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On Summing Soft Pions*

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A simple derivation of Weinberg's result on the chiral-symmetric coupling of soft pions to a high-energy process is presented. His formal for soft-virtual-pion corrections is shown to be not invariant under canonical transformation of the pion field. Thus this formula is model dependent and not a consequence of chiral symmetry.

THE production of soft pions during a hadronic collision is not a dominant effect in contrast to the analogous production of soft photons: The amplitude for soft-photon production diverges as ω^{-1} as the photon energy ω vanishes while the pion amplitude approaches a constant in the low-energy limit. Nonetheless, the problem of expressing the chiral-symmetric amplitude for the emission of an arbitrary number of soft pions in closed form is an interesting one. Weinberg[1] has recently discussed this problem in some detail. We give here a very simple derivation of his result for the coupling of soft pions to a high-energy collision process. Weinberg uses this result to obtain a formula for the correction to a high-energy process ensuing from virtual soft-pion exchange. We show that this formula, derived in a modified tree-graph approximation, is not invariant under canonical transformations of the pion field—point transformations that redefine the pion field. Thus its validity depends upon a particular model of pion interaction and is not a general consequence of chiral symmetry. It does not, therefore, have any particular physical significance.

Since soft-pion emission occurs during the external propagation of the hadrons involved in a process governed by an amplitude $T_0(\{p\})$, it is convenient to introduce a multicomponent field Ψ to describe the hadrons and their propagation. We attach the appropriate field components to the external legs of the amplitude T_0, and write an effective Lagrangian

$$\mathcal{L}_H=\mathcal{L}_{\text{free}}(\Psi)+\mathcal{L}_{\text{int}}(T_0,\Psi(x)), \tag{1}$$

which reproduces the basic collision process in first Born approximation. Our task is now to couple soft pions in this effective Lagrangian in a chiral-symmetric manner. We shall assume isotopic spin invariance so that the effective Lagrangian is unchanged by the field transformation

$$\Psi\rightarrow e^{i\omega\cdot\mathbf{T}}\Psi. \tag{2}$$

We shall also require, as does Weinberg, that the components of Ψ that are needed for a particular process are part of a chiral multiplet such that a chiral transformation is represented by the field:

$$\Psi\rightarrow\Psi'=e^{i\nu\cdot\mathbf{X}}\Psi. \tag{3}$$

If, for example, only nucleons appear, then $\mathbf{X}=i\gamma_5\frac{1}{2}\boldsymbol{\tau}$, but the occurrence of a ρ requires an accompanying A_1 to fill out a chiral multiplet. Finally, we consider, as does Weinberg, only high-energy processes so that we can neglect all masses[2] and have chiral invariance of the free Lagrangian,

$$\mathcal{L}_{\text{free}}(e^{i\nu\cdot\mathbf{X}}\Psi)=\mathcal{L}_{\text{free}}(\Psi). \tag{4}$$

The chiral-symmetric coupling of the pion field can now be obtained immediately. We choose the pion field $\boldsymbol{\phi}$ to obey a nonlinear realization of the chiral group such that[3] under a chiral transformation this field undergoes a change $\boldsymbol{\phi}\rightarrow\boldsymbol{\phi}'$ defined by

$$\exp(iF_\pi^{-1}\boldsymbol{\phi}'\cdot\mathbf{X})e^{i\nu\cdot\mathbf{X}}=e^{i\Omega\cdot\mathbf{T}}\exp(iF_\pi^{-1}\boldsymbol{\phi}\cdot\mathbf{X}), \tag{5}$$

in which Ω is an appropriate function of $\boldsymbol{\phi}$ and ν. The chiral-invariant extension of the effective Lagrangian may be expressed as

$$\mathcal{L}=\mathcal{L}_{\text{free}}(\Psi)+\mathcal{L}_{\text{int}}(T_0,\exp(iF_\pi^{-1}\boldsymbol{\phi}\cdot\mathbf{X})\Psi)+\mathcal{L}_\pi, \tag{6}$$

since under a chiral transformation the field

$$\hat{\Psi}=\exp(iF_\pi^{-1}\boldsymbol{\phi}\cdot\mathbf{X})\Psi \tag{7}$$

undergoes a local isotopic spin rotation

$$\begin{aligned}\hat{\Psi}'&=\exp(iF_\pi^{-1}\boldsymbol{\phi}'\cdot\mathbf{X})\Psi',\\&=e^{i\Omega\cdot\mathbf{T}}\hat{\Psi},\end{aligned} \tag{8}$$

