Relativistic Eikonal Expansion*

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A systematic study is made of the relativistic analog of the eikonal expansion. Approximate amplitudes are obtained for the scattering of a high-energy projectile off a single particle and off a compound system. For the two-body problem our results are valid for all momentum transfers, while for the three-body amplitude we obtain the relativistic analog of the Glauber formula.

I. INTRODUCTION

THE study of very-high-energy scattering in relativistic quantum mechanics is interesting both because of its direct application to phenomenology and because one hopes that the underlying theory may be relevant to the real world. Since there has recently been a great deal of interest in both of these applications, it seems worthwhile to make a more systematic study into certain aspects of the problem.

The relativistic eikonal expansion will be discussed for both two- and three-body scattering processes.[1] The spirit of our approach will be very similar to that of a related work on the nonrelativistic problem.[2] Our procedure here will be to develop a systematic expansion for the relativistic Green's functions and to use these approximate Green's functions to evaluate a selected sum of Feynman graphs. These approximate propagators can be used with a functional approach[3–6] or a graphical approach with careful counting.[7–10] This latter procedure becomes very cumbersome when corrections to the leading terms are evaluated. Therefore, we have chosen to present our results using the more convenient functional methods.

In Sec. II, the asymptotic solution of the Klein-Gordon equation will be discussed, and the analogous expansion of the Green's functions and resolvents will be presented. In Sec. III, several eikonal expansions of the T matrix will be presented for one- and two-body scattering, including one which is valid for all momentum transfers. This discussion is extended to the scattering of a projectile from a two-body bound state in Sec. IV. In Sec. V these results are briefly compared with the familiar nonrelativistic ones.

II. ASYMPTOTIC SOLUTION OF KLEIN-GORDON EQUATION

In order to carry out the program outlined in the Introduction, a systematic expansion of the solution to the Klein-Gordon (KG) equation must be developed. The procedure will be to sum the leading terms at high energies using familiar eikonal methods and then develop a perturbation expansion about the zero-order result. In working with this approach, one should be cognizant of the fact that such asymptotic series can fail for (at least) two reasons. The series may not converge and/or the integrals defining the coefficients may not exist. In this latter case, it is sometimes possible to rearrange the expansion. Such a procedure was carried out in I and it also improved the convergence of the series.

Let us first solve the KG equation in a scalar external potential A. The vector case will be treated later. For generality, the equation will be solved off the mass shell and is written as

$$(\Box+m^2+A)\psi=(m^2-p^2)\varphi\,, \tag{1}$$

where

$$\varphi=e^{-ip\cdot x}.$$

If the solution is written in product form,

$$\psi(x)=\rho(x)\varphi(x)\,,$$

then the modulating factor ρ satisfies the equation

$$(\Box+m^2+A-p^2-2ip\cdot\partial)\rho(x)=(m^2-p^2)\,. \tag{2}$$

The boundary condition is that ρ equals 1 if A is zero. In the limit that p gets large, it would seem natural to solve for ρ by treating the term $\Box$ as a perturbation. The lowest-order equation is

$$(m^2+A-p^2-2ip\cdot\partial)\rho_0=(m^2-p^2)\,,$$

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[1] A preliminary version of these results were given by one of us (R. B.) at the International Conference on the Three-Body Problem in Nuclear and Particle Physics, 1969. See *Three-Body Problem*, edited by J. McKee and P. Rolph (North-Holland, Amsterdam, 1970).

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and its solution is easily seen to be

$$\rho_0(x)=1-i\int_0^\infty d\tau\, A(x-2p\tau)\exp[-i\chi_s(x,\tau)], \quad (3)$$

where

$$\chi_s(x,\tau)=\tau(m^2-p^2)+\int_0^\tau d\tau'\, A(x-2p\tau'). \quad (4)$$

On the mass shell, the integral defining ρ_0 becomes a perfect derivative, and one finds

$$\rho_0(x)=\exp\left[-i\int_0^\infty d\tau'\, A(x-2p\tau')\right],$$

the familiar-looking eikonal result.

The equation satisfied by ρ_1, the first-order correction to ρ_0, is

$$(m^2+A-p^2-2ip\cdot\partial)\rho_1=-\Box\rho_0,$$

and its solution is

$$\rho_1(x)=-i\int_0^\infty d\tau\,\exp[-i\chi_s(x,\tau)]\Box\rho_0(x-2p\tau). \quad (5)$$

Since the τ integrals scale with the magnitude of p, this is easily seen to be an expansion in inverse powers of p_0, or, equivalently, $p_0+|\mathbf{p}|$.

