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Summing Soft Pseudoscalar Pions*

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Momentum-transfer-dependent damping of the nucleon form factors due to the virtual exchange of soft nonchiral pseudoscalar-pion pairs is estimated. When chirality is enforced, by embedding the calculation in the nonlinear σ model, these soft effects are shown to cancel to all orders in the strong coupling. By themselves, the contributions described here do not easily fit the experimental data.

I. INTRODUCTION

THE question of soft-pion summations and their possible role in high-energy hadronic processes has recently become a matter of some interest, as present experiments begin to display evidence of inelastic production,[1] while models of soft virtual mesons (neutral pions and vector mesons) have been used to construct elastic amplitudes which exhibit damping at large energies and/or momentum transfers.[2] The specific problem of summing soft charged pions has long been an open and difficult question. Several years ago, a treatment of soft-pion pairs in pseudoscalar-pseudoscalar theory was given[3] in terms of a small generalization of that type of Bloch-Nordsieck calculation[4] which has since come to be known as "eikonal." In that world of soft-pseudoscalar-pion pairs, one found an expected damping; and in addition the curious result that the same analysis applied to the chiral-invariant nonlinear σ model[5] produced a null result, with all soft effects canceling out to all orders in the strong coupling. Subsequently, another calculation was described[6] in which a "chiral cancellation" of soft-pion effects was observed, although only within the context of the rather special approximations involved. In a recent paper, Weinberg[7] has given a general proof of the cancellation of all soft-pion effects in a suitably defined chiral model. He has also provided a framework within which one may describe the effects of soft chiral pions together with chiral-violating nonsoft structure; it is interesting that the latter results display damping independent of energy and/or momentum transfer. A distinction between the calculations of Refs. 3, 6, and 7 is that in the first cases soft pions were not assumed to be chiral at the start; the effects observed depended upon momentum transfer, and canceled only when chirality was demanded. Another distinction is that purely pionic loops were not ruled out in the earlier papers.

These calculations suggest that soft chiral pions play no appreciable role in providing the strong damping seen in recent high-energy experiments. Since strict chiral invariance cannot really be true, it is of some interest to ask if soft nonchiral pions might generate results which could be tested by experiment, even if such results may depend somewhat upon the precise definition of "soft" and upon the form of the nonchiral interaction assumed. It is the purpose of this paper to describe the special form of soft-pion physics which can be extracted from the simplest pseudoscalar theory, and to argue that the results are not suggested by present experiments, although better accuracy would certainly be useful in ruling them out. In passing, it is easy to show that embedding this calculation in the nonlinear σ model leads to a complete cancellation of all soft effects. Finally, the explicit forms found are of some interest in themselves, for they mark the limit of combinatoric solubility in eikonal-type calculations.

II. DERIVATION

The interaction Lagrangian initially assumed is $\mathcal{L}'=ig\bar{\psi}\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}\psi$, using standard notation to describe $SU(2)$ nucleons and pions. It is useful to begin by contrasting soft-pion emission in this theory with soft-photon emission in quantum electrodynamics (QED). In QED, the invariant amplitude for the emission of n soft photons each of energy ω is proportional to a factor ω^{-n}, arising from the propagator of the charged particle emitting the photon. In contrast, if one soft pseudoscalar pion is emitted, the corresponding matrix element is of order $(\omega)^0$; but if two soft pions are produced the dominant part of the amplitude behaves as ω^{-1}. One essential difference between soft-pion production in pseudoscalar theory and in partially conserved axial-vector-current (PCAC) models[8] is that in the latter an amplitude representing n soft pions contributes factors of order ω^0 for all n, while in pseudoscalar theory the dominant part of each amplitude behaves as $\omega^{-n/2}$ if n is even, and as $\omega^{-(1/2)(n-1)}$ if n is odd. In other words, the most important contributions (more precisely, the ones extracted here) will come from the emission of soft-pion pairs.

