Unitarity and Fixed Poles of Weak Amplitudes in a Perturbation-Theory Model*

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We show that weak unitarity equations in general require the existence of fixed simple poles in weak amplitudes at right-signature nonsense-nonsense points and are consistent with the presence of higher-order fixed poles with constant or polynominal residues. These fixed poles are exhibited in a perturbation-theory model.

I. INTRODUCTION

It has long been known that because of the linear unitarity equations which they obey, weak and electromagnetic amplitudes can have fixed poles in the angular momentum plane at right-signature points in contrast to purely strong-interaction amplitudes. The actual presence or absence of such poles in weak amplitudes is, however, controlled by dynamics and there is no known general method for locating them or for computing their residues. The only known exceptions are the current-algebra sum rules[1] which assert the presence of fixed $J=1$ poles in the double-flip t-channel helicity amplitudes of Compton scattering with the residues determined by local current commutation relations. Nevertheless, current-algebra sum rules do not provide a general framework for determining all fixed-pole residues and, furthermore, the dynamical assumptions[2] (infinite-momentum limit or unsubtracted dispersion relations) necessary for deriving the sum rules may fail; i.e., possible subtraction constants could contribute additional unknown pieces to the fixed-pole residues. Hence empirical data are usually used[3] to infer the presence of fixed poles and to determine the residues via finite-energy sum rules. Clearly, then, we are very far from a complete theoretical understanding of the phenomenon of fixed poles.

The purpose of the present paper is to explore the restrictions imposed on the fixed poles and their residues in various weak amplitudes by the unitarity equations which they obey. Linear unitarity equations, though much less restrictive than strong-interaction unitarity, do nevertheless place consistency constraints on fixed poles and their residues in various weak amplitudes. More importantly, they may be used, together with the known current-algebra fixed poles,[1] to uncover the presence of additional fixed poles which cannot be determined by current algebra alone. We have in mind in particular the s- or u-channel unitarity equations relating the photoproduction and Compton amplitudes. We show in Sec. II that in general fixed poles in photoproduction at sense-nonsense points lead via unitarity to fixed poles in Compton amplitudes at nonsense-nonsense points. Furthermore, the absorptive part of the fixed-pole residue in the Compton amplitude is also determined. It is permissible, however, for the Compton amplitude to have additional fixed poles besides those determined from unitarity with the condition that their residues lack absorptive parts. To determine these additional poles, dynamical models are necessary. To this end, we consider in Sec. III a perturbation-theory model which verifies our previous conclusions and shows in addition that higher-order fixed poles are to be expected at nonsense-nonsense points. The residues of these higher-order poles lack absorptive parts and are, of course, model dependent. The details of the perturbation calculations are to be found in the two Appendices.

II. UNITARITY AND FIXED POLES

We illustrate in this section, by means of an example, that fixed poles at nonsense-nonsense right-signature points may be required by the unitarity equations. Consider the photoproduction amplitude T_μ and Compton scattering amplitudes $T_{\mu\nu}$ of scalar particles given, respectively, by

$$T_\mu = i\int d^4x\, e^{ikx}\langle p|J_\mu(x)|k',p'\rangle = \sum_i I_\mu{}^i A_i\,, \tag{1}$$

$$T_{\mu\nu} = i\int d^4x\, e^{ikx}\langle p|T[J_\mu(x),J_\nu(0)]|p'\rangle = \sum_i I_{\mu\nu}{}^i B_i\,, \tag{2}$$

where

$$I_\mu{}^i = [(k'-p')_\mu,\ (k+p)_\mu,\ (k-p)_\mu]\,, \tag{3}$$

$$\begin{aligned} I_{\mu\nu}{}^i = [&(k'-p')_\mu(k'-p')_\nu,\ (k'-p')_\mu(k-p)_\nu,\\ &(k'-p')_\mu(k+p)_\nu,\ (k-p)_\mu(k'-p')_\nu,\\ &(k-p)_\mu(k-p)_\nu,\ (k-p)_\mu(k+p)_\nu,\\ &(k+p)_\mu(k'-p')_\nu,\ (k+p)_\mu(k-p)_\nu,\\ &(k+p)_\mu(k+p)_\nu,\ g_{\mu\nu}]\,. \end{aligned} \tag{4}$$

We are particularly interested in the s-channel helicity amplitudes $T_+{}^s$ and $T_{++}{}^s$ defined by

$$T_+{}^s = \epsilon_\mu{}^+ T_\mu\,, \tag{5}$$

$$T_{++}{}^s = \epsilon_\mu{}^+ T_{\mu\nu}\epsilon_\nu{}^{+*}\,, \tag{6}$$

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[1] J. B. Bronzan *et al.*, Phys. Rev. Letters **18**, 32 (1967); Phys. Rev. **157**, 1448 (1967); V. Singh, Phys. Rev. Letters **18**, 36 (1967).

