Interaction of a Gauge-Independent Spin-One Field*

ROGER PALMER AND YASUSHI TAKAHASHI
Theoretical Physics Institute and Department of Physics, University of Alberta, Edmonton, Alberta, Canada
(Received 14 May 1970)

In a recent paper we considered a massive spin-1 field that was gauge invariant, at least in the free-field case. We pointed out that when we have an interaction in order to maintain the gauge invariance we had to impose certain conditions on the source functions. The object of this paper is to construct the interaction Hamiltonian and to consider the consequences of these conditions.

I. INTRODUCTION

THE gauge-invariant spin-1 field can be described by a skew-symmetric second-rank tensor obeying the equations

$$\begin{aligned}(\Box-m^2)\psi_{\mu\nu}&=0\,,\\ \psi_{\mu\nu}+\psi_{\nu\mu}&=0\,,\\ \partial_\mu\psi_{\mu\nu}&=0\,.\end{aligned}\tag{1.1}$$

These equations can be combined into the single equation

$$\Lambda_{\sigma\rho\mu\nu}(\partial)\psi_{\mu\nu}=0\,,\tag{1.2}$$

where

$$\begin{aligned}\Lambda_{\sigma\rho\mu\nu}(\partial)=&\tfrac{1}{2}(\Box-m^2)(\delta_{\mu\sigma}\delta_{\nu\rho}-\delta_{\mu\rho}\delta_{\nu\sigma})\\ &-\tfrac{1}{2}(\delta_{\nu\rho}\partial_\mu\partial_\sigma-\delta_{\nu\sigma}\partial_\mu\partial_\rho-\delta_{\mu\rho}\partial_\nu\partial_\sigma+\delta_{\mu\sigma}\partial_\nu\partial_\rho)\\ &-\tfrac{1}{2}m^2(\delta_{\mu\sigma}\delta_{\nu\rho}+\delta_{\mu\rho}\delta_{\nu\sigma})\,.\end{aligned}\tag{1.3}$$

We have shown[1] that this formulation leads to a completely gauge-invariant free-field theory and that if we make the identification

$$\psi_{\mu\nu}=\tfrac{1}{2}i\epsilon_{\mu\nu\sigma\rho}\partial_\sigma V_\rho\,,\tag{1.4}$$

the theory is identical to the Proca field when the gauge is fixed.

If there is interaction present the situation becomes much more complicated. The propagator is[2,3]

$$\langle T^*(\psi_{\mu\nu}(x),\bar\psi_{\sigma\rho}(x'))\rangle_0=id_{\mu\nu\sigma\rho}(\partial)\Delta_C(x-x')\,,\tag{1.5}$$

where[4]

$$\bar\psi_{\sigma\rho}=g_{\sigma\sigma'}g_{\rho\rho'}\psi_{\sigma'\rho'}{}^\dagger\tag{1.6}$$

and

$$\begin{aligned}d_{\mu\nu\sigma\rho}(\partial)=&\tfrac{1}{2}(\delta_{\mu\sigma}\delta_{\nu\rho}-\delta_{\mu\rho}\delta_{\nu\sigma})\\ &-(1/2m^2)(\delta_{\nu\rho}\partial_\mu\partial_\sigma-\delta_{\nu\sigma}\partial_\mu\partial_\rho\\ &+\delta_{\mu\sigma}\partial_\nu\partial_\rho-\delta_{\mu\rho}\partial_\mu\partial_\sigma)\\ &-[(\Box-m^2)/2m^2](\delta_{\mu\sigma}\delta_{\nu\rho}+\delta_{\mu\rho}\delta_{\nu\sigma})\,.\end{aligned}\tag{1.7}$$

The Klein-Gordon divisor $d_{\mu\nu\sigma\rho}(\partial)$ can be expressed in terms of the Klein-Gordon divisor of the Proca field, $d_{\sigma\lambda}(\partial)$:

$$\begin{aligned}d_{\sigma\rho\lambda\tau}=&\tfrac{1}{2}[d_{\sigma\lambda}(\partial)d_{\rho\tau}(\partial)-d_{\sigma\tau}(\partial)d_{\rho\lambda}(\partial)]\\ &-[(\Box-m^2)/2m^2](\delta_{\sigma\lambda}\delta_{\rho\tau}+\delta_{\sigma\tau}\delta_{\rho\lambda})\,,\end{aligned}\tag{1.8}$$

where

$$d_{\sigma\lambda}(\partial)=\delta_{\sigma\lambda}-m^{-2}\partial_\sigma\partial_\lambda\,,\tag{1.9}$$

