Simple and General Derivation of the Angular Condition on Current Algebras at Infinite Momentum

P. P. DIVAKARAN AND N. MUKUNDA
Tata Institute of Fundamental Research, Bombay 5, India
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We present a simple derivation of the angular condition on current algebra at infinite momentum, wide enough to cover the possible presence of spacelike states. The use of the Breit frame, which is not in general possible if spacelike states are involved, is completely avoided.

IN their remarkable work on the representations of current algebras at infinite momentum, Chang, Dashen, and O'Raifeartaigh[1] have proved the theorem that, within a factorizable representation, either there is an infinite mass degeneracy, or there are spacelike states. This theorem, in fact, followed the discovery[2] of a number of representations all of which had one or the other of the above-mentioned features, and is a direct consequence of the covariance constraints.

It should be realized, however, that the treatment of the angular condition as given, for example, in CDO'R, is incomplete when spacelike states are present. It is very easy to see that in the passage from the physical Hilbert space H to the "space of quantum numbers" $\hat{H}$ on which the representations at infinite momentum are defined, the (mass)^2 operator M^2 remains essentially invariant. Therefore, if spacelike states occur in representations on $\hat{H}$, they must already be present in H. However, the usual methods of derivation of the angular condition itself, of which we shall take CDO'R as the typical example, are valid only if no spacelike states are present in H. To see this, it may be recalled that the angular condition is simply the requirement that *in any collinear frame*,[3] the matrix element of a vector operator vanishes except when the helicities of the two states are the same or differ by 1—i.e., one uses the Wigner-Eckart theorem in a frame in which the helicity is the same as J_3 (choosing the common direction of the momenta as the 3-axis). But, in general, it is impossible to transform a pair of spacelike vectors to a collinear form.[4]

We present here a method of deriving the angular condition which overcomes this difficulty, i.e., which is valid even when spacelike states are present in H. The method completely avoids the use of such devices as collinear frames and standard decelerations and is, consequently, extremely simple and straightforward. This simplicity is achieved by choosing, not the helicity operator, but an appropriate linear combination of the components of the Pauli-Lubanski operator as the operator whose eigenvalues are changed in a definite (bounded) way under the action of a four-vector. It turns out that the form of the general angular condition is formally the same as that derived by CDO'R, though some of the operators on $\hat{H}$ in terms of which it is stated have a different physical interpretation. The angular condition of CDO'R is

$$I_k(I_k(I_k(F(k_\perp))))=J_k(I_k(F(k_\perp)))\,, \tag{1}$$

where

$$I_k(\theta)=[M^2,[\mathcal{J}_3,\theta]]-2k_\perp\cdot[M\mathcal{J}_\perp,\theta]-k_\perp^2[\mathcal{J}_3,\theta]_+\,, \tag{2}$$

$$J_k(\theta)=[M^2,[M^2,\theta]]+2k_\perp^2[M^2,\theta]_+ +k_\perp^4\theta\,, \tag{3}$$

and where M is the mass operator and $\mathcal{J}_a$ are the (Hermitian) generators of the "helicity group" at infinite momentum, satisfying the commutation relations of $SU(2)$. If Eq. (1) is assumed to be valid with M and $\mathcal{J}_a$ interpreted as above even when M^2 is not a positive operator, then because of the second term in Eq. (2), Hermitian conjugation will lead to an independent constraint, which allows[5] only trivial solutions for $F(k_\perp)$. As stated above, we find that the angular condition can be written, *in all cases*, in the form of Eq. (1), but when $M\mathcal{J}_\perp$ acts on a spacelike state, $i\mathcal{J}_\perp$ are to be interpreted as two Hermitian operators which, along with $\mathcal{J}_3$, have commutators characteristic of a unitary representation of $O(2,1)$.

