Box Diagram for the Interaction of Pseudoscalar Bosons with Spin-$\frac{1}{2}$ Fermions*

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We find explicit expressions for the invariant-amplitude contributions of the box diagrams in the interaction of pseudoscalar bosons with spin-$\frac{1}{2}$ fermions. The spin complications are reduced using unitarity such that we can express the A and B amplitudes as single-variable integrals over closed-form analytic functions. The problem is solved for the elastic-process amplitudes and is useful in calculations including fourth-order terms. Previous calculations used mainly numerical methods; we rely on exact partial-fraction expansions to obtain amplitudes from a few key analytic functions. The amplitudes are calculated independently for both the s and t channels and are tested by three different consistency checks. Various useful dispersion-integral forms are given, with emphasis on those suitable for partial-wave expansions. The closed forms obtained also yield "natural" subtraction points for the s-variable integrals. In writing the crossed box amplitude for convenient partial-wave expansion, we find a relation free from any formal dependence on a subtraction. This is due to an improved convergence of the absorptive parts in the new functional dependence which arises. The partial-wave contributions are also given explicitly.

I. INTRODUCTION

THE aim of this paper is to find explicit expressions for the invariant-amplitude contributions of the box diagrams (direct and crossed) in the conventional interaction of pseudoscalar bosons with spin-$\frac{1}{2}$ fermions. This is a technical problem solved in principle since the classical papers of Mandelstam[1] and Cutkosky[2] have yielded simple recipes for finding the absorptive parts and double-spectral functions of box diagrams. Nevertheless, when spin is involved as in $\pi N \rightarrow \pi N$ scattering, the constant numerator of the spinless case[1] is replaced by a complicated angular dependence folded in with spin matrices. Expressing the aforementioned contributions as single-variable integrals over closed-form analytic functions is not so easy and, to our knowledge, has not been published to date. These contributions are of particular interest in calculations such as those using Padé approximants which rely on the lowest-order terms of the perturbation expansions.[3]

The problem has been treated recently by Kubis and Gammel,[4] who have reduced the absorptive parts to computerizable integrals over complicated expressions. These absorptive parts will be given here by simple analytic functions similar to the ones arising in Mandelstam's spinless case.[1] They will be written for the simplest case, which is the one most commonly of interest, i.e., pseudoscalar bosons of identical mass μ and spin-$\frac{1}{2}$ fermions of equal mass m. The method will be seen readily applicable to more diversified mass cases and certain other spin cases.

We take it for granted that the amplitudes can be expressed correctly by dispersion integrals.[5] Inspired by Grisaru,[6] we use unitarity to compute absorptive parts, finding this more transparent and less prone to errors of sign than the Cutkosky δ-function technique.[7] The method used here relies chiefly on partial-fraction expansions and manipulations of four-momentum conservation to bring about easy angular integrations. In Sec. II, the absorptive parts are thus calculated for both the s and t channels and are checked for correctness by three methods: first, the obvious one of a consistent value for the double-spectral function from the s and t channels, a second one involving the cancellation of false singularities appearing because of the spin, and finally the correct asymptotic behavior in t and s.

In Sec. III we will cast the amplitudes in various useful dispersion relation forms insisting on those suitable for partial-wave expansions. We verify the need for a subtraction for dispersion integrals in s, although no arbitrariness remains when the t-channel information is exploited.[1] In this respect the closed forms obtained for the absorptive parts will yield natural subtraction points. An interesting result appears when the crossed box amplitude is written in the form most convenient for partial-wave expansion: The subtraction needed originally is shown to be implied only formally. This is due to the improved convergence of the absorptive parts in their new functional dependence in the integration variable occurring after denominator manipulation and passage from double to single dispersion integrals. The dispersion integrals can then be written formally as if no subtraction had existed. The partial-wave contributions of the direct and crossed box diagrams to the A and B amplitudes are also written explicitly.

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II. ABSORPTIVE PARTS

The matrix element is written as

$$S_{fi}=\delta_{fi}+i(2\pi)^4\delta^4(P_f-P_i)\left[\frac{M^2}{(2\pi)^{12}4p_{10}p_{20}k_{10}k_{20}}\right]^{1/2}\times\bar{\psi}_f(p_2)T\psi_i(p_1)\,, \quad (2.1)$$

where

$$T=\mathbf{A}(s,t)+\tfrac{1}{2}(\not{k}_1+\not{k}_2)\mathbf{B}(s,t)\,; \quad (2.2)$$

p_1 and p_2 are the four-momenta of the incident and outgoing fermions of mass M, respectively, while k_1 and k_2 are the four-momenta of the incident and outgoing pseudoscalar bosons of mass μ, respectively. As usual,

$$\begin{aligned} s&=(p_1+k_1)^2\,, \quad t=(k_1-k_2)^2\,,\\ u&=(p_1-k_2)^2\,, \quad s+t+u=2M^2+2\mu^2\equiv\Sigma\,. \end{aligned} \quad (2.3)$$

We follow the Feynman rules and metric convention of Schweber,[8] where

$$H_I=G\bar{\psi}(x)\gamma_5\psi(x)\phi(x)\,. \quad (2.4)$$

Of all the fourth-order diagrams involved in the elastic fermion-boson scattering described by the above Hamiltonian, the direct-box-diagram amplitude $T_{\mathrm{BxD}}^{(4)}$ and the crossed-box-diagram amplitude $T_{\mathrm{BxC}}^{(4)}$ [see Figs. 1(a) and 1(b)] are the most difficult to calculate because of the numerator in the following[9]:

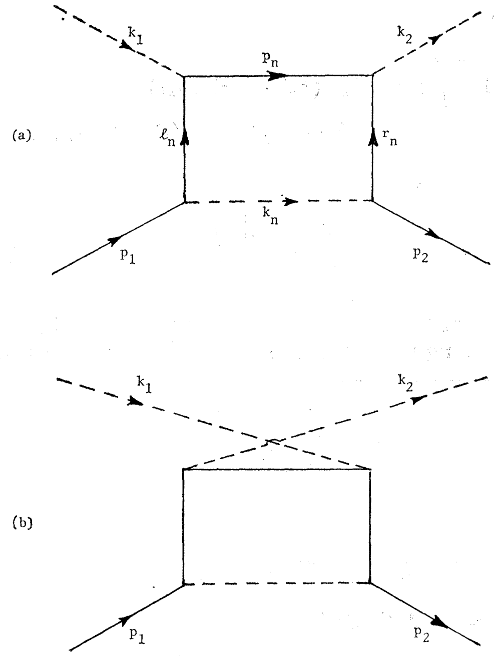

FIG. 1. The direct (a) and crossed (b) box diagrams for the interaction of pseudoscalar bosons with spin-$\frac{1}{2}$ fermions.