and $\mathcal{L}_{\text{int}}$ is invariant under such rotations[3a]. The chiral-invariant pion Lagrangian may be written in the form[4]

$$\mathcal{L}_\pi=-\tfrac{1}{4}F_\pi^2\,\mathrm{tr}(\partial_\mu\exp(-iF_\pi^{-1}\boldsymbol{\tau}\cdot\boldsymbol{\phi}))\times(\partial^\mu\exp(iF_\pi^{-1}\boldsymbol{\tau}\cdot\boldsymbol{\phi})). \tag{9}$$

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[1] S. Weinberg, Phys. Rev. D 2, 674 (1970).

[2] The problem becomes very much more difficult if the masses are not neglected because they break the algebraic chiral symmetry and thus give rise to additional soft pion emission. This is evident from the structure of the result (10) which relates the coupling of the pion field to the extent to which T_0 does not commute with X.

[3] S. Coleman, J. W. Wess, and B. Zumino, Phys. Rev. **177**, 2239 (1968). With our normalization, $F_\pi\simeq 94$ MeV.

[3a] It should be remarked that this invariance requires that $\mathcal{L}_{\text{int}}$ contain no derivatives, for the isotopic rotation W is a function of $\boldsymbol{\phi}(x)$. The particle momenta just before and after the basic collision process should parametrize T_0. Derivatives occur in $\mathcal{L}_{\text{int}}$ if proper account is taken of the change in these momenta brought about by particle recoil during the external emission of soft pions. However, any derivative can be neglected in the soft-pion limit since it gives terms proportional to the pion momenta. The situation here is analogous to that discussed in Sec. 6 of L. S. Brown and R. L. Goble, Phys. Rev. **173**, 1505 (1968).

[4] P. Chang and F. Gürsey, Phys. Rev. **164**, 1752 (1967).

The pion coupling to the basic collision amplitude is obtained from the first Born approximation to the effective interaction Lagrangian. Putting aside inessential factors such as spinor wave functions, the field operators are replaced by $e^{\pm ipx}$, and we obtain

$$S=i\int(dx)e^{-iqx}\prod_f \exp[-iF_\pi^{-1}\boldsymbol{\phi}(x)\cdot\mathbf{X}^{(f)}]T_0 \times\prod_i \exp[iF_\pi^{-1}\boldsymbol{\phi}(x)\cdot\mathbf{X}^{(i)}], \quad (10)$$

where q is the momentum transfer of the basic process, and the two products refer to initial and final particles. This is precisely Weinberg's result (3.29).

Weinberg's derivation of this result is rather lengthy. This is because, in essence, he employs the field $\hat{\Psi}$ defined by Eq. (7) rather than the field Ψ. Either field may be used since the two differ by a canonical transformation. However, since $\hat{\Psi}(x)$ undergoes a local isospin rotation involving $\boldsymbol{\phi}(x)$ when a chiral transformation is performed, such a transformation does not leave $\mathfrak{L}_{\text{free}}$ invariant as it contains derivatives. Thus, although the use of $\hat{\Psi}$ makes $\mathfrak{L}_{\text{int}}$ invariant, complicated terms that involve $\partial_\mu\boldsymbol{\phi}$ as well as $\boldsymbol{\phi}$ must be added to $\mathfrak{L}_{\text{free}}$ to obtain invariance. Carrying out the canonical transformation to the field Ψ, as we have done, exhibits the result immediately.

The correction to the basic collision process brought about by the exchange of virtual soft pions is obtained by taking the vacuum expectation value of the result (10). We exploit the translation invariance of the vacuum and adopt a schematic notation to write this expectation value as

$$S=\langle\exp[iF_\pi^{-1}\boldsymbol{\phi}(0)\cdot\Delta\mathbf{X}]\rangle S_0. \quad (11)$$

Weinberg computes the expectation value in the tree-graph approximation. It follows directly from the work of Boulware and Brown[5] that this entails the evaluation of

$$\exp\left[F_\pi^{-1}\Delta\mathbf{X}\cdot\frac{\delta}{\delta\boldsymbol{\zeta}(0)}\right]\exp\left(\frac{i}{\hbar}W_{\text{cl}}[\boldsymbol{\zeta}]\right)\bigg|_{\zeta=0}, \quad (12)$$

where $W_{\text{cl}}[\boldsymbol{\zeta}]$ is the classical action for the meson Lagrangian (9) with the addition of an external source $\boldsymbol{\zeta}(x)$.