It is clear that, in addition to spin 1, both spin 0 and spin 2 will propagate through the last term in (1.8). It is easy to see that if the interaction is such that the equation of motion becomes

$$\Lambda_{\mu\nu\sigma\rho}(\partial)\psi_{\sigma\rho}(x)=J_{\mu\nu}(x)\,.\tag{1.10}$$

then to retain (1.4) we must insist that

$$\begin{aligned}J_{\mu\nu}(x)+J_{\nu\mu}(x)&=0\,,\\ \partial_\mu J_{\mu\nu}(x)&=0\,.\end{aligned}\tag{1.11}$$

This means that there exists some vector or pseudovector current or their linear combination J_ρ, such that

$$J_{\mu\nu}=\tfrac{1}{2}i\epsilon_{\mu\nu\sigma\rho}\partial_\sigma J_\rho\,,\tag{1.12}$$

where J_ρ is not necessarily conserved. Thus the gauge invariance leads to restrictions on the type of interaction Lagrangian possible.

In this paper we investigate some of the properties of this spin-1 meson theory. In Sec. II we calculate the interaction Hamiltonian for the interaction of the meson field with a baryon field and in Sec. III we illustrate certain properties of the S matrix.

II. DETERMINATION OF INTERACTION HAMILTONIAN

Consider the Lagrangian

$$\mathfrak{L}=\mathfrak{L}_0+\mathfrak{L}_D+\mathfrak{L}_{\rm int}\,,\tag{2.1}$$

where $\mathfrak{L}_0$ is the spin-1-meson free Lagrangian given in Ref. 1,

$$\mathfrak{L}_D=-\overline{\mathbf{B}}(x)(\gamma_\mu\partial_\mu+M)\mathbf{B}(x)$$

and

$$\mathfrak{L}_{\rm int}=G\epsilon_{\mu\nu\sigma\rho}[\partial_\sigma\psi_{\mu\nu}(x)][\overline{\mathbf{B}}(x)\gamma_\rho\mathbf{B}(x)]\,.\tag{2.2}$$

For simplicity we shall consider the real-field case, i.e., $\bar\psi_{\mu\nu}=\psi_{\mu\nu}$. Then this Lagrangian leads to the equation of motion

$$\Lambda_{\mu\nu\sigma\rho}(\partial)\psi_{\sigma\rho}(x)=\mathbf{J}_{\mu\nu}(x)\,,\tag{2.3}$$

* Work supported in part by the National Research Council of Canada.
[1] Y. Takahashi and R. Palmer, Phys. Rev. D 1, 2974 (1970).
[2] See, e.g., Y. Takahashi, *Introduction to Field Quantization* (Pergamon, Oxford, England, 1969).
[3] Throughout this paper we have adopted the notation of Ref. 2.
[4] $g_{\mu\nu}=1$ for $\mu=\nu=1, 2, 3$; $=-1$ for $\mu=\nu=4$; $=0$ otherwise.

with

$$\mathbf{J}_{\mu\nu}(x)=G\epsilon_{\mu\nu\sigma\rho}\partial_\sigma[\bar{\mathbf{B}}(x)\gamma_\rho\mathbf{B}(x)]$$
$$=G\partial_\sigma[\epsilon_{\mu\nu\sigma\rho}\bar{\mathbf{B}}(x)\gamma_\rho\mathbf{B}(x)] \tag{2.4}$$
$$\equiv G\partial_\sigma\mathbf{J}_{\mu\nu\sigma}(x). \tag{2.5}$$

This $\mathbf{J}_{\mu\nu}(x)$ satisfies the restriction (1.11) and also $\bar{\mathbf{J}}_{\mu\nu}(x)=\mathbf{J}_{\mu\nu}(x)$.