In the case of a vector potential B_μ, the solution to the KG equation follows in the same way as in the scalar case. It is convenient to choose the perturbation in a different way, however. The equation to be solved is

$$[M^2-(i\partial-B)^2]\psi=(m^2-p^2)\varphi. \quad (6)$$

Introducing again the product form for ψ, the equation to be solved for ρ is

$$(M^2+2p\cdot B-p^2-2ip\cdot\partial)\rho-(m^2-p^2) \\ =(-\Box+B^2-i\partial\cdot B-iB\cdot\partial)\rho.$$

Choosing the operator on the right to be the perturbation, the lowest-order solution for ρ is

$$\rho_0(x)=1-i\int_0^\infty d\tau\, 2p\cdot B(x-2p\tau)\exp[-i\chi_v(x,\tau)], \quad (7)$$

where

$$\chi_v(x,\tau)=\tau(m^2-p^2)+\int_0^\tau d\tau'\, n\cdot B(x-n\tau'), \quad (8)$$

and

$$n_\mu=2p_\mu/s^{1/2}.$$

It may be convenient to include the B^2 terms in the lowest-order problem in which case χ_v is changed in an obvious manner.

The first-order solution is easily found, and the expansion in this vector case is in inverse powers of p_0, just as in the scalar-exchange problem.

The Green's function for the linearized equation can be read off from the solutions for ρ_0 and ρ_1. Using these propagators and the formal approach developed in I will allow considerable flexibility in expanding the scattering matrix. Since considerable insight has been developed in the nonrelativistic problem in the use of such expansion methods to improve convergence, this flexibility may prove very useful in the relativistic discussions.

The eikonal Green's function in an external field C is

$$G_p[C]=-i\int_0^\infty d\tau\,\delta(y-x-2p\tau) \\ \times\exp[-i\chi(y,\tau)-2i\tau p^2], \quad (9)$$

where p is the momentum about which the exact propagator has been expanded. It is not necessarily on the mass shell. The eikonal phase χ is either χ_s or χ_v.

The resolvent for the KG equation in the eikonal approximation is

$$\psi_P=R_P\varphi_p=(1+G_p[C]C)\varphi_p.$$

The integrations are easily carried out and one finds the expected result

$$\psi_P=\rho_0\varphi_p,$$

where ρ_0 has been given above.

The exact propagator g and the eikonal propagator G_p in an external potential are related by

$$G_p^{-1}-g^{-1}=N_p, \quad (10)$$

and in the scalar case, $N_P=(p-i\partial)^2$. This is the perturbation that must be used to improve the eikonal expansion. For example, one finds

$$g=(1-G_PN_P)^{-1}G_p,$$

and the exact resolvents can be written as

$$r=1+gC=R_P+gN_P(R_p-1)$$

and

$$r^T=1+Cg=R_P{}^T+(R_P{}^T-1)N_Pg. \quad (11)$$

III. T-MATRIX EXPANSIONS

A. Single Particle in External Field

Using the previously derived relations it is a simple matter to obtain various forms of the eikonal expansion for the scattering matrix. First, consider the form for T expressed in terms of the final-momentum eikonal resolvent $R_f{}^T$,

$$T=r^TC=R_f{}^TC+(R_f{}^T-1)N_fgC.$$

Now if g is expanded eikonally about the initial momentum, that is,

$$gC=(1+gN_i)(R_i-1),$$

then the T matrix achieves the familiar form

$$T=R_f{}^TC+(R_f{}^T-1)N_f(R_i-1) \\ +(R_f{}^T-1)N_fgN_i(R_i-1). \quad (12)$$

This is the form derived in I. The first term is the usual eikonal amplitude and the second term is the Saxon-Schiff correction. The third term is the remainder expressed in terms of the exact Green's function g. One can also proceed by expanding the resolvent in terms of R_i rather than R_f^T.