The expectation value (11) computed without approximation is invariant under a canonical transformation $\boldsymbol{\phi}\to\boldsymbol{\Phi}$ of the form

$$\boldsymbol{\phi}=\mathbf{f}(\boldsymbol{\Phi}) \quad (13)$$

when proper account is taken of the alteration of the meson Lagrangian. However, an invariance property of the full theory may not hold in an approximation, and indeed Weinberg's evaluation of the correction (11) is not invariant under the canonical transformation[6] (13). It depends upon a particular model of pion interaction, and does not follow from chiral symmetry without additional assumptions.

It is best to verify our assertion with a simple example: We consider the correction to a scalar vertex $\bar{\psi}\psi$ which entails the evaluation of the vacuum value of

$$U(\boldsymbol{\phi})=\exp(F_\pi^{-1}\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\phi}). \quad (14)$$

Now different parametrizations of this unitary matrix provide canonical transformations of the type that we are considering.[7] As a particular case, we consider

$$U(\boldsymbol{\phi})=\bar{U}(\boldsymbol{\Phi})=F_\pi^{-1}\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\Phi}+\sigma(\boldsymbol{\Phi}), \quad (15)$$

with

$$\sigma(\boldsymbol{\Phi})=(1-F_\pi^{-2}\boldsymbol{\Phi}^2)^{1/2}. \quad (16)$$

The new pion Lagrangian can be written as

$$\bar{\mathfrak{L}}_\pi=-\tfrac{1}{2}[(\partial_\mu\boldsymbol{\Phi})^2+(\partial_\mu\sigma)^2]. \quad (17)$$

Invariance under this canonical change requires that

$$\langle U(\boldsymbol{\phi})\rangle=\langle\bar{U}(\boldsymbol{\Phi})\rangle_{\text{new}}, \quad (18)$$

in which the second expectation value is computed with the new Lagrangian $\bar{\mathfrak{L}}_\pi$. It is sufficient to examine the first few orders of perturbation theory in F_π^{-1} to exhibit the lack of invariance. If we neglect terms of order F_π^{-6} and higher, then there are no effects of meson interaction in the tree approximation that Weinberg employs. Hence, to this order the tree approximation is identical with a free field theory, and we can readily compute

$$\langle U(\boldsymbol{\phi})\rangle^{\text{tree}}\cong\langle 1-\tfrac{1}{2}F_\pi^{-2}\boldsymbol{\phi}^2+(1/4!)F_\pi^{-4}(\boldsymbol{\phi}^2)^2\rangle_0 \cong 1+\tfrac{3}{2}F_\pi^{-2}i\Delta(0)+\tfrac{5}{8}F_\pi^{-4}[i\Delta(0)]^2, \quad (19a)$$

while

$$\langle\bar{U}(\boldsymbol{\Phi})\rangle_{\text{new}}{}^{\text{tree}}\cong\langle 1-\tfrac{1}{2}F_\pi^{-2}\boldsymbol{\phi}^2-\tfrac{1}{8}F_\pi^{-4}(\boldsymbol{\phi}^2)^2\rangle_0 \cong 1+\tfrac{3}{2}F_\pi^{-2}i\Delta(0)-(15/8)F_\pi^{-4}[i\Delta(0)]^2. \quad (19b)$$

Here $\Delta(x)$ is the free, zero-mass propagator,

$$\langle iT(\phi_a(x)\phi_b(x'))\rangle_0=\delta_{ab}\Delta(x-x'). \quad (20)$$

Since the numerical coefficients of the terms of order F_π^{-4} differ, the invariance requirement (18) is violated by the tree-graph approximation.

The invariance of the full theory is recovered if account is taken of terms that are not of a tree-graph character. In particular, it is not difficult to compute the first correction to the two-point propagators

$$\langle iT(\phi_a(x)\phi_b(x'))\rangle=\delta_{ab}\Delta(x-x')[1-\tfrac{2}{3}F_\pi^{-2}i\Delta(0)] \quad (21a)$$

[5] D. G. Boulware and L. S. Brown, Phys. Rev. **172**, 1628 (1968). The original remark that the tree-graph approximation is related to a formal semiclassical approximation is due to Y. Nambu, Phys. Letters **26B**, 626 (1968).