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[1] C. P. Wang, Phys. Rev. **180**, 1463 (1969).
[2] H. M. Fried and T. K. Gaisser, Phys. Rev. **179**, 1491 (1969); R. Perrin, *ibid.* **162**, 1343 (1967); G. Mack, *ibid.* **154**, 1617 (1967).
[3] H. M. Fried, Winter Lectures at Karpac, Poland, 1967 (unpublished).
[4] F. Bloch and A. Nordsieck, Phys. Rev. **52**, 54 (1937).
[5] M. Gell-Mann and M. Lévy, Nuovo Cimento **16**, 705 (1960).
[6] H. M. Fried, in *Lectures in Theoretical Physics*, edited by W. E. Brittin *et al.* (Gordon and Breach, New York, 1969), Vol. XI.
[7] S. Weinberg, Phys. Rev. D **2**, 674 (1970).
[8] S. Weinberg, Phys. Rev. Letters **18**, 188 (1967).

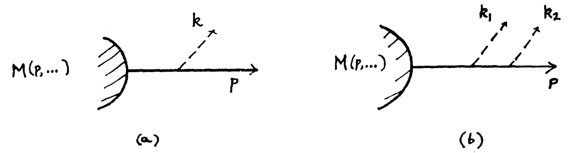

FIG. 1. (a) Emission of a single soft pion from an external nucleon line during the process characterized by $M(p,\ .\ .\ .\ )$. (b) Corresponding emission of a pair of soft pions.

These remarks may be illustrated[9] by the simplest relevant Feynman graphs of Fig. 1, which depict the emission of soft pions from an external nucleon line of momentum p, appearing in some otherwise unspecified process denoted by $M(p,\ldots)$. The one-pion graph of Fig. 1(a) produces a soft-pion contribution proportional to $M(p,\ldots)(\gamma\cdot k\gamma_5/2p\cdot k)\tau_\alpha u(p)$, a result of order ω^0. On the other hand, if two soft pions are emitted as in Fig. 1(b), the corresponding contribution is $M(p,\ldots)\times[\tau_\beta\tau_\alpha/2p\cdot(k_1+k_2)]u(p)$, a result of order ω^{-1}. Including the graph corresponding to the permutation of these pion lines, one obtains the symmetric result $M(p,\ldots)\times[\delta_{\alpha\beta}/p\cdot(k_1+k_2)]u(p)$, which carries the pictorial interpretation of a soft-pion pair (both neutral or oppositely charged) emitted at the same point, as in Fig. 2. The similarity of this expression to the corresponding soft-photon forms suggests that it will be possible to sum all such terms explicitly, and this is true; however, because pion pairs are involved, the result is more complicated than might be anticipated on the basis of statistical independence. One has not only the graphs of Fig. 3(a), corresponding to the exchange of virtual-soft-pion pairs between nucleon lines, here illustrated for the vertex function, but also the graphs of Fig. 3(b).

As always in soft or eikonal summations, the functional method is the most appropriate. The calculation begins with an examination of $G[\pi]$, the nucleon Green's function[10] in the presence of an external c-number field $\pi_\alpha(x)$,

$$[m_0+\gamma\cdot\partial-ig\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}]G[\pi]=1\,, \tag{1}$$

and is based upon the fortunate circumstance that the soft-pion-pair approximation to $G[\pi]$ will not involve the τ_α in any significant way. For, if we set

$$G[\pi]=[m_0-\gamma\cdot\partial+ig\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}]\mathcal{G}[\pi]\,,$$

the $\mathcal{G}[\pi]$ will satisfy

$$(m_0^2-\partial^2+g^2\boldsymbol{\pi}^2-ig\gamma_5\gamma_\mu\tau_\alpha[\partial_\mu,\pi_\alpha])\mathcal{G}[\pi]=1\,. \tag{2}$$

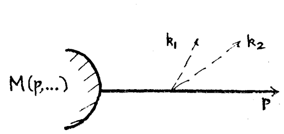

FIG. 2. Equivalent Feynman graph for the emission of a soft-pseudoscalar-pion pair.

[9] Much of the discussion of this paragraph is due to D. R. Yennie.

[10] A very general formulation has been given by E. S. Fradkin, Nucl. Phys. **76**, 588 (1966).

In the soft-pion limit, the contribution of the $[\partial_\mu,\pi_\alpha]$ term of (2) will be of order ω relative to the pion-pair contribution of the $g^2\boldsymbol{\pi}^2$ term. It may be noted that the $[\partial_\mu,\pi_\alpha]$ dependence leads to forms of the order ω^0 because, upon iteration, the operator $m_0{}^2-\partial^2$ will produce (after mass renormalization) factors of $[m^2+(p+k)^2]^{-1}\sim\omega^{-1}$ for small k.