To simplify the writing of subsequent formulas, it is convenient to introduce a parameter a in the phase integrals by defining

$$C^i=\int_a^\infty d\tau\, C(x-2p_i\tau)$$

and

$$C^f=\int_a^\infty d\tau\, C(x+2p_f\tau)\,.$$

The first two terms in the expansion for T become

$$T=Ie^{-iC^i}+(e^{-iC^f}-1)N_i(e^{-iC^i}-1)\,, \tag{13}$$

where $I=-i\partial/\partial a$, and the limit $a\rightarrow 0$ is to be taken once the derivative has been performed.

The form of the scattering amplitude given by Lévy and Sucher can be derived by considering the identity [*Note added in proof.* A similar derivation has been given by D. R. Harrington, Phys. Rev. D **2**, 2512 (1970).]

$$T=\int_0^1 d\lambda \frac{\partial}{\partial\lambda}T(\lambda C)=\int_0^1 d\lambda\, r^T(\lambda C)Cr(\lambda C)\,.$$

If r^T and r are expanded about p_f and p_i, respectively, the Lévy-Sucher form emerges as the lowest-order term and the corrections are easily written down. The lowest-order result is

$$T_{\rm LS}=\int dx\int_0^1 d\lambda\, C(x)e^{i2\Delta\cdot x-i\lambda LC}\,, \tag{14}$$

where

$$LC=\int_0^\infty d\tau[C(x+2p_f\tau)+C(x-2p_i\tau)]$$
$$=C^f+C^i\,,$$

and the integral over λ is easily evaluated. The momentum transfer 2Δ has been introduced for (p_f-p_i). The symmetry of this form for the T matrix in the initial and final momenta might lead one to expect that it would be more accurate for large momentum transfers than the conventional eikonal. Unfortunately, this does not seem to be the case. A rough estimate of the corrections to the Lévy-Sucher form of T leads one to expect them to be as large as the corrections to the ordinary eikonal although special cancellations in particular problems cannot be ruled out.

B. Two-Particle Scattering

The scattering of two relativistic particles can be derived by using the above results for the scattering of individual particles in external fields and connecting them in an appropriate way. This can be done in a straightforward graphical approach or by using functional techniques. The functional approach of Schwinger,[3] as used by Abarbanel and Itzykson[4] for the eikonal problem, is quite convenient, especially when evaluating the corrections to the eikonal result.

The scattering of particles one and two via the exchange of quanta with propagator and coupling constant Dg^2 can be written as

$$T_{12}=T_1\left(\frac{ig^2D\delta}{\delta C}\right)T_2(C)\bigg|_{C=0}\,. \tag{15}$$

Vertex corrections and pair effects have, of course, been neglected.

As an example of the above procedure, let us derive the Lévy-Sucher form for T_{12}. Using the result in Eq. (14) for T_1, and the fact that

$$\frac{\delta}{\delta C}T_2(C)=r^T(C)r(C)\,, \tag{16}$$

one finds

$$T_{12}=ig^2\int_0^1 d\lambda\, r_2^T(\lambda g^2L_1D)Dr_2(\lambda g^2L_1D)\,.$$

The use of the eikonal approximation for r_2^T and r_2 now yields

$$T_{12}=ig^2\int d\lambda dxdy\, D(x-y)e^{i2\Delta_1\cdot x+i2\Delta_2\cdot y-i\lambda g^2L_1DL_2}\,, \tag{17}$$

where the phase L_1DL_2 is a function of the relative coordinate $z=x-y$ only, and is an integral over the proper time along both sides of the ladder:

$$\begin{aligned}L_1DL_2=&\int_0^\infty d\tau d\sigma[D(z+2P_{1f}\tau-2P_{2f}\sigma)\\&+D(z-2P_{1i}\tau+2P_{2i}\sigma)+D(z+2P_{1f}\tau+2P_{2i}\sigma)\\&+D(z-2P_{1i}\tau-2P_{2f}\sigma)]\\\equiv&\, D^{ff}+D^{ii}+D^{fi}+D^{if}\,.\end{aligned} \tag{18}$$

This is the Lévy-Sucher form for T_{12}. The corrections corresponding to the Saxon-Schiff term in the ordinary eikonal are easily written down.