$$\bar{\psi}(p_2)T_{\mathrm{BxD}}^{(4)}\psi(p_1)=\frac{-iG^4}{(2\pi)^4}\int\frac{d^4k_nN}{[(p_2-k_n)^2-M^2][(p_1+k_1-k_n)^2-M^2][(p_1-k_n)^2-M^2][k_n^2-\mu^2]}\,, \quad (2.5)$$

where

$$N=\bar{\psi}(p_2)\gamma_5(\not{p}_2-\not{k}_n+M)\gamma_5(\not{p}_1+\not{k}_1-\not{k}_n+M)\times\gamma_5(\not{p}_1-\not{k}_n+M)\gamma_5\psi(p_1)\,. \quad (2.6)$$

The crossed box diagram is easily given by $-k_2\leftrightarrow k_1$, so that we can write the invariant amplitudes as

$$\begin{aligned} A_{\mathrm{BxD}}&=A(s,t)\,, \quad B_{\mathrm{BxD}}=B(s,t)\\ \text{and}\quad A_{\mathrm{BxC}}&=A(u,t)\,, \quad B_{\mathrm{BxC}}=-B(u,t)\,. \end{aligned} \quad (2.7)$$

The box-diagram numerator implies the k_n in a complicated way even after the γ algebra is done and the Dirac equation for the spinors is exploited. The integration over k_n does not seem feasible directly. On the other hand, the analytic structure of such integrals was studied by several authors, and in particular Mandelstam[1] showed that of the fourth-order amplitudes, only the box diagram can be written as a double dispersion relation. For the case of interest here, the basic analytic structure is not altered since it is determined by the denominator. However, the spin complication brings in a single-dispersion-integral term as well as a double dispersion term.[1,4] So mere evaluation of the double-spectral function by the Cutkosky δ-function method, though even this is not trivial,[4] will not yield all the information needed. We still need to know the absorptive parts for the t channel at least. When we treat the crossed box diagram, the s-channel absorptive part will come in handy also.

Let us first evaluate the absorptive parts in the s channel. The discontinuity across the cut in the s channel is given by the unitarity condition such that only second-order terms need enter explicitly:

$$\bar{\psi}(p_2)[A_s+\tfrac{1}{2}(\not{k}_1+\not{k}_2)B_s]\psi(p_1)$$

$$=\int_{\phi_s}\sum_{n\text{-spin}}\frac{G^4}{(2\pi)^2}\frac{M}{4}\times\frac{\bar{\psi}(p_2)\frac{1}{2}(\not{k}_2+\not{k}_n)\psi(p_n)\bar{\psi}(p_n)\frac{1}{2}(\not{k}_1+\not{k}_n)\psi(p_1)}{[(p_n-k_2)^2-M^2][(p_n-k_1)^2-M^2]}\,, \quad (2.8)$$

[8] S. S. Schweber, *Introduction to Relativistic Quantum Field Theory* (Row, Peterson, New York, 1961).

[9] Reference 4. These authors have taken $G=1$ and differ from us by a factor $[1/(2\pi)^{12}]^{1/2}$ in their Eq. (2.8), since we use Schweber's rules throughout.

where

$$\int_{\phi} = 4\int\int d^4p_n d^4k_n\, \delta(p_n{}^2-M^2)\theta(p_{n0}) \times\delta(k_n{}^2-\mu^2)\theta(k_{n0})\delta^4(P_i-p_n-k_n)\,. \quad (2.9)$$

In the c.m. frame

$$\int_{\phi_s} = \frac{|\mathbf{p}_n|}{\sqrt{s}}\int d\Omega_n\,. \quad (2.10)$$

A_s and B_s stand for the imaginary parts of $A(s,t)$ and $B(s,t)$ along the cut in the s variable wlth t fixed. After spin summation and manipulation of Dirac matrices,

$$\bar{\psi}(p_2)[A_s+\tfrac{1}{2}(\mathbf{k}_1+\mathbf{k}_2)B_s]\psi(p_1) = \frac{G^4}{8(2\pi)^2}\int_{\phi_s}\bar{\psi}(p_2) \times\frac{[(s-M^2)M-M^2(\mathbf{k}_1+\mathbf{k}_2)]+[\mathbf{k}_2\mathbf{p}_n\mathbf{k}_1]}{[(p_n-k_2)^2-M^2][(p_n-k_1)^2-M^2]}\psi(p_1)\,. \quad (2.11)$$

The first bracket in the numerator of the above is independent of the four-vector p_n, and its contribution can be established easily since[1]

$$\mathcal{I}_s(s,t) = \int_{\phi_s}\frac{1}{[(p_n-k_2)^2-M^2][(p_n-k_1)^2-M^2]} = \frac{\pi}{2p_s{}^3\sqrt{s}}\frac{1}{\sqrt{\kappa_s}}\ln\left[\frac{t/2p_s{}^2-(Z_s{}^2-1)-\sqrt{\kappa_s}}{t/2p_s{}^2-(Z_s{}^2-1)+\sqrt{\kappa_s}}\right], \quad (2.12)$$

where

$$p_s{}^2=[s-(M+\mu)^2][s-(M-\mu)^2]/4s\,, \quad (2.13)$$

$$Z_s{}^2-1=(4p_s{}^4)^{-1}[M^2+2\mu^2-s] \times[M^2+2\mu^2-s+4p_s{}^2]\,, \quad (2.14)$$

$$\kappa_s=t^2/4p_s{}^4-(t/p_s{}^2)(Z_s{}^2-1)\,. \quad (2.15)$$

The more interesting term is the one in the second bracket of (2.11), where we need to evaluate

$$I_s{}^\alpha=\int_{\phi_s}\frac{p_n{}^\alpha}{[(p_n-k_2)^2-M^2][(p_n-k_1)^2-M^2]}\,. \quad (2.16)$$

This four-vector can be described most generally as

$$I_s{}^\alpha=x'X^\alpha+y'Y^\alpha+zQ^\alpha+wR^\alpha\,, \quad (2.17)$$

where

$$\begin{aligned} &X^\alpha=P^\alpha+(P^2/K^2)^{1/2}K^\alpha\,, && \begin{cases}P^\alpha=\tfrac{1}{2}(p_1{}^\alpha+p_2{}^\alpha)\\ K^\alpha=\tfrac{1}{2}(k_1{}^\alpha+k_2{}^\alpha)\end{cases}\\ &Y^\alpha=P^\alpha-(P^2/K^2)^{1/2}K^\alpha\,, && \\ &Q^\alpha=\tfrac{1}{2}(k_1{}^\alpha-k_2{}^\alpha)=\tfrac{1}{2}(p_2{}^\alpha-p_1{}^\alpha)\,, &&\\ &R^\alpha\sim\epsilon^{\alpha\beta\gamma\delta}P_\beta K_\gamma Q_\delta\,. && \end{aligned} \quad (2.18)$$

This orthogonal base allows us to conclude that x', y', z, and w are functions of the four invariants X^2, Y^2, Q^2, and R^2. Projecting on R^α, we obtain, writing the integral (2.16) in invariant form,

$$wR^2=\int\frac{4d^4p_nd^4k_n\,\delta(p_n{}^2-M^2)\delta(k_n{}^2-\mu^2)\theta(p_{n0})\theta(k_{n0})\delta^4(P+K-p_n-k_n)p_n\cdot R}{[(p_n-K+Q)^2-M^2][(p_n-K-Q)^2-M^2]}\,. \quad (2.19)$$

Whereas the left-hand side is even under $R^\alpha\to-R^\alpha$, the right-hand side is odd; hence $w=0$. This fact is independent of the exchange masses in the denominator.