The baryon field $\mathbf{B}$ obeys the Dirac equation

$$-(\gamma_\mu\partial_\mu+M)\mathbf{B}(x)=\mathbf{J}(x), \tag{2.6}$$

where

$$\mathbf{J}(x)=-G\epsilon_{\mu\nu\sigma\rho}[\partial_\sigma\boldsymbol{\psi}_{\mu\nu}(x)]\gamma_\rho\mathbf{B}(x). \tag{2.7}$$

The interaction Hamiltonian can be calculated using the general method outlined in Ref. 2. The first stage of the calculation is to define the two auxiliary fields $\psi_{\mu\nu}(x/\sigma)$ and $B(x/\sigma)$ in terms of the Heisenberg field operators. The Heisenberg operators are defined by

$$\boldsymbol{\psi}_{\mu\nu}(x)=Z_3{}^{1/2}\psi_{\mu\nu}(x)+G\int_{-\infty}^{\infty}d^4x'$$
$$\times\partial_\sigma d_{\mu\nu\alpha\beta}(\partial)\Delta^{(\mathrm{ret})}(x-x')\mathbf{J}_{\alpha\beta\sigma}(x'), \tag{2.8}$$

$$\mathbf{B}(x)=Z_2{}^{1/2}B(x)-\int_{-\infty}^{\infty}d^4x'S^{(\mathrm{ret})}(x-x')\mathbf{J}(x'). \tag{2.9}$$

Then the auxiliary fields are defined by

$$\psi_{\mu\nu}(x,\sigma)=Z_3{}^{1/2}\psi_{\mu\nu}(x)+G\int_{-\infty}^{\sigma}d^4x'\partial_\sigma d_{\mu\nu\omega\rho}(\partial)\mathbf{J}_{\omega\rho\sigma}(x'),$$
$$B(x,\sigma)=Z_2{}^{1/2}B(x)-\int_{-\infty}^{\sigma}d^4xS(x-x')\mathbf{J}(x'). \tag{2.10}$$

From Eqs. (2.8)–(2.10) we see that $B(x/\sigma)=\mathbf{B}(x)$ and

$$\psi_{\mu\nu}(x/\sigma)=\boldsymbol{\psi}_{\mu\nu}(x)+G\int_{-\infty}^{\infty}d^4x'$$
$$\times[\theta(x_0-x_0'),\partial_\alpha d_{\mu\nu\sigma\rho}(\partial)]\Delta(x-x')\mathbf{J}_{\sigma\rho\alpha}. \tag{2.11}$$

We now make use of Eq. (1.7) and find that the second term vanishes, i.e.,

$$\psi_{\mu\nu}(x/\sigma)=\boldsymbol{\psi}_{\mu\nu}(x). \tag{2.12}$$

Also,

$$\partial_\tau\psi_{\mu\nu}(x/\sigma)=\partial_\tau\boldsymbol{\psi}_{\mu\nu}(x)+G\int_{-\infty}^{\infty}d^4x'$$
$$\times[\theta(x_0-x_0'),\ \partial_\tau\partial_\alpha d_{\mu\nu\sigma\rho}(\partial)]\Delta(x-x')\mathbf{J}_{\sigma\rho\alpha}(x'). \tag{2.13}$$

After a lengthy calculation, we find that in this case not all the normal dependent terms disappear. We are left with the expression

$$\partial_\tau\psi_{\mu\nu}(x/\sigma)=\partial_\tau\boldsymbol{\psi}_{\mu\nu}(x)+Gn_\tau n_\alpha\mathbf{J}_{\mu\nu\alpha}(x), \tag{2.14}$$

where n_μ is a unit timelike vector, i.e.,

$$n_\mu n_\mu=-1. \tag{2.15}$$

Using Eqs. (2.14) and (2.11), we can express the currents in terms of the auxiliary fields:

$$J(x/\sigma)=-G\epsilon_{\mu\nu\tau\rho}(\partial_\tau\psi_{\mu\nu}(x/\sigma))\gamma_\rho B(x/\sigma)$$
$$+G^2\epsilon_{\mu\nu\tau\rho}\epsilon_{\mu\nu\alpha\sigma}n_\tau n_\alpha\bar{B}(x/\sigma)$$
$$\times\gamma_\sigma B(x/\sigma)\gamma_\rho B(x/\sigma), \tag{2.16}$$

$$J_{\mu\nu}(x/\sigma)=G\partial_\sigma\epsilon_{\mu\nu\sigma\rho}\bar{B}(x/\sigma)\gamma_\rho B(x/\sigma). \tag{2.17}$$