It is also easy to write down the two-particle amplitude using the standard eikonal expansion given in Eq. (13). For small momentum transfer, only the first term in the expansion of each T matrix need be kept, and one obtains the usual eikonal approximation. However, it is possible to derive an expression which is accurate for all momentum transfers by retaining the first two terms in the expansion of each T matrix. The important point is that at large momentum transfers it is advantageous for all of the momentum to be transferred via the exchange of one hard quanta, rather than having two or more quanta each carry across a fraction of the transferred momentum. This is because the propagators of the exchanged quanta fall off like a power in momentum space, so $D(4\Delta^2)D(0)\gg D(\Delta^2)^2$ for large values of Δ^2. This is analogous to the situation in nonrelativistic potential scattering which was discussed in I.

Expanding each T matrix about the initial momentum, we find

$$\begin{aligned}T_{12}=I_1I_2e^{-ig^2D^{ii}}&+I_1(e^{-ig^2D^{if}}-1)N_{2i}(e^{-ig^2D^{ii}}-1)\\&+I_2(e^{-ig^2D^{fi}}-1)N_{1i}(e^{-ig^2D^{ii}}-1)\\&+[(e^{-ig^2D^{ff}}-1)e^{-ig^2(D^{fi}+D^{if})}\\&\qquad+(e^{-ig^2D^{fi}}-1)(e^{-ig^2D^{if}}-1)]\\&\qquad\times N_{1i}N_{2i}(e^{-ig^2D^{ii}}-1)+\cdots. \qquad (19)\end{aligned}$$

In writing the last term in Eq. (19), we have made use of the fact that D^{fi} and D^{if} cannot carry a large momentum transfer since that would imply that there are at least two large scatterings. As a result, the derivatives in the N's need not be applied to these terms. By expanding the single-particle T matrices about either or both of the final momenta, one obtains three other expressions for T_{12} which are analogous to Eq. (19). It is convenient to take the average of the four terms. One then finds

$$\begin{aligned}T_{12}\cong&\tfrac{1}{4}I_1I_2[e^{-ig^2D^{ii}}+e^{-ig^2D^{if}}+e^{-ig^2D^{fi}}+e^{-ig^2D^{ff}}]\\&+\tfrac{1}{4}\Delta^2I_1[(e^{-ig^2D^{if}}-1)(e^{-ig^2D^{ii}}-1)\\&\qquad+(e^{-ig^2D^{ff}}-1)(e^{-ig^2D^{fi}}-1)]\\&+\tfrac{1}{4}\Delta^2I_2[(e^{-ig^2D^{fi}}-1)(e^{-ig^2D^{ii}}-1)\\&\qquad+(e^{-ig^2D^{ff}}-1)(e^{-ig^2D^{if}}-1)]\\&+\tfrac{1}{4}\Delta^4(e^{-ig^2D^{ff}}-1)e^{-ig^2(D^{fi}+D^{if})}(e^{-ig^2D^{ii}}-1). \qquad (20)\end{aligned}$$

For small momentum transfers, only the first term in Eq. (20) is important and it reduces to the ordinary eikonal approximation. For momentum transfers large compared to the masses, all terms in Eq. (20) are comparable. In the latter case, Eq. (20) has a simple physical interpretation. The two particles come in along their initial directions, building up the phase g^2D^{ii}. They both undergo a single large scattering and leave building up the final phase g^2D^{ff}. The only difference between Eq. (20) and the nonrelativistic case discussed in I is that here it is possible for either line to emit a soft quanta before the large scattering which is not absorbed by the other line until after this scattering. Such events give rise to the phases g^2D^{fi} and g^2D^{if}.

The errors in Eq. (20) can be estimated using the techniques developed in I. If μ is the mass of the exchanged quanta, then the error is of order $\mu^2/s+\Delta\mu/s$.

IV. BOUND-STATE SCATTERING

Our previous discussion can easily be extended to the scattering of compound systems. The types of application that we have in mind are models of high-energy hadron-hadron scattering in which the hadrons are viewed as compound objects, and the scattering of elementary projectiles off nuclei. In the former case, one of course wants to treat the bound-state wave functions covariantly.