The soft-pion-pair limit may now be obtained by replacing (2) by

$$[m^2-\partial^2+g^2\boldsymbol{\pi}^2]\mathcal{G}[\pi]=1\,, \tag{3}$$

and solving (3) in a Bloch-Nordsieck or no-recoil approximation. The particular form of approximation used can vary a little, depending upon the type of application and complexity desired, and we adopt here the simplest possible forms, which yield the same sort of approximation to form factors and scattering amplitudes as those discussed in the first paper of Ref. 2; that is, the solution to (3), with $A(x)\equiv g^2\boldsymbol{\pi}^2(x)$, may be written in exact parametric form as

$$\langle p|\mathcal{G}|p'\rangle=i\int_0^\infty d\xi\, e^{-i\xi(m^2+p^2)}\langle p|\mathfrak{F}(\xi)|p'\rangle\,,$$

where

$$\frac{\partial}{\partial\xi}\langle p|\mathfrak{F}(\xi)|p'\rangle=-iA\left(2\xi p+i\frac{\partial}{\partial p}\right)\langle p|\mathfrak{F}(\xi)|p'\rangle\,, \tag{4}$$

with $\langle p|\mathfrak{F}(0)|p'\rangle=\langle p|p'\rangle$. The no-recoil approximation now introduced neglects in (4) that operator $i\partial/\partial p$, which produces the variation of the nucleon line's momentum p upon the emission or absorption of the pion pairs characterized by $A(x)$. Because $A(x)=g^2\boldsymbol{\pi}^2(x)$, this statement is really made independently of the magnitude of the Fourier momentum components of $\tilde{\pi}(k)$, which need not be small if those of $\tilde{A}(k)$ are small. The momenta of $\tilde{\pi}(k)$ will subsequently be limited by an upper cutoff inserted by hand in the final stages of the calculation, in a manner familiar from previous work.[2] Thus, the approximation used here defines an approximate propagator

$$\langle p|\mathcal{G}|p'\rangle=i\int_0 d\xi\, e^{-i\xi(m^2+p^2)}\times\exp\left[-ig^2\int_0^\xi d\xi'\,\boldsymbol{\pi}^2(2\xi'p)\right]\langle p|p'\rangle\,, \tag{5}$$

and all the linkage operations, which provide the soft-virtual-pion-pair structure, will again appear in the form

$$\exp\left(-i\frac{\delta}{\delta\pi_\alpha}\Delta_c{}^{(\pi)}\frac{\delta}{\delta\pi_\alpha{}'}\right)\exp\left\{-ig^2\int_0^\infty d\xi[\boldsymbol{\pi}^2(2\xi p)+\boldsymbol{\pi}'^2(-2\xi p')]\right\}\Bigg|_{\pi=\pi'=0}\,, \tag{6}$$

corresponding to the exchange of an infinite number of soft-pion pairs between nucleon lines of momenta p and p', respectively. The emission of arbitrary numbers of pions for any S-matrix element built out of this model may be easily calculated by performing the requisite functional derivatives with respect to the sources before the latter are set equal to zero, as in (6). We shall consider here only elastic processes, and, in particular, the nucleon electromagnetic form factors whose non-soft-pion-pair dependence is, in this model approximation, merely multiplied by the renormalized combinations of (6). One finds an expression for the latter in the form $\exp f(t)$, $f(t) \equiv F(t) - F(0)$, with

$$F(t) = -\tfrac{3}{2}\,\mathrm{Tr}\,\ln[1-\Delta_c^{(\pi)}Q^{(p)}\Delta_c^{(\pi)}Q^{(-p')}] \qquad (7)$$

and

$$\langle x|Q^{(p)}|y\rangle = -2g^2\delta(x-y)\int_0^\infty d\xi\;\delta(x-2\xi p)\,. \qquad (8)$$

This result is really an implicit one, in the sense that an integral equation must first be solved before $F(t)$ can be known. This particular generalization[11] of the ordinary eikonal forms[12] occurs because of the quadratic exponents of (6), and represents the most complicated type of functional integration which can be put into closed form.