We now calculate the interaction Hamiltonian using the equations

$$[B(x/\sigma),\mathcal{H}_{\mathrm{int}}(x'/\sigma,n)]=id(\partial)\Delta(x-x')J(x'/\sigma),$$
$$[\psi_{\mu\nu}(x/\sigma),\mathcal{H}_{\mathrm{int}}(x'/\sigma,n)]$$
$$=id_{\mu\nu\sigma\rho}(\partial)\Delta(x-x')J_{\sigma\rho}(x'/\sigma). \tag{2.18}$$

Thus,

$$\mathcal{H}_{\mathrm{int}}(x/\sigma,n)=-G\epsilon_{\mu\nu\tau\rho}[\partial_\tau\psi_{\mu\nu}(x/\sigma)]\bar{B}(x/\sigma)\gamma_\rho B(x/\sigma)$$
$$+\tfrac{1}{2}G^2n_\tau n_\alpha J_{\mu\nu\alpha}(x/\sigma)J_{\mu\nu\tau}(x/\sigma). \tag{2.19}$$

The general theory states that the interaction Hamiltonian $\mathcal{H}_{\mathrm{int}}(x,n)$ can be obtained by simply replacing the auxiliary fields in (2.19) by the original field operators. Making this substitution and setting

$$-i\bar{B}\gamma_\rho B=J_\rho, \tag{2.20}$$

we finally obtain

$$\mathcal{H}_{\mathrm{int}}(x,n)=-\mathcal{L}_{\mathrm{int}}(x)+G^2[J_\rho J_\rho+(nJ)^2]. \tag{2.21}$$

We have also performed the calculation of the interaction Hamiltonian for other interaction Lagrangians which lead to source functions consistent with the restrictions (1.11).[5] In particular, we have performed the calculation including the isospin index explicitly. This calculation proceeds exactly as above and we obtained an analogous result.

III. S MATRIX

To second order in G, the S matrix for the interaction described by the Lagrangian (2.1) is

$$S=1+i\int d^4x'\!:\mathcal{L}_{\mathrm{int}}(x'):$$
$$-i\int d^4x'(\tfrac{1}{2}G^2)n_\tau n_\alpha J_{\mu\nu\alpha}(x')J_{\mu\nu\tau}(x')$$
$$+\frac{(-i)^2}{2!}\int d^4x'\int d^4x''T(\mathcal{L}_{\mathrm{int}}(x')\mathcal{L}_{\mathrm{int}}(x'')). \tag{3.1}$$

[5] For example, one can include a term $\lambda\bar{\psi}_{\mu\nu}\psi_{\mu\nu}$, so long as $\lambda\neq m$.

The fourth term in this expression yields

$$\frac{(-i)^2}{2!}\int d^4x'\int d^4x''T^*(\mathcal{L}_{\text{int}}(x')\mathcal{L}_{\text{int}}(x''))$$

$$+\frac{(-i)^2}{2!}\int d^4x'\int d^4x''iG^2\epsilon_{\mu\nu\sigma\rho}\epsilon_{\mu\nu\sigma'\rho'}n_{\sigma'}n_\sigma\delta^4(x-x')$$

$$\times T^*(J_\rho(x')J_{\rho'}(x''))$$

$$=\frac{(-i)^2}{2!}\int d^4x'\int d^4x''T^*(\mathcal{L}_{\text{int}}(x'),\,\mathcal{L}_{\text{int}}(x''))$$

$$+i\int d^4x'(\tfrac{1}{2}G^2)n_\sigma n_{\sigma'}J_{\mu\nu\sigma}(x')J_{\mu\nu\sigma'}(x')\,. \quad (3.2)$$

So,

$$S=1+i\int d^4x':\mathcal{L}_{\text{int}}(x'):+\frac{(-i)^2}{2!}\int d^4x'\int d^4x''$$

$$\times T^*(\mathcal{L}_{\text{int}}(x'),\,\mathcal{L}_{\text{int}}(x''))+\cdots. \quad (3.3)$$

We can see that in an S-matrix calculation, at least to second order, we can replace $\mathcal{H}_{\text{int}}$ by $(\mathcal{H}_{\text{int}})_{\text{eff}}=-\mathcal{L}_{\text{int}}$, so long as we also replace the T product by the T^* product. This has been called Matthew's rule. The same result also holds for the other interactions considered in Sec. II.