Projecting on Q^α, we obtain

$$zQ^2=\int\frac{4d^4p_nd^4k_n\,\delta(p_n{}^2-M^2)\delta(k_n{}^2-\mu^2)\theta(p_{n0})\theta(k_{n0})\delta^4(P+K-p_n-k_n)p_n\cdot Q}{[(p_n-K+Q)^2-M^2][(p_n-K-Q)^2-M^2]}\,. \quad (2.20)$$

We conclude similarly that $z=0$, though here we exploit the fact that the exchange masses are equal, which will not be the case for the t channel. Finally

$$I_s{}^\alpha=x_sP^\alpha+y_sK^\alpha\,, \quad (2.21)$$

with

$$x_s=\frac{(I_s\cdot P)K^2-(I_s\cdot K)(P\cdot K)}{P^2K^2-(P\cdot K)^2}\,,\qquad y_s=\frac{(I_s\cdot K)P^2-(I_s\cdot P)(P\cdot K)}{P^2K^2-(P\cdot K)^2}\,. \quad (2.22)$$

The problem is thus reduced to evaluating $I_s\cdot P$ and $I_s\cdot K$. This is done easily by a partial-fraction expansion under the integral sign using the mass-shell condition and momentum conservation. Thus

$$I_s\cdot K=\tfrac{1}{2}(K^2+Q^2) \times\int_{\phi_s}\frac{1}{[(p_n-K+Q)^2-M^2][(p_n-K-Q)^2-M^2]} -\frac{1}{4}\int_{\phi_s}\frac{1}{[(p_n-K+Q)^2-M^2]} -\frac{1}{4}\int_{\phi_s}\frac{1}{[(p_n-K-Q)^2-M^2]}\,. \quad (2.23)$$

The last two integrals can be evaluated easily; each gives a term

$$J(s)=\frac{\pi}{p_s\sqrt{s}}\ln\left(\frac{Z_s+1}{Z_s-1}\right), \tag{2.24}$$

where

$$Z_s=(M^2+2\mu^2-s+2p_s^2)/2p_s^2. \tag{2.25}$$

Hence

$$I_s\cdot K=\tfrac{1}{2}(K^2+Q^2)\mathcal{I}_s(s,t)-\tfrac{1}{2}J(s). \tag{2.26}$$

To evaluate $I_s\cdot P$ it is better to express the integrand in term of k_n rather than p_n, using $p_n=P+K-k_n$; then

$$I_s\cdot P=\int_{\phi_s}\frac{P^2+P\cdot K-k_n\cdot P}{[(k_n-P-Q)^2-M^2][(k_n-P+Q)^2-M^2]}. \tag{2.27}$$

Using a partial-fraction expansion with the mass-shell condition again for the last term of the numerator, we get

$$I_s\cdot P=[P^2+P\cdot K-\tfrac{1}{2}(Q^2+K^2)]\mathcal{I}_s(s,t)+\tfrac{1}{2}J(s). \tag{2.28}$$

Remembering the kinematical relations

$$\begin{gathered}P^2=\tfrac{1}{4}(4M^2-t),\quad K^2=\tfrac{1}{4}(4\mu^2-t),\\ Q^2=M^2-P^2=\mu^2-K^2,\\ P\cdot K=\tfrac{1}{4}[2(s-M^2-\mu^2)+t],\\ P^2K^2-(P\cdot K)^2=-\tfrac{1}{4}s(t+4p_s^2),\end{gathered} \tag{2.29}$$

substituting all this in (2.21) and in (2.11), we obtain

$$A_s=\frac{G^4}{8(2\pi)^2}\left[\frac{M(s-M^2)(s-M^2+\mu^2)}{s(t+4p_s^2)}\right]\{[s-M^2+\tfrac{1}{2}(t-4\mu^2)]\mathcal{I}_s(s,t)+J(s)\}, \tag{2.30a}$$

$$A_s\equiv(s-M^2)(s-M^2+\mu^2)A_s^F(s,t), \tag{2.30b}$$

$$B_s=\frac{G^4}{8(2\pi)^2}\left\{(s-M^2-\mu^2)\mathcal{I}_s(s,t)-\left[\frac{(s+M^2-\mu^2)(s-M^2)}{s(t+4p_s^2)}\right]\{[s-M^2+\tfrac{1}{2}(t-4\mu^2)]\mathcal{I}_s(s,t)+J(s)\}\right\}. \tag{2.31}$$

Proceeding similarly in the t channel,

$$A_t+\tfrac{1}{2}(\boldsymbol{k}_1+\boldsymbol{k}_2)B_t=\frac{G^4}{8(2\pi)^2}\int_{\phi_t}\frac{\tfrac{1}{2}(\boldsymbol{k}_1+\boldsymbol{k}_2)[2l_n\cdot p_1-2M^2]+(\boldsymbol{l}_n-M)[2k_1\cdot l_n-2k_1\cdot p_1]}{[(l_n+k_1)^2-M^2][(l_n-p_1)^2-\mu^2]}, \tag{2.32}$$

where

$$\int_{\phi_t}=4\int d^4r_nd^4l_n\,\delta(l_n^2-M^2)\delta(r_n^2-M^2)\theta(r_{n0})\theta(l_{n0})\delta^4(k_1-k_2+l_n+r_n). \tag{2.33}$$