When one considers variations of the original Lagrangian, these results can change completely. The most striking example of complete cancellation occurs in the nonlinear σ model, defined by the Gell-Mann and Lévy Lagrangian

$$\begin{aligned}-\mathcal{L}_2 = \bar\psi[m_0+\gamma\cdot\partial-g_0(\sigma+i\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi})]\psi\\ +\tfrac{1}{2}[(\partial\boldsymbol{\pi})^2+(\partial\sigma)^2+\mu_0^2\boldsymbol{\pi}^2+(\mu_0^2+2\lambda_0/f_0^2)\sigma^2]\\ +\lambda_0[(\boldsymbol{\pi}^2+\sigma^2)^2-(2/f_0)\sigma(\sigma^2+\boldsymbol{\pi}^2)]\,,\end{aligned} \qquad (9)$$

where $f_0 = g_0/2m_0$, and the operator σ must be delicately defined in terms of the pion operator $\boldsymbol{\pi}$, according to the nonlinear relation

$$\boldsymbol{\pi}^2+\sigma^2 = \sigma/f_0\,. \qquad (10)$$

Proceeding as above, the nucleon Green's function will satisfy

$$(m_0+\gamma\cdot\partial-ig_0\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}-g_0\sigma)G = 1\,, \qquad (11)$$

and upon setting

$$G = (m_0-g_0\sigma-\gamma\cdot\partial+ig_0\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi})\mathcal{G}\,,$$

there results

$$\{m_0^2-\partial^2-g_0\gamma_\mu[\partial_\mu, m_0\sigma+i\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}] + g_0^2(\sigma^2+\boldsymbol{\pi}^2-\sigma/f_0)\}\mathcal{G} = 1\,. \qquad (12)$$

[11] See, for example, the Appendix of the paper by C. Sommerfield, Ann. Phys. (N. Y.) **26**, 1 (1964). Graphs with nucleon self-energy insertions have not been included in (6) and (7).

[12] H. Abarbanel and C. Itzykson, Phys. Rev. Letters **23**, 53 (1969); M. Lévy and J. Sucher, Phys. Rev. **186**, 1656 (1969); R. Blankenbecler and R. Sugar, *ibid.* **183**, 1387 (1969); S. Chang and S. Ma, Phys. Rev. Letters **22**, 1334 (1969); G. Erickson and H. Fried, J. Math. Phys. **6**, 414 (1965).

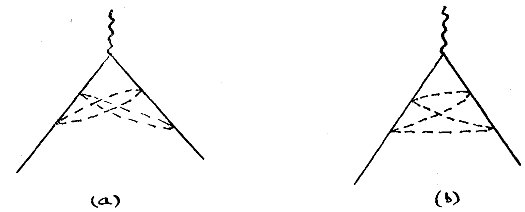

FIG. 3. (a) Typical vertex graph corresponding to the exchange of a "crossed" pion pair between nucleon lines. (b) Corresponding graph which results upon "opening up" the pion lines of each pair.

Because (10) must also hold for the c-number source functions, the entire soft-pion-pair effect calculated above disappears, to all orders in g_0. The resulting amplitudes contain soft-pion dependence of order ω^0, in agreement with the forms of Ref. 8.

III. EVALUATION

The content of Eqs. (7) and (8) may be expressed in terms of the quadrature

$$F(t) = 6g^4\int_0^1 d\lambda\int_0^\infty\int_0^\infty d\xi d\eta\;\bar\Delta_\lambda(-2\eta p', 2\xi p)\times\Delta_c(2\xi p+2\eta p') \qquad (13)$$

when $\bar\Lambda_\lambda$ satisfies the integral equation

$$\begin{aligned}\bar\Delta_\lambda(-2\eta p', 2\xi p) &= \Delta_c(2\eta p'+2\xi p)+4\lambda g^4\int_0^\infty\int_0^\infty d\xi' d\eta'\,\Delta_c(2\eta p'+2\xi' p)\\ &\quad\times\Delta_c(2\xi' p+2\eta' p')\bar\Delta_\lambda(-2\eta' p', 2\xi p)\,;\end{aligned} \qquad (14)$$

here $\Delta_c(x) = \Delta_c(-x)$ denotes the pion propagator, and $-t = (p-p')^2$ is a positive momentum transfer. Rather than attempt to solve (14), we shall consider the quadratures (13) of the iterates of (14), and of these evaluate exactly only the simplest for all $-t$, while estimating the remainder for both zero and asymptotically large momentum transfer.