In a recent paper by Kyriakopoulos,[6] an interaction Hamiltonian was calculated for a similar spin-1 meson theory. He had to impose the symmetry of his tensor field as a separate condition, however, and so was unable to use the general methods we have employed. In addition, in S-matrix calculations his $(\mathcal{H}_{\text{int}})_{\text{eff}}$ does not reduce to $-\mathcal{L}_{\text{int}}$. No attempt was made in Kyriakopoulos's paper to preserve gauge invariance and so he also considered an interaction term $\bar{B}\sigma_{\mu\nu}T_{\mu\nu}B$, where $T_{\mu\nu}$ is the antisymmetric tensor field. This type of term is, of course, not permitted by the restrictions (1.11).

An interesting result holds for internal meson lines appearing in the S matrix. For an illustration, let us consider baryon scattering with one-boson exchange:

$$S^{(2)}=+G^2\int\int d^4x'd^4x''\partial_\sigma'J_\rho(x')$$

$$\times\langle T^*(\epsilon_{\sigma\rho\mu\nu}\bar{\psi}_{\mu\nu}(x'),\,\epsilon_{\lambda\tau\xi\eta}\psi_{\lambda\tau}(x''))\rangle\partial_\xi''J_\eta(x'')\,. \quad (3.4)$$

The contraction term is

$$\begin{aligned}\langle T^*(\epsilon_{\sigma\rho\mu\nu}\epsilon_{\lambda\tau\xi\eta}\bar{\psi}_{\mu\nu}(x')\psi_{\lambda\tau}(x''))\rangle &\\ =\epsilon_{\sigma\rho\mu\nu}\epsilon_{\lambda\tau\xi\eta}\langle T^*(\bar{\psi}_{\mu\nu}(x'),\psi_{\lambda\tau}(x''))\rangle_0 &\\ =\epsilon_{\sigma\rho\mu\nu}\epsilon_{\lambda\tau\xi\eta}[id_{\mu\nu\lambda\tau}(\partial')\Delta_C(x'-x'')]. &\quad (3.5)\end{aligned}$$

[6] E. Kyriakopoulos, Phys. Rev. **183**, 1318 (1969).

Therefore,

$$S^{(2)}=G^2\int\int d^4xd^4x'$$

$$\times\partial_\sigma J_\rho\epsilon_{\sigma\rho\mu\nu}[id_{\mu\nu\lambda\tau}(\partial)\Delta_C(x-x')]$$

$$\times\epsilon_{\lambda\tau\xi\eta}\partial_\xi'J_\eta(x') \quad (3.6)$$

$$\begin{aligned}=G^2\int\int d^4xd^4x'&\\ \times 2i[\partial_\xi J_\eta(x)-\partial_\eta J_\xi(x)]\Delta_C(x-x')\partial_\xi'J_\eta(x')&\\ +[(2i)/m^2]\{\partial_\sigma J_\xi(x)[\partial_\eta\partial_\sigma\Delta_C(x-x')]\partial_\xi'J_\eta(x')&\\ -\partial_\sigma J_\eta(x)[\partial_\xi\partial_\sigma\Delta_C(x-x')]\partial_\xi'J_\eta(x')&\\ +\partial_\xi J_\rho(x)[\partial_\eta\partial_\rho\Delta_C(x-x')]\partial_\xi'J_\eta(x')&\\ -\partial_\eta J_\rho(x)[\partial_\xi\partial_\rho\Delta_C(x-x')]\partial_\xi'J_\eta(x')&\\ +[\partial_\xi J_\eta(x)-\partial_\eta J_\xi(x)][\Box\Delta_C(x-x')]\partial_\xi'J_\eta(x')\}\,.&\quad (3.7)\end{aligned}$$

Integrating by parts, we can cast (3.7) into the form

$$S^{(2)}=G^2\int d^4x\int d^4x'$$

$$\times(-2i)(\Box\delta_{\mu\nu}-\partial_\mu\partial_\nu)\Delta_C(x-x')J_\mu(x)J_\nu(x')\,. \quad (3.8)$$

We observe that the characteristic factor $(\Box\delta_{\mu\nu}-\partial_\mu\partial_\nu)$ of a gauge-invariant theory appears in the S matrix and we further notice that all the terms involving $1/m^2$ as a coefficient have disappeared so that we can take the $m\to 0$ limit for such processes.