[2] See, e.g., S. L. Adler and R. F. Dashen, *Current Algebra* (Benjamin, New York, 1968); B. Renner, *Current Algebra* (Pergamon, London, 1968).

[3] See, e.g., G. C. Fox and D. Z. Freedman, Phys. Rev. **181**, 1628 (1969).

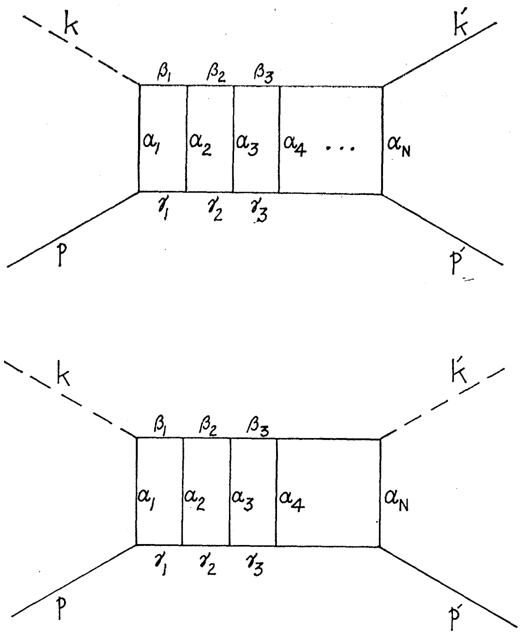

FIG. 1. Ladder graphs in a $\lambda\phi^3$ model for Compton and photoproduction amplitudes. The dashed line denotes a photon and the straight line denotes a scalar meson.

where $\epsilon_\mu{}^+$ is the s-channel c.m. photon polarization vector with helicity $+1$. The s-channel helicity amplitudes can be expressed in terms of the invariant amplitudes

$$T_+{}^s=-\sqrt{2}\,|\mathbf{p}|\,\sin\theta\, A_1\,, \tag{7}$$

$$T_{++}{}^s=-2\,|\mathbf{p}|^2\sin^2\theta\, B_2-\cos^2\tfrac{1}{2}\theta\, B_{10}\,, \tag{8}$$

where $|\mathbf{p}|$ is the magnitude of the three-momentum in the c.m. system and θ is the scattering angle.

The s-channel unitarity equation for $T_{++}{}^s$ just above the two-particle threshold and below the three-particle threshold takes the form

$$T_{++}{}^s(s_+)-T_{++}{}^s(s_-)=\int\rho(s)T_+{}^s(s_+)T_+{}^s(s_-)d\Omega\,, \tag{9}$$

where $\rho(s)$ is the two-particle phase space, $s_\pm$ denote points just above and below the two-particle cut, and the integration is over the solid angle. Defining partial-wave helicity amplitudes $T_{++}{}^J$ and $T_+{}^J$ by

$$T_{++}{}^s(s,\,\cos\theta)=\sum_J\,(2J+1)T_{++}{}^J(s)D_{1,1}{}^J(\cos\theta)\,, \tag{10}$$

$$T_+{}^s(s,\,\cos\theta)=\sum_J\,(2J+1)T_+{}^J(s)D_{1,0}{}^J(\cos\theta)\,, \tag{11}$$

the s-channel unitarity equation decomposes, upon using orthogonality relations for the D^J functions, into component equations

$$T_{++}{}^J(s_+)-T_{++}{}^J(s_-)=\rho(s)T_+{}^J(s_+)T_-{}^J(s_-)\,. \tag{12}$$

Using the Sommerfeld-Watson transform to define analytically continued partial-wave amplitudes $T_{++}(s,J)$ and $T_+(s,J)$, Eq. (12) becomes