The first term of F, of order g^4, is given by

$$F_0(t) = 6g^4\int_0^\infty\int_0^\infty d\xi d\eta\;\Delta_c^2(2\xi p+2\eta p')\,. \qquad (15)$$

The necessary upper cutoff to the virtual-pion momenta can be inserted in a variety of ways, of which perhaps the neatest is the procedure of the first paper of Ref. 2, inserting a factor $e^{-i\alpha k^2}$ into each virtual-pion momentum integrand, and performing the continuation $\alpha \to -i\beta$, $\beta = \mu_c^{-2}$, at the end of the calculation. This method introduces an effective, covariant cutoff μ_c to the pion momenta, while reproducing the same over-all phase of the result (in this case it is real) which would be obtained by using $k^2+2k\cdot p$ denominators without a cutoff, rather than the $2k\cdot p$ denominators of this model. The computation of (15) is elementary, and yields for the corresponding contribution to F the amount $F_0 = C_0\phi_0(t_1)$, where $t_1 = -t/m^2$, m and μ are the nucleon and pion masses, respectively,

$$C_0 = \frac{3}{16\pi^2}\left(\frac{g^2}{4\pi}\right)^2\left(\frac{\mu_c}{m}\right)^2 I_0\,,$$

$$\begin{aligned}\phi_0(t_1) &= \int_0^1 dx[1+x(1-x)t_1]^{-1}\\ &= [\tfrac{1}{4}t_1(\tfrac{1}{4}t_1+1)]^{-1/2}\ln[(\tfrac{1}{4}t_1)^{1/2}+(\tfrac{1}{4}t_1+1)^{1/2}]\,,\end{aligned} \tag{16}$$

and

$$I_0 = \beta\int_0^\infty db_1\int_0^\infty db_2\, e^{-\mu^2(b_1+b_2)}(b_1+\beta)^{-1}(b_2+\beta)^{-1} \times(b_1+b_2+2\beta)^{-1} \simeq 2\ln 2\,,$$

where the last approximation to I_0 is obtained by neglecting the ratio of pion to cutoff mass, $(\mu/\mu_c)^2 \ll 1$. For large $-t$, $\phi_0(t_0) \sim (2/t_1)\ln t_1$, so that F_0 alone produces a damping of the nonsoft form factors by a multiplicative factor which varies[13] from unity at $t=0$ to a fixed constant e^{-C_0} as $-t\to\infty$. Choosing $\mu_c \sim m$, $g^2/4\pi \sim 14$, one may estimate $C_0 \sim 5$, and then find that there is not enough variation in $\exp[+f_0(t)]$ to fit the $10 < -t < 20$-(BeV)^2 data[14] (it drops by a factor of 2 compared to the corresponding dipole-fit decrease by a factor of 4). If C_0 is increased to ~ 10, there will be a sufficient dropoff in this momentum range; but then a multiplicative constant >1 will be required to get the correct numerical value, and it is difficult to see how such a constant could arise from other soft or nonsoft damping mechanisms at large $-t$. One concludes that the term $\exp[+f_0(t)]$ alone will give a description of the large-momentum-transfer form factors and the pp scattering data no better than existing power-decrease fits.[15] When combined with other damping mechanisms, a rough fit to the form-factor data can be made (e.g., $(m^2/m_\rho^2)[1+|t|/m_\rho^2]^{-1} \times e^{f_0(t)}$, with $C_0 \sim 5$). However, the wide-angle experiments are not known with sufficient precision to be able to distinguish competing mechanisms.

Quadratures over the remaining iterates of (14) are somewhat more complicated to evaluate. They are defined, by renormalization, to give an exponential of zero at $t=0$, and may also be estimated to give little contribution asymptotically. For this, it is convenient to make use of an alternate form of cutoff which has found previous application in zero-mass problems[16] and which, when applied to (15), yields a result equivalent to that of (16). If, for simplicity, we discard the pion mass at the very beginning, the pion propagator $\Delta_c(x) = (i/4\pi^2)(x^2+i\epsilon)^{-1}$ may be replaced by the regularized form $(i/4\pi^2)(x^2+l^2)^{-1}$, where l is understood to have a small positive imaginary part until, and $\mathrm{Re}\,l$ may be set equal to μ_c^{-1} at, the end of the calculation. Writing the renormalized soft-pion-pair exponent in the form

$$f(t) = C_0[\phi_0(t_1)-\phi_0(0)] + \sum_{n=0}^\infty C_n[\phi_n(t_1)-\phi_n(0)]\,,$$