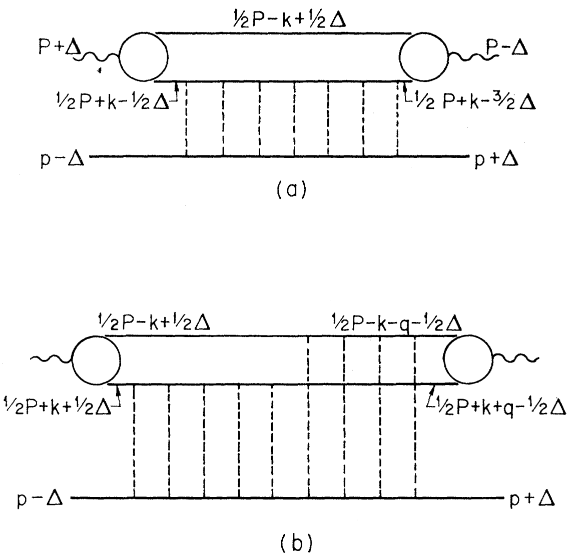

FIG. 1. (a) A typical single-scattering diagram. (b) A typical double-scattering diagram.

In order to keep our discussion as simple as possible, we shall restrict ourselves to the interaction of an elementary projectile with a two-body bound state. However, the generalization to more complicated systems should be obvious. Similarly, we shall restrict ourselves to the region of small momentum transfer, since the general formulas are rather lengthy.

Typical diagrams are shown in Fig. 1. In the language of Glauber, Fig. 1(a) corresponds to single scattering and Fig. 1(b) to double scattering. In each case the lines of the exchanged quanta are to be crossed in all possible ways. The circles in Fig. 1 represent the form factors of the bound state χ. They are related to the Bethe-Salpeter wave functions Ψ by

$$\begin{aligned}\Psi(k;P)&=G_1G_2\chi\\&=[(\tfrac{1}{2}P+k)^2-m^2]^{-1}[(\tfrac{1}{2}P-k)^2-m^2]^{-1}\chi(k,P),\end{aligned}$$

where P and k are, respectively, the total and relative momenta of the bound particles. We shall refer to the bound particles as 1 and 2 and to the projectile as particle 3. To simplify the kinematics, we have taken the bound particles to have the same mass M.

Let us start by considering scattering via the exchange of spin-one quanta. It is convenient to leave the integration over the loop momentum k defined in Fig. 1 until the end. Then each of the scattering particles can be thought of as having a definite momentum. The relative momentum between each of the bound particles and the projectile will be large except for these values of k on the order of $\pm(p-\frac{1}{2}P)$. However, for such values of k the bound-state wave functions will be down by at least a factor of $1/s$, so there will be no contribution to the asymptotic behavior of the amplitude. As a result, all of the techniques developed for the two-body problem can be used directly. Each of the scattering particles propagates in a straight line, emitting and absorbing quanta whose momenta are small compared to its own momentum.

We shall always work in a Lorentz frame in which the vectors $\mathbf{P}$ and $\mathbf{p}$ are parallel, and we shall take the z axis to be in their direction. We denote a general four-vector by $(k_0;k_z,\boldsymbol{\kappa})$, where $\boldsymbol{\kappa}\equiv(k_x,k_y)$ is a two-dimensional vector in the x-y plane. Since $p\cdot\Delta=P\cdot\Delta=0$, one has $\Delta=(0;0,\boldsymbol{\Delta})$.

The sum of all single-scattering diagrams, which are illustrated in Fig. 1(a), can be written down directly. We find

$$H_1=\frac{-i}{(2\pi)^4}\int d^4k\,\Psi^*(k,P+\Delta)\Psi(k-\Delta,P-\Delta)4p\cdot(\tfrac{1}{2}P+k)\times[(\tfrac{1}{2}P-k)^2-m^2][E_{13}(2\boldsymbol{\Delta})+E_{23}(2\boldsymbol{\Delta})]$$
$$\equiv p\cdot P\,F(2\boldsymbol{\Delta})[E_{13}(2\boldsymbol{\Delta})+E_{23}(2\boldsymbol{\Delta})]\,, \tag{21}$$

where

$$E_{i3}(2\boldsymbol{\Delta})=-i\int d^2b\,e^{2i\boldsymbol{\Delta}\cdot\mathbf{b}}(e^{i\delta_{i3}(\mathbf{b})}-1)\,, \tag{22}$$

and

$$\delta_{i3}(\mathbf{b})=4p\cdot(\tfrac{1}{2}P+k)g^2\int_{-\infty}^{\infty}d\tau_1d\tau_2\times D_{i3}[x-2p\tau_1-2(\tfrac{1}{2}P+k)\tau_2]$$
$$=-\frac{g^2}{(2\pi)^2}\int d^2q\,e^{-i\mathbf{q}\cdot\mathbf{b}}(\mu_{i3}{}^2+\mathbf{q}^2)^{-1}. \tag{23}$$

The mass of the quanta exchanged between the ith bound particle and the projectile is denoted by μ_{i3}. F is recognized as the form factor for the emission of a single spin-one quantum by the bound state.

