in the future, it would be possible to deduce the phase shift δ_0^0 by fitting the t dependence of various decay moments to the model with a more flexible form involving more parameters for $F_c(t)$, thereby removing the assumptions of spin independence and perhaps $m_{\pi\pi}$ independence. However, much higher statistics (especially for $-t<\mu^2$) would also permit a "nonevasive" polynomial *extrapolation* (with sufficient number of parameters) which is, in principle, model independent. However, we believe that the use of a reasonably "good" absorption model with somewhat better (say, doubled) data would still give a value for δ_0^0 deduced from the "s-wave" term more reliable than that obtained by nonevasive polynomial extrapolation.

In conclusion, we have seen that while an absorption model can be used on present data to give a set of reasonable δ_0^0 from A_0, these results are perhaps too sensitive to the amount of depolarization predicted by *any* absorption model to be by themselves definitive. However, these results can be used to discriminate between the twofold ambiguous solutions for δ_0^0 obtained from A_1 in the region $500 \leq m_{\pi\pi} \leq 700$ MeV. With present data, it is not possible to make a similar statement on the twofold ambiguity in the higher-$m_{\pi\pi}$ region. We believe that a simple doubling of the present data for $-t>\mu^2$ would be helpful in settling, perhaps once and for all, the ambiguity at least in the low-$m_{\pi\pi}$ region. *Even more* statistics would be necessary to look into the same problem at the higher-$m_{\pi\pi}$ regions because of the possibility of important d-wave effects and other processes contributing to the same three-body final state; also, it is desirable for discrimination to have good data for $-t<\mu^2$.

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Unitarity Upper Bounds and Elastic Scattering*

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A derivation is given in terms of the eikonal formalism of a recently derived upper bound on the absorptive part of a spin-zero elastic scattering amplitude. The multivaluedness inherent in the problem is explored, and considerably smaller upper bounds are obtained by imposing conditions of reasonability on the profile function. Experimental differential cross sections for $\pi^{\pm}p$ and pp elastic scattering over a wide energy range are found to fall on a universal curve, which lies just below our most stringent upper bound over a range of three decades in $d\sigma/dt$.

It was shown by MacDowell and Martin[1] that unitarity puts a bound on $[dA(t)/dt]_0/A(0)$, the slope of the absorptive part $A(t)$ of the elastic scattering amplitude in the forward direction. In a recent Letter, Singh and Roy[2] extended those considerations of unitarity to obtain an upper bound on the non-spin-flip absorptive amplitude $A(t)/A(0)$ apparently for general (negative) values of t. The purpose of this comment is threefold: (1) to give a simpler derivation of the latter result, in terms of the eikonal formalism; (2) in this framework to show that the functional character of the solution is multivalued; and (3) to show that the assumption of reasonable additional restrictions on the profile function gives a considerably reduced upper bound, which lies close to experimental values of $d\sigma/dt$ over three decades.

As is made clear in Refs. 1 and 2, these bounds are useful mainly in the energy regions where there is strong absorption and many partial waves contribute. This is the region where the eikonal approximation is appropriate. We therefore use this approximation. In terms of the impact parameter $b=l/k$, the momentum transfer q, assumed to be transverse, where $t=-q^2$, and the imaginary part of the partial-wave amplitude $\Gamma(b)$ (the profile function), we must examine the quantity

$$F(q)=A(q^2)/A(0)$$

$$=\int d^2b\,\Gamma(b)\,e^{iq\cdot b}\bigg/\int d^2b\,\Gamma(b) \tag{1a}$$

$$=\int_0^\infty bdb\,\Gamma(b)J_0(qb)\bigg/\int_0^\infty bdb\,\Gamma(b)\,. \tag{1b}$$

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† General Electric Company Predoctoral Fellow.

[1] S. W. MacDowell and A. Martin, Phys. Rev. **135**, B960 (1964).

[2] V. Singh and S. M. Roy, Phys. Rev. Letters **24**, 28 (1970); **24**, 699(E) (1970).