quadrature over the nth iterate of (14) yields

$$\begin{aligned}C_n\phi_n(t) = \frac{6g^{2(n+3)}}{n+1}\int_0^\infty\cdots\int_0^\infty d\xi_1 d\eta_1\cdots d\xi_{n+1}d\eta_{n+1}\\ \times\Delta_c(2\xi_1 p+2\eta_1 p')\Delta_c(2\eta_1 p'\times 2\xi_2 p)\cdots\\ \times\Delta_c(2\xi_{n+1}p+2\eta_{n+1}p')\Delta_c(2\eta_{n+1}p'+2\xi_1 p)\,.\end{aligned} \tag{17}$$

In the limit of large $-t$, (17) can be written as $\phi_n = t_1^{-(n+1)}$, and

$$C_n = \frac{6g^4}{n+1}\left(\frac{g}{16\pi^2 m^2}\right)^{2(n+1)}\int_0^\infty d\xi_1\cdots \times\int_0^\infty d\eta_{n+1}\left[\xi_1\eta_1+\frac{l^2}{m^2}\right]^{-1}\cdots\left[\eta_{n+1}\xi_1+\frac{l^2}{m^2}\right]^{-1}, \tag{18}$$

where we have rotated each of the ξ_i contours by $-i\pi$ in order to cast the multiple integrals of (18) into positive-definite form. These integrals may be estimated in a simple way by transferring the regulator dependence from the denominators to the lower limits of integration, to obtain the quantities

$$\int_{l/m}^\infty d\xi_1\cdots\int_{l/m}^\infty d\eta_{n+1}[\xi_1\eta_1]^{-1}[\eta_1\xi_2]^{-1}\cdots[\eta_{n+1}\xi_1]^{-1} = \left(\frac{m}{l}\right)^{2(n+1)},$$

where we may make the identification $l \sim \mu_c^{-1}$. In this way, one finds $C_n \sim [6g^4/(n+1)]G^{n+1}$, with $G = (g\mu_c/16\pi^2 m)^2$. Summing over all $n \geqslant 1$, the large $-t$ behavior of the unsubtracted (17) has the form

$$-6g^4[G/t_1+\ln(1-G/t_1)] \sim O(G^2/t_1^2)$$

[13] This discussion assumes that the cutoff μ_c^2 is really a constant, and not dependent upon momentum transfer. However, there is no *a priori* reason why, for large momentum transfer, μ_c^2 cannot be thought of as proportional to $-t$, for this really depends upon the details of the nonsoft part of the form factors, and is left unspecified by this analysis. Were this the case, $c_0 \sim t_1$ in the asymptotic region, and the form factors would be supplied with an exponentially decreasing factor, which would tend to fall off much too fast to reproduce the data.

[14] D. Coward *et al.*, Phys. Rev. Letters **20**, 292 (1968).

[15] T. Gaisser, Phys. Rev. D **2**, 1337 (1970), and the first paper of Ref. 2.

[16] H. M. Fried, Phys. Rev. **174**, 1725 (1968).

and gives a negligible contribution. If one thinks[13] of $\mu_c^2 \sim |t|$ for large $-t$, then $G/t_1 \sim$const, and these terms could contribute a finite amount to the soft exponent. In neither case do they represent a significant addition to the t dependence of the lowest-order exponent $F_0(t)$ calculated above.

The net effect of these higher iterates is to produce, at large $-t$, a finite addition to the renormalized exponent of amount $-\sum_{n=1}^{\infty} F_n(0)$, where

$$F_n(0)=C_n\phi_n(0)=\frac{6g^4}{n+1}\left(\frac{g}{16\pi^2 m^2}\right)^{2(n+1)} I_n\,,$$

with

$$I_n=\int_0^\infty d\xi_1\cdots\int_0^\infty d\eta_{n+1}\left[(\xi_1+\eta_1)^2+\frac{l^2}{m^2}\right]^{-1}\cdots \times\left[(\eta_{n+1}+\xi_1)^2+\frac{l^2}{m^2}\right]^{-1}. \quad (19)$$