In the c.m. frame

$$\int_{\phi_t}=\frac{|\boldsymbol{l}_n|}{\sqrt{t}}\int d\Omega_n.$$

Equation (2.32), after a judicious regrouping of terms, becomes

$$A_t+\tfrac{1}{2}(\boldsymbol{k}_1+\boldsymbol{k}_2)B_t=\frac{G^4}{8(2\pi)^2}\int_{\phi_t}\frac{M(s-M^2)-\boldsymbol{K}\mu^2}{[(l_n+k_1)^2-M^2][(l_n-p_1)^2-\mu^2]}-\frac{\boldsymbol{K}}{(l_n+k_1)^2-M^2}$$
$$-\frac{M}{(l_n-p_1)^2-\mu^2}-\frac{(s-M^2)\boldsymbol{l}_n}{[(l_n-p_1)^2-\mu^2][(l_n+k_1)^2-M^2]}+\frac{\boldsymbol{l}_n}{(l_n-p_1)^2-\mu^2}. \tag{2.34}$$

The first three terms have no dependence on $\boldsymbol{l}_n$ in the numerator and are given successively by

$$\mathcal{I}_t(s,t)=\int_{\phi_t}\frac{1}{[(l_n+k_1)^2-M^2][(l_n-p_1)^2-\mu^2]}, \tag{2.35}$$

$$\mathcal{I}_t(s,t)=\frac{4\pi}{(t-4M^2)[t(t-4\mu^2)]^{1/2}}\frac{1}{\sqrt{\kappa_t}}\ln\left(\frac{\cos\psi-Z_pZ_k-\sqrt{\kappa_t}}{\cos\psi-Z_pZ_k+\sqrt{\kappa_t}}\right), \tag{2.36}$$

where

$$Z_k=\frac{2\mu^2-t}{[(t-4M^2)(t-4\mu^2)]^{1/2}},\quad Z_p=\frac{4M^2-2\mu^2-t}{t-4M^2}, \tag{2.37}$$

$$\cos\psi=\frac{t+2s-2M^2-2\mu^2}{[(t-4M^2)(t-4\mu^2)]^{1/2}},\quad \kappa_t=Z_p^2+Z_k^2-1+\cos^2\psi-2Z_pZ_k\cos\psi,$$

and

$$J_k(t)=\int_{\phi t}\frac{1}{(l_n+k_1)^2-M^2}=\frac{2\pi}{[t(t-4\mu^2)]^{1/2}}\ln\left(\frac{Z_k+1}{Z_k-1}\right), \tag{2.38}$$

$$J_p(t)=\int_{\phi t}\frac{1}{(l_n-p_1)^2-\mu^2}=\frac{2\pi}{[t(t-4M^2)]^{1/2}}\ln\left(\frac{Z_p+1}{Z_p-1}\right). \tag{2.39}$$

The fourth term is similar to $I_s{}^\alpha(s,t)$ of the s channel (2.16). To exploit the symmetry as before, it is convenient to define

$$P'=\tfrac{1}{2}(-p_2+k_2),\quad K'=\tfrac{1}{2}(p_1-k_1),\quad Q'=\tfrac{1}{2}(-p_2-k_2)=\tfrac{1}{2}(-p_1-k_1), \tag{2.40}$$

$$I_t{}^\alpha(s,t)=4\int\frac{d^4r_nd^4l_n\,\delta(r_n{}^2-M^2)\delta(l_n{}^2-M^2)\theta(r_{n0})\theta(l_{n0})\delta^4(K'+P'-r_n-l_n)l_n{}^\alpha}{[(l_n-K'+Q')^2-\mu^2][(l_n-K'-Q')^2-M^2]}. \tag{2.41}$$

Here the exchange masses are not identical so there will be a term in $Q^{\alpha\prime}$. We evaluate as before by partial-fraction expansion $I_t\cdot P'$, $I_t\cdot K'$, and $I_t\cdot Q'$ (the latter using the difference of the partial fractions instead of the sum); they yield $I_t\cdot P$, $I_t\cdot K$, and $I_t\cdot Q$ such that coming back to the base P, K, Q, we get

$$I_t{}^\alpha(s,t)=x_tP^\alpha+y_tK^\alpha+z_tQ^\alpha, \tag{2.42}$$

with

$$x_t=\frac{K^2(I_t\cdot P)-(P\cdot K)(I_t\cdot K)}{P^2K^2-(P\cdot K)^2},\quad y_t=\frac{P^2(I_t\cdot K)-(P\cdot K)(I_t\cdot P)}{P^2K^2-(P\cdot K)^2}.$$

z_t is of no interest since Q does not contribute to the amplitudes;

$$\begin{aligned}I_t\cdot K&=\tfrac{1}{4}(t-2\mu^2)\mathcal{G}_t(s,t)+\tfrac{1}{2}J_P(t),\\ I_t\cdot P&=\tfrac{1}{4}(4M^2-2\mu^2-t)\mathcal{G}_t(s,t)-\tfrac{1}{2}J_k(t).\end{aligned} \tag{2.43}$$

The fifth term in (2.34) can be found using the orthogonal basis (2.18) and defining

$$J_t{}^\alpha=4\int\frac{l_n{}^\alpha d^4l_nd^4k_n\,\delta(r_n{}^2-M^2)\delta(l_n{}^2-M^2)\theta(r_{n0})\theta(l_{n0})\delta(2Q+r_n+l_n)}{[l_n-\frac{1}{2}(X+Y)+Q]^2-\mu^2}=a'X^\alpha+b'Y^\alpha+cQ^\alpha+dR^\alpha. \tag{2.44}$$

By an argument similar to the one for (2.19), d vanishes and by symmetry we can show that $a'=b'$, so that

$$J_t{}^\alpha=aP^\alpha+cQ^\alpha. \tag{2.45}$$

Then with

$$\begin{aligned}2(J_t\cdot Q-J_t\cdot P)&=\int_{\phi t}\left(1-\frac{2M^2-\mu^2}{2M^2-\mu^2+2l_n\cdot Q-2l_n\cdot P}\right),\\ -2(J_t\cdot Q+J_t\cdot P)&=\int_{\phi t}\left(1-\frac{2M^2-\mu^2-4Q^2}{2M^2-\mu^2+2r_n\cdot Q+2r_n\cdot P}\right),\end{aligned} \tag{2.46}$$

we get

$$J_t\cdot P=\tfrac{1}{4}(4M^2-2\mu^2-t)J_p(t)-2\pi\left(\frac{t-4M^2}{4t}\right)^{1/2},\quad J_t\cdot Q=-\tfrac{1}{4}tJ_p(t),\quad a=J_t\cdot P/P^2,\quad c=J_t\cdot Q/Q^2. \tag{2.47}$$