According to the methods of Refs. 1 and 2, the bound $F_B(q)$ is obtained when $F(q)$ is expressed as a function of ρ, defined by the relationship

$$\rho = q^2\sigma_{\mathrm{tot}}^2/4\pi\sigma_{\mathrm{el}}\,. \tag{2}$$

Calling ρ/q^2 a new quantity B^2, we obtain, in terms of impact parameters, and assuming that the scattering is entirely imaginary, the relationship

$$B^2 = \left[\int d^2b\,\Gamma(b)\right]^2 \Big/ \left(\pi\int d^2b\,[\Gamma(b)]^2\right) \tag{3a}$$

$$= 2\left[\int_0^\infty bdb\,\Gamma(b)\right]^2 \Big/ \int_0^\infty bdb\,[\Gamma(b)]^2. \tag{3b}$$

B is seen to have dimensions of length and to be of zero degree in Γ. $F(q)$ also is of zero degree in Γ. The question raised in Refs. 1 and 2 can thus be posed as follows: For all functions $\Gamma(b)$, $\Gamma(b)>0$, of given spatial extent B defined by (3), what is the maximum value of the (normalized to 1 at $q=0$) Fourier transform $F(q)$? The corresponding problem in three dimensions concerns form factors of nuclear charge distributions, and it is there a matter of common experience that $F(q)<1$.

At first sight, the formal solution to this problem, obtained by finding the extremum of the quantity Q,

$$Q = F(q) + \lambda[2\langle\Gamma\rangle^2/\langle\Gamma^2\rangle - B^2], \tag{4}$$

with respect to variations of the function $\Gamma(b)$, is identical to that obtained by Singh and Roy.[2] λ is a Lagrange multiplier and $\langle\Gamma\rangle$ indicates integration of $\Gamma(b)$ over b with weight b. The resulting form[3] for the function $\Gamma_B(b)$ which maximizes Q is, after elimination of λ in favor of a new constant $b_{\max}$,

$$\Gamma_B(b) \propto J_0(qb) - J_0(qb_{\max})\,,\quad b \le b_{\max} \tag{5a}$$

$$= 0\,,\quad b > b_{\max}\,. \tag{5b}$$

The scale of Γ is assumed to be such that $\Gamma<1$, because of unitarity, but it is otherwise immaterial because Q is of zero degree in Γ.[4] The resulting functional form of $F_B(q)$, the upper bound on $F(q)$, is

$$F_B(q) = [\langle J_0^2\rangle - \langle J_0\rangle J_0(qb_{\max})]/[\langle J_0\rangle - \langle 1\rangle J_0(qb_{\max})] \tag{6a}$$

$$= J_1^2(qb_{\max})/J_2(qb_{\max}) - J_0(qb_{\max})\,. \tag{6b}$$

The condition that (5) is consistent with the definition (3b) of the length parameter B gives the relationship

$$B^{-2} = [F_B(q) - J_0(qb_{\max})]/[b_{\max}^2 J_2(qb_{\max})]\,. \tag{7}$$

Equations (6b) and (7) give the parametric dependence of F_B and B^2 on $b_{\max}$, and thus implicitly give the dependence of F_B on $(qB)^2 \equiv \rho$. Singh and Roy,[2] who obtained these same equations, calculate explicitly the first few terms of the expansion of F_B as a power series in ρ, by eliminating $b_{\max}$ from the above relationships. They also give numerical values of $F_B(\rho)$ for larger values of ρ, obtained by an exact solution of these equations. Their result is plotted as the curve labeled B in Fig. 1. We cite the agreement in results between the eikonal approximation and the partial-wave treatment of Ref. 2 as a justification for further use of this approximation.

We now examine in more detail the character of solutions of Eqs. (6b) and (7). It is not surprising that, since each of these equations has many branches, the expression of F_B in terms of $\rho=(qB)^2$ involves the problem of multivaluedness. To understand what this means physically, we first note that the variational calculation has not ensured that $\Gamma(b)\ge 0$. This can be achieved by writing $\Gamma(b)=f^2(b)$ in the variational expression (4). The extremum condition then becomes

$$f_B(b)\{f_B^2(b) - c[J_0(qb) - J_0(qb_{\max})]\} = 0\,. \tag{8}$$

$\Gamma(b)$ is thus given by (6b) for these ranges of b where $\Gamma_B(b)>0$, and by $\Gamma_B(b)=0$ where (5) would give a negative value. The possibility of additional contributions to $\Gamma_B(b)$, interspersed with regions where $\Gamma_B=0$, causes the multivaluedness. Sketches of $\Gamma(b)$ appropriate to the various ranges of $(qB)^2$ and to two separate branches are inset in Fig. 1. The solid curve, down to point A, is obtained when the parameter $qb_{\max}$ covers the range $0<qb_{\max}<x_1$, where x_1 gives the first minimum of $J_0(x)$. This curve corresponds to a $\Gamma(b)$ which decreases monotonically. For the solid curve labeled B, $qb_{\max}$ lies in the range $x_2<qb_{\max}<x_3$, where x_2 and x_3 mark the next maximum and minimum of J_0, and $\Gamma(b)$ is zero over some intermediate range. *This is the curve given in Ref. 2.* It thus corresponds to a profile function which has a gap. Another broken curve, sketched in Fig. 1 starting at B, $\rho\simeq 8$, is the solution which has two gaps. It lies above the curve of Ref. 2 for $\rho \gtrsim 8$.