Consider an operator $Z(p',p)$ defined by

$$\begin{aligned}Z(p',p)&=\epsilon^{\mu\nu\rho\sigma}p'_\mu p_\nu J_{\rho\sigma}\\&=2(\omega'\mathbf{p}-\omega\mathbf{p}')\cdot\mathbf{J}-2(\mathbf{p}'\times\mathbf{p})\cdot\mathbf{K}\,,\end{aligned} \tag{4}$$

where $J_{\rho\sigma}$ are the generators of the Lorentz group, $\mathbf{J}$ and $\mathbf{K}$ being the generators of rotations and boosts, and $p_\mu=(\omega,\mathbf{p})$ and p'_μ are arbitrary numerical four-vectors. Acting on a state of momentum p_μ or p'_μ, $Z(p',p)$ is

[1] S. J. Chang, R. Dashen, and L. O'Raifeartaigh, Phys. Rev. Letters **21**, 1026 (1968); Phys. Rev. **182**, 1805 (1969); **182**, 1819 (1969). These papers will be collectively referred to as CDO'R.

[2] References can be found in CDO'R.

[3] The use of the Breit frame is not essential.

[4] It may be mentioned here that there is no similar problem in transcribing the commutator of the currents to $\hat{H}$, at least if it is assumed that the currents do not connect states with energies of opposite sign. In this case, a proper modification of the by now standard procedure leads to the relation $[F_i(k_\perp),F_j(k_\perp')]=if_{ij}{}^kF_k(k_\perp+k_\perp')$. Along with the positive-energy timelike states, only the positive-energy spacelike states are represented in $\hat{H}$.

[5] In the special case of M^2 being a negative operator, this additional constraint is given by $I_k(I_k(I_k(F(-k_\perp))))=J_k(I_k(F(-k_\perp)))$. The k-independent angular conditions of CDO'R (for factorizable representations) have then to be supplemented by a new set of equations which arise from the equation above. It is straightforward work to show that together they allow only the trivial solution $F_i(k_\perp)=\tau_i$, i.e., the current algebra degenerates into the corresponding algebra of charges. This only shows that in the presence of spacelike states, the angular conditions must be rederived and properly interpreted.

simply a contraction of the Pauli-Lubanski operator with p'_μ or p_μ, respectively; therefore, acting on a state of momentum p_μ or p'_μ, it leaves the momentum unchanged, and affects only the helicity labels:

$$[Z(p',p),P_\lambda]|p_\mu\rangle=[Z(p',p),P_\lambda]|p'_\mu\rangle=0. \quad (5)$$

The derivation of the angular condition depends on the fact that if j^λ is an arbitrary constant four-vector, it satisfies the condition

$$[Z(p',p),[Z(p',p),[Z(p',p),j^\lambda]]] = 4[(p'p)^2-p'^2p^2][Z(p',p),j^\lambda]. \quad (6)$$

The derivation of Eq. (6) is straightforward. Using the commutation relation

$$[J_{\mu\nu},j^\lambda]=i(g_\mu{}^\lambda j_\nu-g_\nu{}^\lambda j_\mu),$$

one sees, in turn, that

$$[Z(p',p),j^\lambda]=-2i\epsilon^{\lambda\mu\nu\rho}p'_\mu p_\nu j_\rho\equiv-2iU^\lambda, \quad (7)$$

$$[Z(p',p),U^\lambda]=2i\epsilon^{\lambda\mu\nu\rho}p'_\mu p_\nu U_\rho\equiv 2iV^\lambda, \quad (8)$$

and that

$$[Z(p',p),V^\lambda]=2i[p'^2p^2-(p'p)^2]U^\lambda. \quad (9)$$

Equations (7)–(9) together give Eq. (6). The projections $p^\mu j_\mu$ and $p'^\mu j_\mu$ do not alter the eigenvalues of $Z(p',p)$, while the two independent components of U^λ do shift these eigenvalues.