Substituting (2.35), (2.38), (2.39), (2.42), and (2.45) into (2.34), we obtain

$$\begin{aligned}A_t=\frac{G^4}{8(2\pi)^2}M\Bigg\{&\left[1+\frac{1}{4s(t+4p_s{}^2)}[(4\mu^2-t)(4M^2-2\mu^2-t)-(t-2\mu^2)(2s-2M^2-2\mu^2+t)]\right](s-M^2)\mathcal{G}_t(s,t)\\ &+\frac{4\pi}{[t(t-4M^2)]^{1/2}}+\left[\frac{2\mu^2}{t-4M^2}-\frac{(s-M^2)(2s-2M^2-2\mu^2+t)}{2s(t+4p_s{}^2)}\right]J_p(t)+\left[\frac{(t-4\mu^2)(s-M^2)}{2s(t+4p_s{}^2)}\right]J_k(t)\Bigg\},\end{aligned} \tag{2.48}$$

$$B_t=\frac{G^4}{8(2\pi)^2}\left\{\left[-\mu^2+\frac{(s-M^2)}{4s(t+4p_s^2)}[4\mu^2(t-2M^2)-2(4M^2-2\mu^2-t)(s-M^2-\mu^2)]\right]\mathcal{J}_t(s,t)\right.$$
$$\left.+\left[-1+\frac{(s-M^2)(2s-2M^2-2\mu^2+t)}{2s(t+4p_s^2)}\right]J_k(t)+\left[\frac{(s-M^2)(4M^2-t)}{2s(t+4p_s^2)}\right]J_p(t)\right\}. \quad (2.49)$$

The double-spectral function is a first indirect check on those absorptive parts obtained independently. We notice that[10]

$$\mathcal{J}_{st}(s,t)=\mathcal{J}_{ts}(s,t)\equiv\rho(s,t)=4\pi^2t^{-1/2}\{t[s-(M+\mu)^2][s-(M-\mu)^2]+4[s-M^2-2\mu^2][\mu^4-M^2(s-M^2+2\mu^2)]\}^{-1/2}. \quad (2.50)$$

We then verify explicitly that the double-spectral functions obtained via the s- and t-channel absorptive parts are identical. Thus

$$\rho_A=\frac{G^4}{8(2\pi)^2}\left[\frac{M(s-M^2)(s-M^2+\mu^2)[s-M^2+\frac{1}{2}(t-4\mu^2)]}{s(t+4p_s^2)}\right]\rho(s,t)\,, \quad (2.51a)$$

$$\rho_A\equiv(s-M^2)(s-M^2+\mu^2)\rho_A{}^F(s,t)\,, \quad (2.51b)$$

$$\rho_B=\frac{G^4}{8(2\pi)^2}\left[(s-M^2-\mu^2)-\frac{(s-M^2)(s+M^2-\mu^2)[s-M^2+\frac{1}{2}(t-4\mu^2)]}{s(t+4p_s^2)}\right]\rho(s,t)\,. \quad (2.51c)$$

Encouraged by this consistency, we verify the asymptotic behavior to be

$$\begin{aligned}
&A_s(s,t_f)\xrightarrow[s\to\infty]{}\text{const},\quad B_s(s,t_f)\xrightarrow[s\to\infty]{}0\,,\\
&A_t(s_f,t)\xrightarrow[t\to\infty]{}0,\quad B_t(s_f,t)\xrightarrow[t\to\infty]{}0\,,\\
&\rho_A(s,t_f)\xrightarrow[s\to\infty]{}\text{const},\quad \rho_B(s,t_f)\xrightarrow[s\to\infty]{}0\,,\\
&\rho_A(s_f,t)\xrightarrow[t\to\infty]{}0,\quad \rho_B(s_f,t)\xrightarrow[t\to\infty]{}0\,.
\end{aligned} \quad (2.52)$$

This confirms the need for a subtraction in a dispersion integral which uses A_s. We notice by the way that $B_s\to 0$ as $s\to\infty$ because of the additional coefficient $(s-M-\mu^2)$ of the term in $\mathcal{J}_s(s,t)$ which is not present for A_s and which has a canceling effect.

The last two checks are primarily dependent on the behavior of terms in $\mathcal{J}_s(s,t)$ or $\mathcal{J}_t(s,t)$ which dominate asymptotic behavior and which alone contribute to the double-spectral functions. The next test checks indirectly the relative weight of terms in $J(s)$, $J_p(t)$, and $J_k(t)$. It is based on the fact that the particular kinematics has introduced a "false" singularity with $s(t+4p_s^2)$ in the denominator of the absorptive parts, whereas we know from general consideration[1,2] that the singularity structure of (2.5) should allow the absorptive parts to have singularities only in the double-spectral region. For example, expressing $A_s(s,t)$ as a dispersion integral in t, we note that the term in (2.30) of the form

$$\frac{s-M^2+\frac{1}{2}(t-4\mu^2)}{s(t+4p_s^2)}\mathcal{J}_s(s,t)\equiv g(s,t)=\frac{1}{2\pi i}\oint\frac{g(s,t')}{t'-t}dt' \quad (2.53)$$

has a pole for $t=-4p_s^2$ as well as the cut in $t>0$. The residue of $g(s,t)$ for $t=-4p_s^2$ cancels exactly the term in $J(s)$ of (2.30a) leaving, as should be, A_s as a dispersion integral in t on the double-spectral cut. We can treat A_t and B_t similarly, writing them as dispersion integrals in s over the double-spectral region. The cancellation by residues is more tedious because of the two poles in s of the denominator $s(t+4p_s^2)$. Care must also be taken for A_t, which needs a subtraction in s, although this can be obviated by regrouping (2.48) such that it can be written as

$$A_t(s,t)=A_t{}^a(t)+\frac{s-M^2}{s(t+4p_s^2)}\times[\tfrac{1}{2}(t-4\mu^2)A_t{}^b(t)+(s-M^2+\mu^2)A_t{}^c(s,t)], \quad (2.54)$$

where

$$A_t{}^a(t)=\frac{G^4M}{8(2\pi)^2}\left(\frac{4\pi}{[t(t-4M^2)]^{1/2}}+\frac{2\mu^2}{t-4M^2}J_p(t)\right), \quad (2.55a)$$

$$A_t{}^b(t)=\frac{G^4M}{8(2\pi)^2}[J_k(t)-J_p(t)]\,, \quad (2.55b)$$

$$A_t{}^c(s,t)=\frac{G^4M}{8(2\pi)^2}\times\{[(s-M^2)+\tfrac{1}{2}(t-4\mu^2)]\mathcal{J}_t(s,t)-J_p(t)\}\,. \quad (2.55c)$$

With the removal of $A_t{}^a(t)$, which is independent of s, and after factorizing $(s-M^2)$, the rest of (2.54) can be written without a subtraction.

[10] We have derived explicitly the numerical factors which confirm the remarks of Rolnick (Ref. 7).