It is physically reasonable to assume that $\Gamma(b)$ is not zero in intermediate regions, and even to postulate that $\Gamma(b)$ must be monotonically decreasing. This we now do. The method given up to now enables us to obtain $F_B(\rho)$ for such a $\Gamma(b)$ only out to $\rho\simeq 7.34$, where $qb_{\max}=x_1$. To go further, we substitute into the variation form (4) the representation

$$\Gamma(b) = \int_b^{b_{\max}} h^2(b)db\,. \tag{9}$$

[3] In the process of functional differentiation of Q, it becomes clear that (3) is the only definition of spatial extent of Γ which leads to an equation for Γ, *and* in which the quantities involved are measured by experiment. Constraints on a moment of Γ, such as $\langle b^2\Gamma\rangle/\langle\Gamma\rangle$, do not lead to an equation for Γ, and thus give only 1 as the upper bound.

[4] If, for a particular choice of scale, Eq. (5) gives $\Gamma(b)>1$ for some range of b, we must replace (5) by $\Gamma(b)=1$ over that range. The resulting bound is of course lower than the one obtained here.

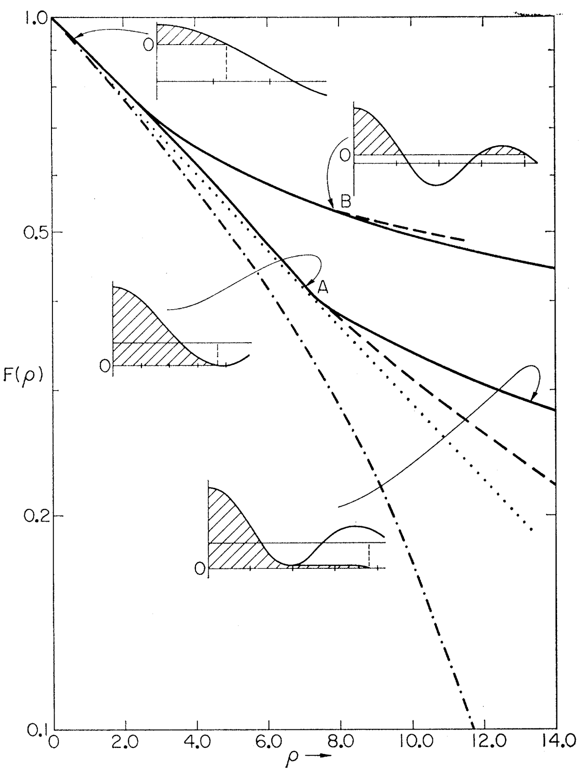

FIG. 1. F, the ratio of the imaginary part of the elastic scattering amplitude to its value at $\theta=0$, plotted against the variable $\rho=q^2(\sigma_{\text{tot}})^2/4\pi\sigma_{\text{el}}$. The full line passing through point A is the upper bound on F, assuming that $\Gamma(b)$ decreases monotonically, and the dashed line starting at A is the upper bound obtained with the further assumption that $-d\Gamma/db$ does not have a local minimum. The full curve passing through point B is the upper bound given in Ref. 2, for which the only restriction on $\Gamma(b)$ is that $\Gamma(b)\geq 0$. The dashed curve starting at B is the higher upper bound obtained when $\Gamma(b)$ is allowed to have two gaps. In the inset figures, the curve is $J_0(x)$ and the hatched area is the unnormalized $\Gamma(b)$ which will produce the upper bound at the indicated value of ρ. The dotted curve is $F_1(\rho)$ [see Eq. (12a)], the actual $F(\rho)$ for a Gaussian $\Gamma(b)$, and the dashed curve below it is $F_2(\rho)$ [see Eq. (12b)] for a square $\Gamma(b)$.