All we have to do now is to take the matrix element of Eq. (6) as $\mathbf{p}'$ and $\mathbf{p}$ go to infinity, and express the resulting limit equation as an equation in $\hat{H}$. To do this, we shall make use of the canonical representation[6,7] of the Lorentz-group generators appropriate to the situation. These representations are distinguished by the fact that they decouple the "internal" degrees of freedom corresponding to the little group [$O(3)$ for timelike states and $O(2,1)$ for spacelike states] from the "orbital" degrees of freedom. Because of Eq. (5), we may, then, ignore the orbital part in taking matrix elements of $Z(p',p)$.

The matrix elements we need to calculate are of the general form

$$\lim_{\kappa\to\infty}\langle\mathbf{p}'_\kappa,n'|[Z(p'_\kappa,p_\kappa),A]|\mathbf{p}_\kappa,n\rangle,$$

where the label n denotes all quantum numbers excepting the momentum $\mathbf{p}$, and including the mass m, and where A is an operator whose infinite-momentum limit exists. We discuss the case of $|\mathbf{p}_\kappa,n\rangle$ being a timelike state and a spacelike state separately.

1. Timelike states. The canonical representation is[6]

$$\mathbf{J}|\mathbf{p},n\rangle=\mathbf{s}|\mathbf{p},n\rangle, \quad \mathbf{K}|\mathbf{p},n\rangle=\frac{\mathbf{p}\times\mathbf{s}}{\omega+m}|\mathbf{p},n\rangle, \quad (10)$$

where $\mathbf{s}$ are the spin matrices corresponding to the spin s of the state and where we have kept only the terms describing the "internal" degrees of freedom. Replacing $\mathbf{p}'$ and $\mathbf{p}$ by $\mathbf{p}'_\kappa=\mathbf{p}'+\kappa\hat{a}$ and $\mathbf{p}_\kappa=\mathbf{p}+\kappa\hat{a}$ ($\hat{a}$ is the unit vector along the third direction), and expanding all κ-dependent quantities in powers of κ^{-1}, we obtain immediately

$$\begin{aligned}\lim\langle\mathbf{p}'_\kappa,n'|AZ(p'_\kappa,p_\kappa)|\mathbf{p}_\kappa,n\rangle &= (m'^2-m^2+k_\perp^2)\lim\langle\mathbf{p}'_\kappa,n'|As_3|\mathbf{p}_\kappa,n\rangle \\ &\quad -2mk_\perp\cdot\lim\langle\mathbf{p}'_\kappa,n'|As_\perp|\mathbf{p}_\kappa,n\rangle. \quad (11)\end{aligned}$$

The limit for the matrix element of $Z(p'_\kappa,p_\kappa)A$ is obtained from the right-hand side of Eq. (11) by the interchange $\mathbf{p}'\leftrightarrow\mathbf{p}$, $m'\leftrightarrow m$, $s'\leftrightarrow s$ and changing the over-all sign. Therefore,

$$\begin{aligned}\lim\langle\mathbf{p}'_\kappa,n'|[Z(p'_\kappa,p_\kappa),A]|\mathbf{p}_\kappa,n\rangle &= (m'^2-m^2)\lim\langle\mathbf{p}'_\kappa,n'|[S_3,A]|\mathbf{p}_\kappa,n\rangle \\ &\quad -2k_\perp\cdot\lim\langle\mathbf{p}'_\kappa,n'|m'S_\perp A-mAS_\perp|\mathbf{p}_\kappa,n\rangle \\ &\quad -k_\perp^2\lim\langle\mathbf{p}'_\kappa,n'|[S_3,A]_+|\mathbf{p}_\kappa,n\rangle. \quad (12)\end{aligned}$$

The operators $\mathbf{S}$ are defined to give, acting on a state, the same result as that of the corresponding matrices $\mathbf{s}$ on the spin index of that state; they generate the group $O(3)$. In the infinite-momentum limit, S_3 goes over into the helicity operator $\mathcal{J}_3$, so that we have the rule of correspondence

$$\lim\langle\mathbf{p}'_\kappa,n'|S_a|\mathbf{p}_\kappa,n\rangle=\langle n'|\mathcal{J}_a|n\rangle, \quad a=1,2,3.$$