III. DISPERSION RELATIONS

We will write the dispersion integrals first for $B_{\mathrm{B\times D}}$ and $B_{\mathrm{B\times C}}$. They are rather straightforward because both $B_s(s,t)$ and $B_t(s,t)$ tend to zero asymptotically. Since we are interested in the physical region, where we have the discontinuity in s for fixed negative t, we can write for s in the complex cut plane and for $t<0$

$$B(s,t)=\frac{1}{\pi}\int_{m_s^2}\frac{B_s(s',t)ds'}{s'-s}. \tag{3.1}$$

This form is not very convenient for partial-wave expansion because t appears in a complicated way under the integral sign. On the other hand, we have in the unphysical region $s<0$ the more convenient form valid for t in the complex plane:

$$B(s,t)=\frac{1}{\pi}\int_{m_t^2}\frac{B_t(s,t')dt'}{t'-t}. \tag{3.2}$$

However, (3.2) can be continued into the physical region in s since B_t is made of analytic functions. Providing we stay on the same branches of these functions, which is ensured by maintaining the small negative imaginary parts,[1] (3.2) continued is equivalent to (3.1) in the physical region of the s channel. This is true because $B(s,t)$ is analytic in the cut s and t complex manifold, and if two analytic functions are equal in a region they are equal everywhere in their analyticity domain.

We verify, starting from either (3.1) or (3.2), that

$$B(s,t)=\frac{1}{\pi^2}\int_D\int\frac{\rho_B(s',t')ds'dt'}{(t'-t)(s'-s)}, \tag{3.3}$$

where D is the region of the real s-t plane for which the term under the square root in (2.50) is positive.

Care needs to be exercised regarding the cancellation of the contribution of kinematic poles if one chooses to write piecemeal single and double dispersion relations. For example, starting with $B_t(s,t)$ [Eq. (2.49)] and factorizing the term $(s-M^2)$, we can write

$$\begin{aligned}B(s,t)&\\=&-\frac{\mu^2}{\pi}\int_{m_t^2}\frac{\mathcal{J}_t(s,t')dt'}{t'-t}-\frac{1}{\pi}\int\frac{J_k(t')}{t'-t}dt'\\&+\frac{(s-M^2)}{\pi}\int\frac{dt'}{t'-t}\,\frac{1}{4s(t'+4p_s^2)}\{[4\mu^2(t'-2M^2)\\&-2(4M^2-2\mu^2-t')(s-M^2-\mu^2)]\mathcal{J}_t(s,t')\\&+2(2s-2M^2-2\mu^2+t')J_k(t')\\&+2(4M^2-t')J_p(t')\}.\end{aligned} \tag{3.4}$$

If we now wish to write the last integral as a double dispersion term, we note that only with the factor $(s-M^2)$ outside will the residues of the term in $\mathcal{J}_t$ at the kinematic poles in s cancel the terms in J_k and J_p inside that integral. We can then write

$$\begin{aligned}B(s,t)&\\=&-\frac{\mu^2}{\pi^2}\int_D\int\frac{\rho(s',t')dt'ds'}{(t'-t)(s'-s)}-\frac{1}{\pi}\int_{m_t^2}\frac{J_k(t')dt'}{t'-t}\\&+\frac{s-M^2}{\pi^2}\int_D\int\frac{dt'ds'}{(t'-t)(s'-s)}\\&\times\left[1-\frac{(s'+M^2-\mu^2)(s'-M^2+\frac{1}{2}t'-2\mu^2)}{s'(t'+4p_{s'}^2)}\right]\\&\times\rho(s',t').\end{aligned} \tag{3.5}$$

The factor $(s-M^2)$ is outside the last integral instead of primed and inside as in (3.3). This is reflected by the presence of the second single integral in (3.5).

The crossed box contribution to the **B** amplitude is easily given by the $s\to u$ interchange either in (3.1) or (3.2) as described by (2.7). However, the angular dependence thus brought into the numerator of (3.2) is inconvenient for partial-wave expansion. This can usually be obviated as follows, using

$$\frac{1}{(t'-t)(s'-u)}=\left(\frac{1}{t'-t}+\frac{1}{s'-u}\right)\frac{1}{[t'+s'-\Sigma+s]}, \tag{3.6}$$

$$B(u,t)=\frac{1}{\pi^2}\int_D\int\frac{\rho_B(s',t')dt'ds'}{(t'-t)(s'-u)}, \tag{3.7}$$

$$\begin{aligned}B(u,t)=&\frac{1}{\pi}\int_{m_t^2}\frac{B_t(\Sigma-s-t',t')dt'}{t'-t}\\&+\frac{1}{\pi}\int_{m_s^2}\frac{B_s(s',\Sigma-s-s')ds'}{s'-u}.\end{aligned} \tag{3.8}$$

One can verify easily that B_t and B_s still tend to zero asymptotically in their new functional dependence on the variable of integration. The Froissart-Gribov formula can then be used on (3.2) and (3.8) to obtain via (2.7) the partial-wave contribution to **B** in the c.m. system of the direct and crossed box amplitudes:

$$\begin{aligned}B_l(s)=&\frac{1}{2\pi p_s^2}\int_{m_t^2}[B_t(s,t')-B_t(\Sigma-s-t',t')]\\&\times Q_l\left(\frac{t'}{2p_s^2}+1\right)dt'\\&+\frac{1}{2\pi p_s^2}\int_{m_s^2}B_s(s',\Sigma-s-s')Q_l\left(\frac{\Sigma-s-s'}{2p_s^2}+1\right)ds'.\end{aligned} \tag{3.9}$$

For the contribution of the direct and crossed box diagrams to the **A** amplitude, we proceed similarly to the **B** amplitude. However, the need for a subtraction for the dispersion integrals in s complicates the issue, but only formally as we shall see. In order to get convenient forms for the partial-wave expansions of the crossed box contribution, we examine both the s and t dispersion integrals.

For negative t, we have valid for s and s_0 on the cut complex plane,

$$A(s,t)=A(s_0,t)+\frac{1}{\pi}(s-s_0)\int_{m_s^2}\frac{A_s(s',t)ds'}{(s'-s_0)(s'-s)} \quad (3.10)$$

or, noticing the discontinuity of $A_s(s,t)$ in the complex t plane,

$$A(s,t)=A(s_0,t)+\frac{1}{\pi^2}(s-s_0)\int_D\int\frac{\rho_A(s',t')ds'dt'}{(s'-s_0)(s'-s)(t'-t)}. \quad (3.11)$$

Alternatively, starting from the t channel for negative s, we have for t in the complex plane

$$A(s,t)=\frac{1}{\pi}\int_{m_t^2}\frac{A_t(s,t')}{t'-t}dt'. \quad (3.12)$$

Formula (3.12) can be continued in s away from its original region of validity since A_t is made of analytic functions. Subject to the arguments used earlier for $B(s,t)$, it will be equal to (3.10) in the physical region of the s channel. This form is handier than (3.10) for a partial-wave expansion. It can also be used to evaluate the hitherto unknown function $A(s_0,t)$ of (3.10) and (3.11), yielding

$$A(s,t)=\frac{1}{\pi}\int_{m_t^2}\frac{A_t(s_0,t')dt'}{t'-t}+\frac{1}{\pi^2}(s-s_0)\times\int_D\int\frac{\rho_A(s',t')ds'dt'}{(s'-s_0)(s'-s)(t'-t)}. \quad (3.13)$$

The same result can of course be obtained directly for the physical t region from (3.12).