$$T_{++}(s_+,J)-T_{++}(s_-,J)=\rho(s)T_+(s_+,J)T_+(s_-,J)\,, \tag{13}$$

where J may now take on complex values and agrees with the physical partial-wave amplitude at even-integer J values ($J\geq 2$). Consider in particular the point $J=0$, which is a sense-nonsense point for $T_+(s,J)$ and a nonsense-nonsense point for $T_{++}(s,J)$. Since $T_{++}(s_+,J)$ is nonlinearly related to T_+, any singular behavior in $T_+(s_+,J)$ must be matched by similar behavior in T_{++} on the left-hand side of the equation. Let us suppose that $T_+(s,J)$ has a fixed pole at $J=0$

$$T_+(s_\pm,J)\sim J^{1/2}\beta_+(s_\pm)/J\,, \tag{14}$$

where $\beta(s_+)$ is the residue and the $J^{1/2}$ factor comes from $D_{1,0}{}^J$.[4] Substituting this into the unitarity equation, we have

$$[T_{++}(s,J)]=\rho(s)\beta_+(s_+)\beta_+(s_-)/J\,, \tag{15}$$

indicating that $T_{++}(s,J)$ must be at least as singular as $1/J$ near $J\sim 0$, i.e., a nonsense-nonsense fixed pole. $T_{++}(s,J)$ may, however, be more singular near $J=0$ since it may take the form

$$T(s,J)=\sum_{n\geq 1}\frac{a_n}{J^n}+\frac{F(s)}{J}\,, \tag{16}$$

where $[F(s)]=\rho(s)\beta_+(s_+)\beta_+(s_-)$ and a_n may be a constant (or, more generally, a polynomial in s), and still be consistent with the unitary equation. The minimal and in some ways the most attractive solution would be for the a_n's to vanish and $F(s)$ would then be completely determined by unitarity and dispersion relations. To examine this possibility, we consider in Sec. III a perturbation-theory model which satisfies all the requirements of this section. We find that in general one cannot expect all the a_n's to vanish; in particular, $a_2\neq 0$ in the ϕ^3 model. Thus there exists, in the model, a double fixed pole in the J plane in addition to the simple pole required by unitarity.

The above considerations can obviously be generalized to other weak amplitudes. For instance, the sense-nonsense fixed pole in Compton amplitude required by current-algebra sum rules would lead via unitarity to at least a fixed simple pole in photon-photon scattering amplitude at the nonsense-nonsense point. It may, however, also have more singular behavior than is required by unitarity.

III. PERTURBATION-THEORY MODEL

We consider, in perturbation theory, a $\lambda\phi^3$ model for the photoproduction and Compton amplitudes where

[4] See, e.g., A. H. Mueller and T. L. Trueman, Phys. Rev. 160, 1296 (1967).

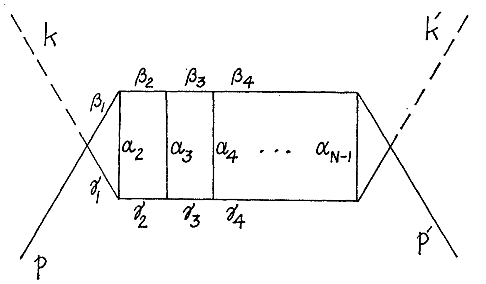

FIG. 2. Contracted diagram of an N-rung ladder graph denoted by $G_N(s)$ in Appendix A.

the electric current is given by $J_\mu{}^3=\epsilon_{3ij}\phi^i\partial_\mu\phi^j$. Our amplitudes are defined by the sum of ladder graphs in the s channel as shown in Fig. 1. They can be shown to satisfy the analytically continued partial-wave unitarity equation. It is known that the photoproduction amplitude, as given by this model, has a fixed pole at $J=0$ in $T_+(s,J)$.[5] Our task here is to determine the behavior near $J=0$ of $T_{++}(s,J)$ as given by the ladder model, and compare with the conclusions of Sec. II. We use the usual techniques of obtaining asymptotic behavior from Feynman integrals,[6] and the existence of fixed poles would manifest itself in fixed asymptotic behavior of various amplitudes. In Appendix A, we find that $T_{++}(s,t)$, for s fixed and negative, and t large, has the behavior

$$T_{++}(s,t) \xrightarrow[t\to-\infty;\, s<0]{} \frac{1}{|\mathbf{p}|^2}\left[t^2B_2-\tfrac{1}{2}tB_{10}\right]\to d\ln t + \tfrac{1}{2}G(s)+C(s)t^{\alpha(s)}, \quad (17)$$

where $G(s)$ is given by the sum of contracted diagrams in Fig. 2 and $\alpha(s)$ is the usual Regge-trajectory function.