Equation (12) can therefore be written in the form

$$\begin{aligned}\lim\langle\mathbf{p}'_\kappa,n'|[Z(p'_\kappa,p_\kappa),A]|\mathbf{p}_\kappa,n\rangle &= \langle n'|\{[M^2,[\mathcal{J}_3,F(k_\perp)]]-2k_\perp\cdot[M\mathcal{J}_\perp,F(k_\perp)] \\ &\quad -k_\perp^2[\mathcal{J}_3,F(k_\perp)]_+\}|n\rangle, \quad (13)\end{aligned}$$

identifying A with the fourth component of the current operator. Further,[8]

$$\lim[(p'_\kappa p_\kappa)^2-p'^2_\kappa p_\kappa^2] = \tfrac{1}{4}[(m'^2-m^2)^2+2(m'^2+m^2)k_\perp^2+k_\perp^4]. \quad (14)$$

Consequently, by taking the infinite-momentum matrix element of Eq. (6), we get the covariance condition given by Eqs. (1)–(3).

2. Spacelike states. In this case, the canonical representation is given by[7]

$$\begin{aligned}&J_1=\frac{s_3'p_1+s_2'\omega}{p_3+\mu}, \quad J_2=\frac{s_3'p_2-s_1'\omega}{p_3+\mu}, \quad J_3=s_3', \\ &K_1=s_1', \quad K_2=s_2', \quad K_3=-\frac{s_1'p_1+s_2'p_2}{p_3+\mu},\end{aligned} \quad (15)$$

[6] Iu. M. Shirokov, Dokl. Akad. Nauk SSSR **94**, 857 (1954); **97**, 737 (1954); Zh. Eksperim. i Teor. Fiz. **33**, 1196 (1957); **33**, 1208 (1957) [Soviet Phys. JETP **6**, 919 (1958); **6**, 929 (1958)]; L. L. Foldy, Phys. Rev. **102**, 568 (1956).

[7] Iu. M. Shirokov (Ref. 6); N. Mukunda, TIFR report, 1969 (unpublished). These representations are correct in the form given only when acting on states with $p_3>-\mu$. In the limit $p_3\to\infty$, this restriction, obviously, does not matter.

[8] Equation (14) makes evident the fact that $J_k(\theta)$ has a very simple geometrical interpretation.

all operators acting on a state of momentum $\mathbf{p}$ and mass $m=(\omega^2-\mathbf{p}^2)^{1/2}=i\mu$, μ being real. The Hermitian matrices s_1', s_2', and s_3' generate the group $O(2,1)$ which is the little group appropriate to a spacelike vector. We have now only to take the matrix element of Eq. (6) between, for example, two spacelike states and take the limit $\kappa\to\infty$ exactly as before. The result corresponding to Eq. (12) is

$$\begin{aligned}\lim\langle\mathbf{p}'_\kappa,n'|[Z(p'_\kappa,p_\kappa),A]|\mathbf{p}_\kappa,n\rangle &= (m'^2-m^2)\lim\langle\mathbf{p}'_\kappa,n'|[S_3',A]|\mathbf{p}_\kappa,n\rangle \\ &\quad -2k_\perp\cdot\lim\langle\mathbf{p}'_\kappa,n'|\mu'S_\perp'A-\mu AS_\perp'|\mathbf{p}_\kappa,n\rangle \\ &\quad -k_\perp^2\lim\langle\mathbf{p}'_\kappa,n'|[S_3',A]_+|\mathbf{p}_\kappa,n\rangle,\end{aligned}\tag{16}$$

where $S_\perp'=(-S_2',S_1',0)$. It follows that the angular condition can still be written in the form of Eq. (1), with $J_k(\theta)$ given by Eq. (3), but $I_k(\theta)$ replaced by