The solution we obtain is that

$$h_B(b)[h_B^2(b)-cJ_1(qb)]=0. \tag{10}$$

The (negative of the) derivative $h^2(b)$ is thus taken to be positive or zero. The solution which provides the continuation of curve A (full line) in Fig. 1 comes from assuming that $h_B^2(b)=cJ_1(qb)$ for $0<qb<x_1$ and for $x_2<qb<x_{\max}$, and $h_B^2(b)=0$ for $x_1<qb<x_2$. The resulting $\Gamma_B(b)$ decreases monotonically, although it has a region where $d\Gamma_B(b)/db=0$. We note that the imposition of a new physical condition on $\Gamma_B(b)$ has produced a much reduced, and therefore more useful, upper bound.

Before discussing experimental cross sections, we emphasize that the curves given are *upper bounds* on functions $F(\rho)$. They are *not* the functions obtained with any fixed, q-independent $\Gamma(b)$. It is, however, a remarkable fact that reasonable q-independent functions $\Gamma(b)$ yield $F(q)$'s which follow the bound fairly closely over the first decade of F^2. Functions containing only one radial scale parameter, such as

$$\Gamma_1(b)=c\exp(-b^2/b_1^2) \tag{11a}$$

or

$$\Gamma_2(b)=d\Theta(b_2-b), \tag{11b}$$

form curves $F_{1,2}(\rho)$ with no adjustable parameters:

$$F_1(\rho)=\exp(-\tfrac{1}{8}\rho) \tag{12a}$$

and

$$F_2(\rho)=2\rho^{-1/2}J_1(\rho^{1/2}). \tag{12b}$$

They are illustrated in Fig. 1. $F_1(\rho)$, the dotted line on our semilogarithmic plot, is almost tangent to our upper bound at $\rho\simeq 7.34$. The square shape $F_2(\rho)$, the dash-dot curve, lies somewhat below the bound, and has its first zero at $\rho\cong 12$. For values of ρ greater than this a (negative) lower bound on $F(q)$ is relevant. It can of course be obtained in the manner discussed here, although no results are included.

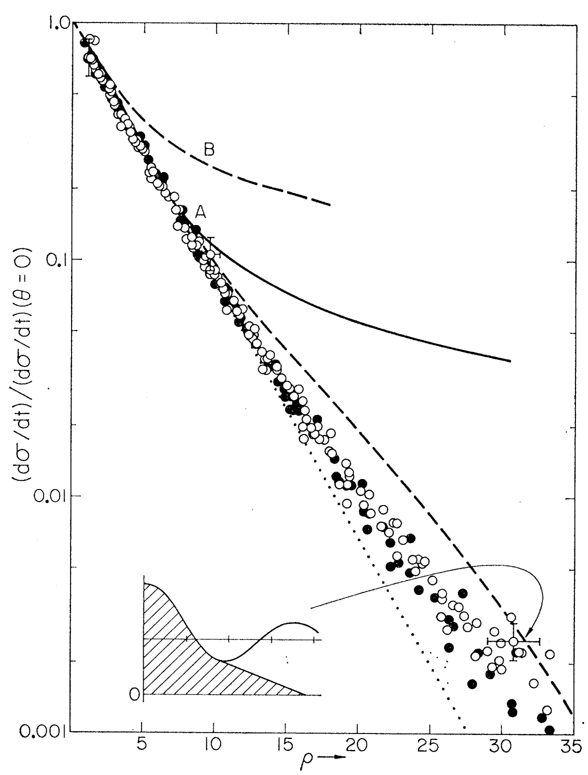

FIG. 2. Experimental values of $(d\sigma/dt)_0$ taken from Ref. 5 are plotted against $\rho=q^2\sigma_{\text{tot}}^2/4\pi\sigma_{\text{el}}$, for pp scattering (closed circles) and $\pi^{\pm}p$ scattering (open circles). Incident momenta range from 6.8 to 19.7 BeV/c. Error bars on all of the points are about equal to the sample values shown. The upper dashed curve is the upper bound given in Ref. 2. The solid curve is the upper bound given by unitarity and the assumption that the profile function $\Gamma(b)$ decreases monotonically, and the dashed curve below it comes from the additional assumption that $-d\Gamma(b)/db$ has no local minimum. The inset figure illustrates the $\Gamma(b)$ which produces the latter bound at the indicated value of ρ. The dotted curve comes from the Gaussian $\Gamma(b)$, Eq. (12a).