The first term on the right-hand side of Eq. (17) gives rise, by the usual Sommerfeld-Watson transform, to a fixed double pole at $J=0$ for $s<0$. For s just above the two-particle threshold, the usual technique for extracting high-energy behavior from Feynman integrals breaks down and consequently the result in Eq. (17) may not hold. In Appendix B, we examine the box diagram in detail near threshold using the technique of Cheng and Wu[7] and find that the same fixed behavior obtains there whereas the Regge term is promoted. We believe that the same situation obtains for all ladder graphs and therefore conclude that a fixed double pole is present in $T_{++}(s,J)$ with constant residue (i.e., no absorptive part in s) consistent with the constraint we derived from the unitarity equation. It would be interesting to investigate experimentally the (possible phenomenological) implications of fixed double and simple poles in hadronic annihilation into two photons.

[5] H. R. Rubinstein, G. Veneziano, and M. A. Virasoro, Phys. Rev. **167**, 1441 (1968).
[6] R. J. Eden *et al.*, *The Analytic S Matrix* (Cambridge U. P., Cambridge, England, 1967).
[7] H. Cheng and T. T. Wu, Phys. Rev. Letters **24**, 759 (1970).

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APPENDIX A

In this Appendix we give the details of the calculations which lead to Eq. (17). It is more convenient to calculate, in perturbation theory, the asymptotic behavior of invariant amplitudes B_i defined in Eq. (2) first and to obtain the corresponding asymptotic behavior of the helicity amplitude $T_{++}{}^s$ from Eq. (8). We shall find that crucial cancellations occur between contributions of B_2 and B_{10} in Eq. (8) which are essential to maintain consistency with the unitarity equations. Thus, unitarity restrictions are much weaker for the invariant amplitudes than for the helicity amplitudes.

The Feynman integral of the N-rung ladder-graph contribution to the Compton amplitude has the following representation:

$$T_{\mu\nu}{}^N=\int\Big(\prod_j d\alpha_j d\beta_j d\gamma_j\Big)\left[\left(k_\mu+\frac{1}{\alpha_1 C}A_\mu\right)\times\left(k_\nu'+\frac{1}{\alpha_N C}A_\nu'\right)+\frac{2F_1 g_{\mu\nu}}{\alpha_1\alpha_N C}\right]e^{D/C}, \quad (\text{A1})$$

$$A_\mu=F_1(k'-p')_\mu+(F_3-F_6)(k+p)_\mu-(F_1+F_3+F_6)(k-p)_\mu, \quad (\text{A2})$$

$$A_\mu'=(F_1+F_4+F_5)(k'-p')_\mu+(F_5-F_9)(k+p)_\mu-F_1(k-p)_\mu, \quad (\text{A3})$$

where α_j, β_j, and γ_j are the Feynman parameters labeled in Fig. 1; D and C are the usual scalar-ladder-graph discriminate and determinant,[6] respectively; and F_i are functions of the Feynman parameters,[8] with

$$F_1=\prod_{i=1}^{N} a_i.$$

The advantage of the representation (A1) is that it allows us to separate, by inspection, the contributions of the ladder graphs to the B_i's. In particular, using (A1) and the fact that D has a simple dependence on t, viz.,

$$D=F_1t-J(\alpha,\beta,\gamma,s)/C, \quad (\text{A4})$$

[8] I. G. Halliday, Nuovo Cimento **51**, 970 (1967).

we can prove the following identity:

$$B_2{}^N = -\frac{1}{2}\frac{\partial}{\partial t} B_{10}{}^N, \tag{A5}$$

with

$$B_{10}{}^N = 2\int \left(\prod_j d\alpha_j d\beta_j d\gamma_j\right)\frac{F_1(\alpha)}{\alpha_1\alpha_N C} e^{D/C}. \tag{A6}$$

Using the usual technique,[6] we easily find the asymptotic behavior of $B_{10}{}^N$, for $N>2$, as

$$B_{10}{}^{N>2} \xrightarrow[t\to-\infty;\, s<0]{} t^{-1}\ln t\, G_N(s) + b_N(s)t^{-1} + \cdots, \tag{A7}$$

where

$$G_N(s) = 2\int \left(\prod_j d\alpha_j d\beta_j d\gamma_j\right)\frac{1}{C^2} e^{D/C}\bigg|_{\alpha 1=\alpha N=0}, \tag{A8}$$

corresponding to the contracted diagrams in Fig. 2.