In arriving at the positive-definite forms of (19), a rotation of each of the ξ_i and η_i contours of (17) by an angle $-i\frac{1}{2}\pi$ has been made. As described in the Appendix, upper and lower bounds on I_n may be found in the form

$$\left(\frac{\pi}{2\sqrt{2}}\frac{m}{l}\right)^{2(n+1)} \geqslant I_n \geqslant \left(\frac{m}{l}\frac{1}{\sqrt{120}}\right)^{2(n+1)}, \quad (20)$$

and since these are similar in structure, we may take an optimistic value of $I_n \sim (m/l)^{2(n+1)}$ to overestimate the sum as

$$\sum_{n=1}^{\infty} F_n(0) \lesssim 6g^4\left[\ln\left(\frac{1}{1-G}\right)-G\right]. \quad (21)$$

If again $\mu_c \sim m$, $g^2/4\pi \sim 14$, then $G \sim (g^2/4\pi)(4\pi)^{-3} \ll 1$, and $\sum_{n=1}^{\infty} F_n(0) \sim 4.6$, which roughly duplicates the contribution of $F_0(0)$, with the effect of giving an essentially zero form factor at infinite momentum transfer. A value of I_n closer to its lower bound would give less damping, which would then be due principally to $F_0(0)$ at infinite momentum transfer. For large but finite $-t$, the fractional falloff is controlled mainly by $F_0(t)$.

IV. SUMMARY

Although a thorough study of these effects cannot be claimed here, it is difficult to reproduce the existing large-momentum-transfer form-factor data using the soft-pseudoscalar-pion-pair contributions alone. There are, of course, other soft-meson processes[2,15] which give a quite respectable fit, and it would require very precise measurements to be able to untangle the different damping mechanisms. Hence, there is at present no compelling argument in favor of such soft pseudoscalar contributions—which, in the light of chiral cancellations and the efficacy of soft chiral pions in other contexts,[8] is no surprise. The use of pseudoscalar theory to illustrate possible momentum-transfer-dependent damping effects is probably somewhat artificial, in the sense that an almost-chiral model of soft pions would really be more appropriate.

ACKNOWLEDGMENTS

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APPENDIX

(a) An upper bound on I_n may be written in the form

$$I_n \leqslant \int_0^\infty d\xi_1\cdots\int_0^\infty d\eta_{n+1}\left(\xi_1^2+\eta_1^2+\frac{l^2}{m^2}\right)^{-1}\cdots \times\left(\eta_{n+1}^2+\xi_1^2+\frac{l^2}{m^2}\right)^{-1},$$

and each integral over the ξ_i variable replaced by

$$\frac{1}{2}\int_{-\infty}^{+\infty} d\xi(\xi^2+\Lambda_a^2)^{-1}(\xi^2+\Lambda_b^2)^{-1}=\tfrac{1}{2}\pi[\Lambda_a\Lambda_b(\Lambda_a+\Lambda_b)]^{-1},$$

where $\Lambda^2=\eta^2+l^2/m^2$. When each factor of $\Lambda_a+\Lambda_b$ is replaced by its smallest value $2l/m$, the remaining η_i integrations can be performed, and yield the upper bound of (20).

(b) A lower bound on I_n may be written in the form

$$I_n \geqslant \int_0^\infty d\xi_1\cdots\int_0^\infty d\eta_{n+1}\left(\xi_1+\eta_1+\frac{l}{m}\right)^{-2}\cdots \times\left(\eta_{n+1}+\xi_1+\frac{l}{m}\right)^{-2},$$

or, changing to a primed set of variables,

$$I_n \geqslant \int_{l/m}^\infty d\eta_1'\cdots\int_{l/m}^\infty d\eta_{n+1}'\int_0^\infty d\xi_1\cdots \times\int_0^\infty d\xi_{n+1}(\xi_1+\eta_1')^{-2}\cdots(\eta_{n+1}'+\xi_1)^{-2}.$$

But

$$(\xi+\eta_a')^2(\xi+\eta_b')^2<\{[\xi+\tfrac{1}{2}(\eta_a'+\eta_b')]^2+\tfrac{1}{2}\eta_a'\eta_b'\}^2\,,$$

and therefore

$$I_n \geqslant \int d\eta_1' \cdots \int d\eta'_{n+1} \int_{\frac{1}{2}(\eta_1'+\eta_2')}^{\infty} dx_1 \cdots \int_{\frac{1}{2}(\eta_{n+1}'+\eta_1')}^{\infty} dx_{n+1} (x_1^2+\tfrac{1}{2}\eta_1'\eta_2')^{-2} \cdots (x_{n+1}^2+\tfrac{1}{2}\eta_{n+1}'\eta_1')^{-2}$$