Because our new upper bound, obtained by requiring that $\Gamma(b)$ decreases monotonically, lies considerably below the one given in Ref. 2, another examination of the agreement with experimental high-energy elastic cross sections is necessary. In Fig. 2, representatisns of $\pi^{\pm}p$ and of pp scattering data[5] taken over a wide range of incident momenta (6 to about 20 GeV/c) are plotted in the manner suggested by these methods,[1,2] i.e., $(d\sigma/dt)/(d\sigma/dt)_0$ versus ρ. They are seen to exhibit the universality discovered by MacDowell and Martin,[1] and emphasized by Singh and Roy,[2] over at least three decades of falloff in $d\sigma/dt$. Our new upper bound is shown as the full curve in Fig. 2, and it closely bounds the data over the first decade. If the data lay above this curve (and the various approximations involved in the comparison were not breaking down), one would have proved that $\Gamma(b)$ was not monotonically decreasing. The converse is not true, of course, but it is an interesting possibility. The results we have illustrated with special models in Fig. 1 suggest that over the first decade of the cross section, any reasonable $\Gamma(b)$ will lie close to the bound, and thus agree with the experiments. In physical terms, once $\Gamma(b)$ has been chosen to give the correct σ_{el} and σ_{tot} (which determine the scale of ρ), it will predict the first decade of $d\sigma/dt$ automatically.[6]

Since the experimental data shown in Fig. 2 exhibit the universality property over the whole range, it is of interest to see what physical constraints are needed on $\Gamma(b)$ to produce an upper bound lying closer to the data. We observe that, in potential theory, $\Gamma(b)$ is always a smoother function than the potential $V(r)$. It would take a most peculiar $V(r)$ to produce a $\Gamma(b)$ which had $d\Gamma(b)/db=0$ over a range of b, the behavior exhibited by our $\Gamma_B(b)$ after point A. For the most reasonable $V(r)$, one finds that $-d\Gamma(b)/db$ is a smooth function, zero at $b=0$ and $b\rightarrow\infty$, with one maximum. We can therefore impose as a more rigorous constraint on $\Gamma(b)$ the property that $-d\Gamma(b)/db$ has no local minima. In terms of $h^2(b)$, in Eq. (9), the function which gives our upper bound on $F(\rho)$ with this additional constraint is that $h^2(b)=cJ_1(x_3)$, for $x_1<qb<x_3$, and $h^2(b)=cJ_1(qb)$, for $0<qb<x_1$. Here x_3 is greater than x_2, where x_2 is the second zero of J_1, and x_1 $(<x_2)$ is defined by $J_1(x_1)=J_1(x_3)$. The resulting $F_B(q)$ is shown as the dashed continuation of curve A in Figs. 1 and 2. It lies not very far above the data, especially at the largest ρ values. The inset in Fig. 2 is a sample profile function.

The fact that data from these separate elastic scattering processes,[7] taken at a variety of incident momenta, form a universal curve is noteworthy. It is remarkable that unitarity plus relatively crude constraints on the profile function should yield such a close upper bound to this curve. It is tempting to conjecture that the additional constraint that $\Gamma(b)$ be differentiable to higher order would produce an even closer upper bound. The differential cross section over its first three decades would then have been determined entirely by unitarity, reasonableness of the profile function, and the assumption that the cross section achieves its upper bound under those conditions. The task of different models would consist only in determining σ_{tot} and σ_{el}, which determine the actual scales of $d\sigma/dt$ and of t. Such a conclusion needs considerable qualification, of course. Additional assumptions made in this approach—that the eikonal approximation can be used at the relatively large values of θ involved, that the elastic amplitude is entirely imaginary, and that spin-flip may be ignored—need examining in a more detailed comparison with experiment. Work on these and other points is in progress.

We wish to acknowledge many helpful comments and discussions with our colleagues.

[5] K. J. Foley, S. J. Lindenbaum, W. A. Love, S. Ozaki, J. J. Russell, and L. C. L. Yuan, Phys. Rev. Letters **11**, 425 (1963). This is a convenient source of data for us, since it includes $d\sigma/dt$ $(t=0)$, needed for both the vertical and horizontal scales in our F-versus-ρ plot.

[6] We feel that this shape insensitivity at small t can have important implications for recent work on forward proton-nucleus elastic scattering and attempted deduction from it of nuclear matter distributions.

[7] The $K^{\pm}p$ experimental cross sections [K. J. Foley *et al.*, Phys. Rev. Letters **11**, 503 (1963)], not included in Fig. 2, behave in the same way. The $\bar{p}p$ cross section, however, appears to be somewhat lower, falling on the Gaussian curve of Fig. 2 over the whole range.