The box diagram ($N=2$), being in this case a singular configuration, has the behavior

$$B_{10}{}^2 \to -dt^{-1}(\ln t)^2 + G_2(s)t^{-1}\ln t + b_2(s)t^{-1} + \cdots, \tag{A9}$$

where d is a constant.

Upon summing over all N, one obtains[9]

$$B_{10} \to dt^{-1}(\ln t)^2 + G(s)t^{-1}\ln t + b(s)t^{-1} + C(s)t^{\alpha(s)-1}. \tag{A10}$$

The first three terms in (A10), if uncanceled, correspond in the J plane to the presence of fixed triple and double poles as well as a simple pole in $T_{++}{}^s$. In particular, the residue of the double pole has the absorptive part in s forbidden by unitarity. From (A5), we get

$$B_2 \to \tfrac{1}{2}dt^{-2}(\ln t)^2 - dt^{-2}\ln t + \tfrac{1}{2}G(s)t^{-2}\ln t + \tfrac{1}{2}[b(s)-G(s)]t^{-2} - \tfrac{1}{2}C(s)t^{\alpha(s)-2}. \tag{A11}$$

Substituting Eqs. (A10) and (A11) into Eq. (2), we obtain

$$T_{++} \to d\ln t + \tfrac{1}{2}G(s) + C(s)t^{\alpha(s)}. \tag{17}$$

We see that the triple pole as well as the unitarity-violating part of the double-pole residue have canceled.

APPENDIX B

Our analysis thus far has been limited to considerations in the region $s<0$. In this Appendix we investigate the box diagram in the region of threshold, $s\simeq 4$. It is seen that no promotion effect occurs for the fixed pole, so that the fixed-double-pole behavior in the present model persists above the s-channel threshold. We make use of an adaptation of the techniques of Cheng and Wu.[7]

For the contribution of the box graph to B_{10}, we find the Feynman parametrization to be given up to constant factors by

$$B_{10}{}^2(s,t) = \int d\alpha_1 d\alpha_2 d\beta_1 d\gamma_1 \delta(1-a_1-\alpha_2-\beta_1-\gamma_1) \times[s\beta_1\gamma_1 + t\alpha_1\alpha_2 + (\alpha_1+\alpha_2)(\beta_1+\gamma_1) - 1 + i\epsilon]^{-1}. \tag{B1}$$

Performing the Mellin transform of $B_{10}(s,t)$ with respect to t at $s=4$, we get

$$\tilde{B}_{10}{}^2(s=4,\,\xi) \simeq -\tfrac{1}{4}\pi\csc(\pi\xi)\,\Gamma^2(\xi)\Gamma^{-1}(2\xi) \times e^{-i\pi\xi}\frac{1}{2\xi+1}\left(\frac{1}{\xi}+\frac{1}{2\xi+1}+\cdots\right), \tag{B2}$$

where we have dropped poles in ξ at $\xi < -\frac{1}{2}$. B_{10} has simple, double, and triple poles at $\xi=0$ and a simple pole at $\xi=-\frac{1}{2}$. These yield high-t behavior t^{-1}, $t^{-1}\ln t$, $t^{-1}(\ln t)^2$, and $t^{-3/2}$, respectively, for B_{10}. The $t^{-3/2}$ term is the promoted Regge term ($t^{-2}\ln t$ away from threshold), whereas the remaining terms agree with the result in (A9) obtained below the threshold. Thus at least for the box diagram, the fixed asymptotic behavior is not affected by threshold effects, and it appears likely that the same holds true for all ladder graphs.

[9] In general the set of ladder graphs that we have considered is not gauge invariant by themselves. One must attach the external photons in all possible ways to the ladder to obtain a gauge-invariant amplitude. However, by the usual scaling arguments (Ref. 6), it is easy to see that only the graphs that we considered will contribute to the fixed behavor; hence the residues of the fixed singularities are gauge invariant. The same is not true of the moving J-plane singularities. One must sum all ladders with the photons attached in all possible ways to preserve gauge invariance and to obtain a simple Regge-pole term in the asymptotic behavior. For a treatment of the analogous problem in photoproduction see Y. Shimizu, Nuovo Cimento Letters **3**, 633 (1970).