$$\geqslant \int d\eta_1' \cdots \int d\eta_{n+1}' \int dx_1 \cdots \int dx_{n+1} (x_1+\tfrac{1}{2}(\eta_1'\eta_2')^{1/2})^{-4} \cdots (x_{n+1}+(\tfrac{1}{2}\eta_{n+1}'\eta_1')^{1/2})^{-4}$$

$$=(\tfrac{1}{3})^{n+1} \int d\eta_1' \cdots \int d\eta_{n+1}' [\tfrac{1}{2}(\eta_1'+\eta_2')+(\tfrac{1}{2}\eta_1'\eta_2')^{1/2}]^{-3} \cdots [\tfrac{1}{2}(\eta_{n+1}'+\eta_1')+(\tfrac{1}{2}\eta_1'\eta_{n+1}')^{1/2}]^{-3}$$

$$\geqslant (\tfrac{1}{3})^{n+1} \int d\eta_1' \cdots \int d\eta_{n+1}' \left\{\frac{1}{\sqrt{2}}[(\eta_1')^{1/2}+(\eta_2')^{1/2}]\right\}^{-6} \cdots \left\{\frac{1}{\sqrt{2}}[(\eta_1')^{1/2}+(\eta_{n+1}')^{1/2}]\right\}^{-6}$$

$$=\left(\frac{8}{3}\right)^{n+1} \int_0^{m/l} dx_1 \cdots \int_0^{m/l} dx_{n+1}\, x_1^{-2} \cdots x_{n+1}^{-2} (x_1x_2)^3[(x_1)^{1/2}+(x_2)^{1/2}]^{-6} \cdots (x_{n+1}x_1)^3[(x_1)^{1/2}+(x_{n+1})^{1/2}]^{-6}$$

$$\geqslant \left(\frac{l}{2m}\right)^{3(n+1)} (\tfrac{1}{3})^{n+1} \left[\int_0^{m/l} dx\, x^4\right]^{n+1} = \left[\left(\frac{l}{2m}\right)^3 \times \tfrac{1}{3} \times \tfrac{1}{5} \left(\frac{m}{l}\right)^5\right]^{n+1},$$

which is the lower bound of (20).

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Threshold Theorem for the Radius of the F_{2V} Nucleon Form Factor*

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We develop a method for expanding the absorptive part of the sidewise dispersion relation for the nucleon F_{2V} radius. We assume that a threshold expansion is valid and we are able to obtain the exact first-order terms in this expansion in the limit of the pion mass going to zero. We are able to prove that the first-order terms in this expansion come solely from the π-N intermediate state. Thus, the expansion provides justification for the retention of only the π-N intermediate state in a first attempt at describing the radius, as in Drell and Silverman. Furthermore, the expansion provides a handhold for attempting to estimate the corrections to the first-order term coming from the 2π-N and higher intermediate states. It must be emphasized that we are not providing justification for the assumption of threshold dominance. We are trying to follow that assumption to its logical conclusion. That is, if the threshold dominance idea is good, then a threshold expansion provides a logical way to calculate low-energy parameters. We have provided such an expansion for the F_{2V} radius.

I. INTRODUCTION

IN this paper we develop a method for expanding the absorptive part of the sidewise dispersion relation[1] for the nucleon F_{2V} radius. We assume that a threshold expansion is valid and we are able to obtain the exact first-order terms in this expansion, in the limit of the pion mass going to zero. We are able to prove that the first-order terms in this expansion come solely from the π-N intermediate state. Thus the expansion provides justification for the retention of only the π-N intermediate state in a first attempt at describing the radius as in Drell and Silverman.[2] Furthermore, the expansion provides a handhold for attempting to estimate the corrections to the first-order term coming from the 2π-N and higher intermediate states.

We now repeat the preceding in a more mathematical way.

The sidewise dispersion relation[1] for the F_{2V} nucleon form factor is

$$F_{2V}(l^2,M^2)=\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty} \frac{d(W^2)\,\mathrm{Abs}\,F_{2V}(l^2,W^2)}{W^2-M^2}, \quad (1)$$

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[1] A. M. Bincer, Phys. Rev. 118, 855 (1960).
[2] S. D. Drell and D. J. Silverman, Phys. Rev. Letters 20, 1325 (1968).