$$I_k'(\theta)=[M^2,[\mathcal{J}_3',\theta]]-2k_\perp\cdot[M\mathcal{J}_\perp',\theta]-k_\perp^2[\mathcal{J}_3',\theta]_+,\tag{17}$$

where the correspondence between the spin operators S_i' and the operators $\mathcal{J}_i'$ is given by $S_3'\to\mathcal{J}_3'$, $-S_2'\equiv\exp(-\frac{1}{2}i\pi S_3')S_1'\exp(\frac{1}{2}i\pi S_3')\to i\mathcal{J}_1'$, and

$$S_1'\equiv\exp(-\tfrac{1}{2}i\pi S_3')S_2'\exp(\tfrac{1}{2}i\pi S_3')\to i\mathcal{J}_2'.$$

In other words, $i\mathcal{J}_\perp'$ are two *Hermitian* operators which, with $\mathcal{J}_3'$, give rise to a unitary representation of $O(2,1)$. Therefore, $M\mathcal{J}_\perp'$ are also *Hermitian* operators satisfying the commutation relations

$$[M\mathcal{J}_+',M\mathcal{J}_-']=2M^2\mathcal{J}_3',$$

the same as in the case of timelike states.

It is obvious that the case of one timelike state and one spacelike state is dealt with in the same way, leading to the conclusion that in the general case, the angular condition is indeed given by Eqs. (1)–(3).

Apart from providing a very simple derivation of the angular conditions, our work shows precisely what conditions *are* obeyed by the explicit models containing spacelike states presented in the literature.[2]

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Schwinger Terms, Hard Pions, and the Structure of the Deuteron

H. J. WEBER*
Department of Physics, University of Virginia, Charlottesville, Virginia 22903
(Received 6 February 1970)

The *npd* vertex, Ward and Jacobi identities are used to evaluate the equal-time commutator (ETC) of the deuteron axial vector with the nucleon field. Schwinger terms are shown to be related to the deuteron wave function. They are important for nuclear short-range correlations. Ward identities yield the pole-term amplitudes for $pp\to d\pi^+$ and $pn\to d\gamma$ at threshold. Pion-induced nucleon-nucleon scattering is traced to the deuteron ETC with the weak axial-vector current. The Bjorken limit with respect to the deuteron yields the nonrelativistic limit of the reaction amplitudes.

RECENTLY, it was found[1] that *soft-pion techniques*[2] *fail* for the reaction $pp\to d\pi^+$ at threshold. The zero-mass pion result gives three times the experimental cross section and decreases by a factor of about 20 in going to the physical pion mass, at fixed pion momentum $\mathbf{k}\neq0$. The reaction mechanism of the inverse process, slow-pion absorption (π^-,NN) on nuclei—the pion in flight or in an atomic orbit—sheds some light on the rapid variation of the amplitude. At the physical pion mass, two nucleons are emitted predominantly back to back.[3] In fact, they can, more easily than a single nucleon, share the *high energy* M_π and *little momentum* $\mathbf{k}$ transferred to the nucleus by the pion and yet satisfy energy-momentum conservation. Hence the soft-pion limit is not applicable because

$$M_\pi^2\gg\mathbf{k}^2,\quad \mathbf{k}_N^2=2M_\pi M\gg\mathbf{p}_F^2\gg\mathbf{k}^2\tag{1}$$

is essential for suppressing single-nucleon emission. Here $\mathbf{p}_F$ denotes the Fermi momentum of the nucleus and $\mathbf{k}_N$ the single-nucleon momentum. This suggests using *hard-pion* techniques[4] for those slow-pion reactions in which the absorbed pion mass induces nuclear rearrangement or disintegration. In contrast, the absorbed pion of radiative pion capture on nuclei[5] may be treated as soft including the application of current algebra[6] because its mass is largely carried away by the

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