We note that if the double dispersion form (3.13) is required, s_0 need not be negative as implied by Kubis and Gammel.[4] This allows us to use the "natural" subtraction points (i.e., M^2 and $M^2-\mu^2$) and simplify the calculations considerably. For example, using $s_0=M^2$ in (3.13) and referring to (2.51b), (2.54), (2.55), we get the simple form

$$A(s,t)=\frac{1}{\pi}\int_{m_t^2}\frac{A_t^a(t')dt'}{t'-t}+\frac{1}{\pi^2}(s-M^2)\int_D\int\frac{(s'-M^2+\mu^2)\rho_A^F(s't')ds'dt'}{(s'-s)(t'-t)}. \quad (3.14)$$

Alternatively with $s_0=M^2-\mu^2$ and

$$s(t+4p_s^2)|_{s=s_0}=(M^2-\mu^2)(t-4\mu^2), \quad (3.15)$$

we find

$$A(s,t)=\frac{1}{\pi}\int_{m_t^2}\frac{A_t^a(t')-\frac{1}{2}[\mu^2/(M^2-\mu^2)]A_t^b(t')}{t'-t}dt'+\frac{1}{\pi^2}(s-M^2+\mu^2)\int_D\int\frac{(s'-M^2)\rho_A^F(s',t')ds'dt'}{(s'-s)(t'-t)}. \quad (3.16)$$

The convergence of the integration in s of (3.10) and (3.11) can be improved, and full use of the two natural subtraction points is achieved with a twice-subtracted dispersion integral. Thus, using (2.30b) and (3.12), we can write

$$A(s,t)=\frac{1}{\pi}\frac{(s-s_1)}{(s_0-s_1)}\int_{m_t^2}\frac{A_t(s_0,t')dt'}{t'-t}+\frac{1}{\pi}\frac{(s-s_0)}{(s_1-s_0)}\int_{m_t^2}\frac{A_t(s_1,t')dt'}{t'-t}+\frac{1}{\pi}(s-s_0)(s-s_1)\int_{m_s^2}\frac{A_s^F(s',t)ds'}{s'-s}. \quad (3.17)$$

With $s_0=M^2$, $s_1=M^2-\mu^2$, (2.51b), (2.54), and some manipulations, we obtain

$$A(s,t)=\frac{1}{\pi}\int_{m_t^2}\frac{A_t^a(t')dt'}{t'-t}+\frac{1}{\pi}\frac{(s-M^2)}{2(M^2-\mu^2)}\int_{m_t^2}\frac{A_t^b(t')dt'}{t'-t}+\frac{1}{\pi^2}(s-M^2)(s-M^2+\mu^2)\int_D\int\frac{\rho_A^F(s',t')ds'dt'}{(s'-s)(t'-t)}. \quad (3.18)$$

For the crossed box contribution A_{BxC} to the amplitude, any of the forms above can be used with $s\leftrightarrow u$. These, however, are ill suited for partial-wave expansions because of complicated angular dependence outside and inside the integrals. With the subtraction in s, it is more complicated to obtain a result similar to (3.8), but we will show that this is only formally the case.

Note that the convenient result (3.8) rested on the new functional dependence of the discontinuities, i.e., $B_s(s',\Sigma-s-s')$ and $B_t(\Sigma-s-t',t')$. In order to obtain a similar result for $A(u,t)$, we will remark first that the substitutions

$$\begin{aligned} t\to\Sigma-s-s'&: \quad A_s(s',t)\to A_s(s',\Sigma-s-s'), \\ s\to\Sigma-s-t'&: \quad A_t(s,t')\to A_t(\Sigma-s-t',t') \end{aligned} \quad (3.19)$$

actually improve the convergence of A_s and A_t in their respective variables of integration s' and t'. Indeed we find, after explicit substitution and asymptotic expansion, the following leading terms:

$$A_s(s',\Sigma-s-s')\underset{s'\to\infty}{\sim}\frac{1}{s'}\ln s' \tag{3.20}$$

and

$$A_t(\Sigma-s-t',t')\underset{t'\to\infty}{\sim}\frac{1}{t'}\ln t'.$$

Comparison with (2.52) shows the improvement in asymptotic behavior. This interesting result (we do not know as yet how general it can be) allows the following manipulations. Starting from (3.13) with $s\leftrightarrow u$ and using relation (3.6), we obtain

$$A(u,t)=\frac{1}{\pi}\int_{m_t^2}\frac{A_t(s_0,t')}{t'-t}dt' +\frac{(u-s_0)}{\pi^2}\left\{\int_D\frac{ds'}{(s'-s_0)(s'-u)}\int\frac{\rho_A(s',t')dt'}{t'-(\Sigma-s-s')} +\int_D\frac{dt'}{t'-t}\int\frac{\rho_A(s',t)ds'}{(s'-s_0)[s'-(\Sigma-s-t')]}\right\}. \tag{3.21}$$

After integration,

$$A(u,t)=\frac{1}{\pi}\int_{m_t^2}\frac{A_t(s_0,t')dt'}{t'-t} +\frac{u-s_0}{\pi}\left\{\int_{m_s^2}\frac{A_s(s',\Sigma-s-s')ds'}{(s'-s_0)(s'-u)} +\int_{m_t^2}\left[\frac{A_t(s_0,t')-A_t(\Sigma-s-t',t')}{s_0-(\Sigma-s-t')}\right]\frac{dt'}{t'-t}\right\}. \tag{3.22}$$

Regrouping the terms in $A_t(s_0,t')$, we get

$$A(u,t)=\frac{1}{\pi}\int_{m_t^2}\frac{A_t(s_0,t')}{t'-(\Sigma-s-s_0)} +\frac{u-s_0}{\pi}\left\{\int_{m_s^2}\frac{A_s(s',\Sigma-s-s')ds'}{(s'-s_0)(s'-u)} +\int_{m_t^2}\frac{A_t(\Sigma-s-t',t')dt'}{(t'-t)(\Sigma-s-t'-s_0)}\right\}. \tag{3.23}$$

