$. The constants f_K and f_π are the usual decay constants involved in K_{l2} and π_{l2} decays.
[3] R. Dashen and M. Weinstein, Phys. Rev. Letters **22**, 1337 (1969); see also A. Sirlin, New York University report (unpublished).
[4] S. Okubo and V. S. Mathur, Phys. Rev. Letters **23**, 1412 (1969); Phys. Rev. D **1**, 2046 (1970).

We would now like to evaluate the best value of Δ under a reasonable hypothesis. For this purpose, we follow the recent suggestion[4] of the present authors that in a broken-$SU^{(+)}(3)\otimes SU^{(-)}(3)$ theory, one may in a specific "physical" domain invoke appropriate continuity arguments for transitions between various subgroups. We have suggested that this transition may be taken to be as smooth as possible. Consider, for simplicity, the model of Gell-Mann *et al.*,[5]

$$H=H_0+\epsilon_0 S_0+\epsilon_8 S_8\,, \tag{7}$$

where H_0 is the part of the Hamiltonian density which is $SU^{(+)}(3)\otimes SU^{(-)}(3)$ invariant, and the scalar densities S_i ($i=0,\ldots,8$) which break this symmetry transform according to the $(3,3^*)\oplus(3^*,3)$ representation. Defining

$$a=\epsilon_8/\sqrt{2}\epsilon_0\,, \tag{8}$$

it can be shown[4] that $a=-1$, 0, and 2 are special points, where $SU^{(+)}(2)\otimes SU^{(-)}(2)$, $SU(3)$, and the chimeral $SU(3)$ symmetries, respectively, become exact. Furthermore, the points $a=-1$ and $a=2$ are essentially singular points of the theory, and the singularity-free range $-1\leq a\leq 2$ corresponds to the physical domain, with the physical value a_0 given by[6]

$$a_0=-2(m_K{}^2-m_\pi{}^2)/(2m_K{}^2+m_\pi{}^2)\simeq-0.89\,. \tag{9}$$

It has also been shown[4] that $m_\pi{}^2$ and $m_K{}^2$, regarded as continuous functions of a and ϵ_0 in the physical domain, are given by

$$m_\pi{}^2=(1+a)m_0{}^2,\quad m_K{}^2=(1-\tfrac{1}{2}a)m_0{}^2\,, \tag{10}$$

where $m_0{}^2$ is in general a function of a and ϵ_0.

Now, suppose that we can solve for Δ exactly from the equation $T(\Delta,m_\pi{}^2,m_K{}^2)=0$. Then the solution for Δ will be a function of a and ϵ_0. For a fixed ϵ_0, we may regard the solution $\Delta=f(a)$ to be a continuous function of a in the physical domain. We shall now assume that this solution $f(a)$ is as smooth a function of a as possible, consistent with the requirements (4), which can be written as

$$\begin{aligned} &f(a=-1)=\tfrac{3}{2}m_0{}^2(a=-1)\,,\\ &f(a=2)=-3m_0{}^2(a=2)\,,\\ &f(a=0)=0,\quad (\partial/\partial a)f(a)|_{a=0}=0\,. \end{aligned} \tag{11}$$

This leads us to the following minimal structure:

$$\Delta(a)=-\tfrac{1}{4}a^2(5-a)m_0{}^2(a)\,. \tag{12}$$

At the physical point (9), we then obtain for the best value

$$\Delta(a_0)\equiv\Delta_0=-\frac{(m_K{}^2-m_\pi{}^2)^2(4m_K{}^2+m_\pi{}^2)}{(2m_K{}^2+m_\pi{}^2)^2}\simeq-0.8m_K{}^2\,. \tag{13}$$

Thus if the hypothesis of maximum smoothness makes sense, we obtain the following "physical" sum rule:

$$(m_K{}^2+m_\pi{}^2)f_+(\Delta_0)+(m_K{}^2-m_\pi{}^2)f_-(\Delta_0)=m_K{}^2f_K/f_\pi+m_\pi{}^2f_\pi/f_K\,. \tag{14}$$

Although the value of Δ_0 given by Eq. (12) still lies in the unphysical region for the K_{l3} decay, it is much closer to the physical region than the argument of the form factors in the soft-pion theorem (1). Thus the sum rule (14) *not only incorporates corrections to the soft-pion result, but also is much easier to test from an experimental point of view*. It may also be pointed out that if $f_K/f_\pi<f_+(\Delta_0)$, as the K^*-dominance model for $f_+(q^2)$ would suggest, one expects a small negative numerical value for $\xi(\Delta_0)\equiv f_-(\Delta_0)/f_+(\Delta_0)$, in agreement with the recent experiments.[7]

Finally, we would like to mention that the discussion based on the model (7) suggests that the sum rule (14) may be valid even if the $SU^{(+)}(3)\otimes SU^{(-)}(3)$ perturbation approach fails. The crucial point is again the assumption of smoothness, so that the sum rule (14) which is exact at the points $a=2$, 0, -1 is expected to continue smoothly to the physical point, independent of the perturbation argument.

The prescription of obtaining corrections to the soft-meson theorems discussed here is evidently quite general and can be applied to a variety of problems. Further applications will be discussed elsewhere.

[5] M. Gell-Mann, R. J. Oakes, and B. Renner, Phys. Rev. **175**, 2195 (1968).

[6] Equation (9) was first derived in Ref. 5.

[7] D. Haidt *et al.* (X2 Collaboration), Phys. Letters **29B**, 691 (1969); **29B**, 696 (1969).