Since we have established the existence of the vector field V_ρ, it is interesting to see how the various relations can be expressed in terms of V_ρ. For example, the commutator

$$[\psi_{\mu\nu}(x),\bar{\psi}_{\mu'\nu'}(x')]=id_{\mu\nu\mu'\nu'}\Delta(x-x') \quad (3.9)$$

becomes

$$[V_\rho(x),\bar{V}_\sigma(x')]$$

$$=(2i/m^2)\delta_{\rho\sigma}\Delta(x-x')+\partial_\rho\partial_\sigma f(x-x')\,. \quad (3.10)$$

We have also found that the energy-momentum operator P_μ, defined by[4]

$$P_\tau=-\frac{1}{2}\int d\sigma_\lambda\bar{\psi}_{\mu\nu}(\partial_\tau-\overleftarrow{\partial}_\tau)\Gamma_{\lambda,\mu\nu\sigma\rho}\psi_{\sigma\rho}\,, \quad (3.11)$$

becomes

$$P_\tau=-\frac{1}{8}\int d\sigma_\lambda[\bar{W}_{\beta\lambda}(\partial_\tau-\overleftarrow{\partial}_\tau)(\partial_\alpha-\overleftarrow{\partial}_\alpha)W_{\alpha\beta}$$

$$+\bar{W}_{\alpha\beta}(\partial_\tau-\overleftarrow{\partial}_\tau)(\partial_\alpha-\overleftarrow{\partial}_\alpha)W_{\beta\lambda}]\,, \quad (3.12)$$

where

$$W_{\alpha\beta}=(\partial_\alpha V_\beta-\partial_\beta V_\alpha). \tag{3.13}$$

It can easily be shown that P_τ is conserved and, as we have shown in Ref. 1, P_0 is positive definite. Expressed in this form, P_τ is manifestly invariant under the gauge transformation

$$V_\rho \rightarrow V_\rho+\partial_\rho\Lambda. \tag{3.14}$$

IV. CONCLUSION

The stimulus for starting this investigation was a paper by Kyriakopoulos[5] in which a similar model was considered but in which the antisymmetry had to be imposed separately. This led to several difficulties in applying the general methods that we have used. In addition our approach has led to a completely gauge-invariant formulation in the free-field case and also in the interaction case if we impose the conditions (1.11) on the source functions. These conditions limit the possible terms in the interaction Lagrangian. For example, a term $g\bar{B}\sigma_{\mu\nu}\psi_{\mu\nu}B$ is not possible for the interaction of a baryon and meson field. We have shown that the terms that are permitted in the interaction Lagrangian lead to effective interaction Hamiltonians for S-matrix calculations that are just equal to $-\mathcal{L}_{\rm int}$, thus obeying Matthew's rule.

It is also interesting to note that for S-matrix terms containing only internal meson lines the $m\rightarrow 0$ limit exists.

PHYSICAL REVIEW D VOLUME 2, NUMBER 12 15 DECEMBER 1970

Erratum

Inclusion of Toller-Angle Dependence in the Multi-Regge Integral Equation, DENNIS SILVERMAN AND CHUNG-I TAN [Phys. Rev. D **1**, 3479 (1970)]. Equations (6) and (8) should contain factors like $[f(t_\pm',\omega_\pm',t_\pm'')/\mu^2]^{\alpha(t_\pm')}$. The denominator μ^2 was missing in the published text.

We would like to stress that the derivation of our integral equation, Eq. (9), involves only *dynamical* assumptions, i.e., the parametrization of production amplitudes. No kinematic approximations to the phase space have been made. Equation (20) then follows as a consequence of multiperipheralism. These results do not correspond to the *weak coupling* limit, contrary to claims made recently in literature.

We would also like to point out that we have been informed recently of the work by E. Byckling and K. Kajantie, Phys. Rev. **187**, 2008 (1969), which covered related topics, but with different emphasis.