Using $s_0=M^2$ or $s_0=M^2-\mu^2$ at this stage would bring many simplifications because of the factorizability of A_s and A_t as shown in (2.30a) and (2.54). But we can do better than this and eliminate the angular dependence of the numerator (i.e., $u-s_0$). The subtraction had been necessary to improve the convergence in s' of $A_s(s',t)$ of (3.10). However, the new functional dependence (3.19) converges much better. We can thus work the subtraction process in reverse, so to speak, by bringing $(u-s_0)$ inside the integrals in (3.23) with the following expansions:

$$\frac{u-s_0}{(s'-s_0)(s'-u)}=\frac{-1}{s'-s_0}+\frac{1}{s'-u},$$
$$\frac{u-s_0}{(t'-t)(\Sigma-s-t'-s_0)}=\frac{1}{\Sigma-s-t'-s_0}+\frac{1}{t'-t}. \tag{3.24}$$

Regrouping, we obtain

$$A(u,t)=\frac{1}{\pi}\int_{m_t^2}\left[\frac{A_t(s_0,t')-A_t(\Sigma-s-t',t')}{[s_0-(\Sigma-s-t')]}\right]dt' -\frac{1}{\pi}\int_{m_s^2}\frac{A_s(s',\Sigma-s-s')}{s'-s_0}ds' +\frac{1}{\pi}\int_{m_s^2}\frac{A_s(s',\Sigma-s-s')}{s'-u}ds' +\frac{1}{\pi}\int_{m_t^2}\frac{A_t(\Sigma-s-t',t')}{t'-t}dt'. \tag{3.25}$$

The first and second integrals cancel since, after consideration of (3.21) and (3.22), we see that they are both equal to

$$\frac{1}{\pi}\int_D\int\frac{\rho_A(s',t')ds'dt'}{(s'-s_0)[s'-\Sigma+s+t']}. \tag{3.26}$$

Hence

$$A(u,t)=\frac{1}{\pi}\int_{m_s^2}\frac{A_s(s',\Sigma-s-s')ds'}{s'-u} +\frac{1}{\pi}\int_{m_t^2}\frac{A_t(\Sigma-s-t',t')}{t'-t}dt'. \tag{3.27}$$

We see that the crossed term $A(u,t)$ can be written formally as if no subtraction had been necessary, in this elegant form free from angular dependence but for the simple denominators. The Froissart-Gribov formula applied to (3.12) and (3.27) allows us to write immediately the partial-wave contribution to the **A** amplitude of the direct and crossed box diagrams in the c.m. frame:

$$A_l(s)=\frac{1}{2\pi p_s^2}\int_{m_t^2}[A_t(s,t')+A_t(\Sigma-s-t',t')] \times Q_l\left(\frac{t'}{2p_s^2}+1\right)dt' -\frac{1}{2\pi p_s^2}\int_{m_s^2}A_s(s',\Sigma-s-s')Q_l\left(\frac{\Sigma-s-s'}{2p_s^2}+1\right)ds'. \tag{3.28}$$

IV. CONCLUSION

A method of general applicability was developed to evaluate, in terms of single dispersion integrals over closed-form analytic functions, the contribution of box diagrams for interactions involving particles with spin. In particular, it was applied to a case of frequent interest: the interaction of spin-$\frac{1}{2}$ fermions of mass M with pseudoscalar bosons of mass μ. The technique for evaluating the absorptive parts with relative ease was a type of partial-fraction expansion yielding numerators free of angular dependence in the phase-space integrals. The correctness of the result was ensured in two different ways. (a) The terms in the absorptive parts which themselves have an imaginary part were checked by explicit calculation of the double-spectral functions via the s and t channels independently. (b) The other terms in the absorptive parts were checked by ensuring their cancellation, upon passing to the double-spectral integral form, by the residues of the absorptive parts at the spurious kinematical singularities.

Several surprising factorizations occurred, bringing out natural positive subtraction points. We underlined the care needed to use them in the presence of kinematical singularities. Another unexpected result derived by explicit calculation lies in the improved convergence of the absorptive parts under the usual substitution made for the convenient evaluation of partial waves. It was shown that this allows one to do away completely with the subtraction in this useful representation.

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Investigation of Some Dual Amplitudes with Cuts

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We study two simple ways of adding an infinite number of direct-channel two-particle thresholds to the Veneziano picture, which usually contains only resonances in the direct channel. We investigate the high-energy behavior, positivity, and l-plane structure of the resulting amplitudes, which turn out to have Regge cuts in addition to Regge poles in the crossed channel.

I. INTRODUCTION

A LONG time had to pass, during which numerous data fits were made, and a deeper understanding of the nature of Regge cuts achieved, before it was realized that cuts in the angular momentum plane could not be neglected in Regge theory. The situation was different with Veneziano theory, which was known from the outset to violate unitarity by giving rise to zero-width resonances.[1,2] In fact, it was introduced as a first step of a bootstrap procedure in which complete crossing symmetry was given priority over unitarity, which was to be treated as an approximation in a further step. Many attempts at unitarization have been made, and it seems that any reasonable unitarization procedure would either result in cuts or have to use cuts as an input.[3,4] The cuts in the models we are planning to investigate are introduced in simple ways, and it will be obvious that unitarization is not the immediate objective. Nevertheless, it will be seen that to study such cuts is not totally unreasonable, since they lead to interesting new ideas.

We will explore the possibility of generating Regge cuts in the crossed channel through the introduction of an infinite number of direct-channel two-particle thresholds with real square-root branch points.[5] We will find that, just as we can imagine a world composed of an infinite number of zero-width direct-channel resonances, which Reggeizes as the direct-channel energy increases giving rise to Regge poles, we can equally well imagine a world composed of an infinite number of direct-channel two-particle thresholds with real branch points, that gives rise to Regge cuts in the crossed channel. It will be seen that the latter picture possesses novel properties similar to those possessed by the former: crossing symmetry, Regge behavior, likelihood of positivity, absence of fixed poles, etc. It also suffers from similar diseases: lack of Regge behavior on the real axis (except on the average), branch points and trajectories can only be real, etc. In fact, the two pictures, instead of being introduced separately, may be interconnected in a "cross-duality" scheme wherein direct-channel resonances give rise to Regge cuts, and direct-channel thresholds give rise to Regge poles in the crossed channel. Furthermore, it appears that the "cross-duality" picture may have certain advantages over the "direct-duality" one, in

[1] For a comprehensive review of narrow-resonance models, see D. Sivers and J. Yellin, Rev. Mod. Phys. (to be published).
[2] For a short review of the Veneziano model, see J. D. Jackson, Rev. Mod. Phys. 42, 12 (1970).
[3] A. Martin, Phys. Letters 29B, 431 (1970).
[4] For a comprehensive list of references to works on unitarization, see Ref. 2.
[5] M. O. Taha, Nuovo Cimento Letters 3, 861 (1970); Phys. Rev. D (to be published).