[6] The set of connected tree graphs with a fixed number of external lines is invariant under a redefinition of the pion field. This follows from the invariances of the full theory and the fact that such connected tree graphs can be uniquely characterized as being of some definite order in Planck's constant $\hbar$. However, when all the tree graphs are tied onto the basic collision process as Weinberg has done, terms of different order in $\hbar$ are combined, this unique characterization is lost, and there is no longer any reason for the approximation to be invariant under a pion field redefinition.

[7] See, for example, L. S. Brown, Phys. Rev. **163**, 1802 (1967).

and

$$\langle iT(\Phi_a(x)\Phi_b(x'))\rangle_{\text{new}}=\delta_{ab}\Delta(x-x')[1+F_\pi^{-2}i\Delta(0)]. \quad (21b)$$

This alteration brings Eqs. (19a) and (19b) into agreement and verifies the invariance to order F_π^{-4}.

This work was begun at the Stanford Linear Accelerator Center and completed at the Aspen Center for Physics. I have enjoyed conversations with Y. P. Yao, S. Weinberg, and D. G. Boulware. I have had particularly fruitful discussions with S. Gasiorowicz.

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Reply to "On Summing Soft Pions"*

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In his paper Brown[1] is correct in pointing out that my use[2] of the tree approximation in evaluating the effects of soft virtual pions introduces a dependence on the definition of the pion field. My statement to the contrary was an error, arising from a misuse of the theorem that the S matrix in the tree approximation is independent of the definition of the fields.

This error does not in my view affect the physical significance of the results. As emphasized in my original paper, there is no known current-algebra justification for using a chiral-invariant Lagrangian to deal with *virtual* pions. These calculations were offered only as a mathematical model, interesting chiefly because a large number of complicated Feynman diagrams could be explicitly summed. What Brown has shown is that this model is somewhat more arbitrary than I had thought.

However, Brown's comments raise some interesting questions. The S matrix for multi-soft-pion processes (like pion-pion scattering) can be calculated with confidence by using a chiral-invariant Lagrangian in the tree approximation, because current-algebra tells us that these S-matrix elements are uniquely determined by the chiral invariance of the Lagrangian, and determined, moreover, to be the lowest order in the "coupling constant" $1/F_\pi$. Is it possible that current algebra could also be used to validate the tree approximation for Feynman amplitudes involving soft pions off the mass shell? It now appears[3] that this may be possible, but only for one particular definition of the pion field, namely, that used in the "nonlinear σ model." If this is so, then it would be natural to calculate the effects of soft virtual pions by using the tree approximation, but only for the nonlinear σ-model definition of the pion field, rather than the Callan, Coleman, Wess, and Zumino[4] (CCWZ) definition. A nonlinear differential equation would then need to be solved in order to take account of the pion-pion interactions, but pionic loops would affect the final result only through a renormalization of F_π. Detailed calculations will have to be performed to see whether these ideas will prove useful.

There is just one point where my paper went beyond mathematical models to make a prediction about the real world. I noted that in the eikonal approximation, the matrix element for the emission of any number of soft real pions in a given process would vanish if the S matrix for that process commutes with the chiral matrix $\mathbf{X}$. The emission of pions by the terms in the S matrix which do not commute with $\mathbf{X}$ might be expected, via unitarity, to suppress those terms, so that the S matrix for any hard-hadron process would approximately commute with $\mathbf{X}$. It is important to note that this argument dealt with *real* soft pions, and is therefore unaffected by Brown's comments. In addition, the model for soft-virtual-pion effects worked out in my paper shows, *even without using the tree approximation*, that soft virtual pions have no effect on the terms in the hard-hadron S matrix which commute with $\mathbf{X}$. Any suppression of the S matrix by virtual-pion exchange can affect only those terms of the S matrix which do not commute with $\mathbf{X}$, and would thus yield an approximately chiral-invariant S matrix. Only the explicit formulas for the suppression factors given in my paper do depend on the use of the tree approximation with the CCWZ pion field.

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[1] Lowell S. Brown, preceding paper, Phys. Rev. D 2, 3083 (1970).

[2] S. Weinberg, Phys. Rev. D 2, 674 (1970).

[3] I. Gerstein, R. Jackiw, B. W. Lee, and S. Weinberg (unpublished).

[4] C. G. Callan, S. Coleman, J. W. Wess, and B. Zumino, Phys. Rev. 177, 2247 (1968).