The sum of the double-scattering diagrams, which are illustrated in Fig. 1(b), is somewhat more complicated. It is given by

$$H_2=\left(\frac{-i}{(2\pi)^4}\right)^2\int d^4kd^4q\,\Psi^*(k,P+\Delta)\Psi(k+q,P-\Delta)\times\langle\tfrac{1}{2}P-k+\tfrac{1}{2}\Delta,\tfrac{1}{2}P+k+\tfrac{1}{2}\Delta,p-\Delta|(-e^KT_1T_2T_3)\times|\tfrac{1}{2}P-k-q-\tfrac{1}{2}\Delta,\tfrac{1}{2}P+k+q-\tfrac{1}{2}\Delta,p+\Delta\rangle\Big|_{A_i=0}\,, \tag{24}$$

where

$$K=ig^2\int d^4xd^4x'\frac{\delta}{\delta A_{3\mu}(x)}\times\left[D_{13}(x-x')\frac{\delta}{\delta A_{1\mu}(x')}+D_{23}(x-x')\frac{\delta}{\delta A_{2\mu}(x')}\right]$$

and

$$T_i=T_i(A_{i\mu})=-2p_iA_i\exp\left[i\int_0^{\infty}d\tau\,2p_i\cdot A_i(x-2p_i\tau)\right].$$

The kets $|p_1p_2p_3\rangle$ are plane-wave states in which the particles are off the mass shell. Carrying out the functional derivatives yields

$$H_2=\left(\frac{-i}{(2\pi)^4}\right)^2\int d^4kd^4q\,\Psi^*(k,P+\Delta)\Psi(k+q,P-\Delta)\times(-i4p_1\cdot p_3)D_{13}(x_{13})4p_2\cdot p_3\int d\tau\,D_{23}(x-2p_3\tau)\times\exp\left[ig^2\left(4p_1\cdot p_3\int d\tau_1d\tau_3D_{13}+4p_2\cdot p_3\int d\tau_2d\tau_3D_{23}\right)\right],$$

where $p_1=\frac{1}{2}P-k$, $p_2=\frac{1}{2}P+k$, and $p_3=p$. In the previous notation, H_2 may be written as

$$H_2=\left(\frac{-i}{(2\pi)^4}\right)^2\int d^4kd^4q\,\Psi^*\Psi 4p\cdot(\tfrac{1}{2}P-k)4p\cdot(\tfrac{1}{2}P+k)\times 2\pi i\delta(2p\cdot q)E_{13}(\mathbf{q}+\boldsymbol{\Delta})E_{23}(\mathbf{q}-\boldsymbol{\Delta})+O(1/s)\,. \tag{25}$$

The coefficient of the two eikonal functions in Eq. (25) is recognized as the amplitude for the emission of one vector quanta by each of the bound particles. To leading order in s, the full amplitude is given by

$$H=H_1+H_2\,. \tag{26}$$

Although Eqs. (21), (25), and (26) are reminiscent of the Glauber equation, they are much more difficult to use in their present form. The difficulty is that the single- and double-scattering terms do not combine to form one simple amplitude as they do in the nonrelativistic case. In order to use Eq. (26) one must know the eikonal phases (or two-body amplitudes) and the amplitudes for the emission of both one and two vector quanta by the bound state. Of course, the amount of information required becomes progressively greater as one considers more complicated compound systems.

There are two important cases in which Eqs. (21) and (25) do simplify. One is when the bound-state wave function can be treated nonrelativistically, as in the scattering of a relativistic projectile from a nucleus. The other is when the binding forces are very short range, as one might expect, for example, in the quark model.

A. Nonrelativistic Bound State

Let us first consider the nonrelativistic limit. We shall work in the Breit frame where $P=((m^2+\mathbf{\Delta}^2)^{1/2};0,0,0)$ and $m=2M-\epsilon$, where ϵ is the binding energy. The nonrelativistic (NR) wave function is related to the Bethe-Salpeter wave function by

$$\Psi_{\rm NR}(\mathbf{k};\pm\mathbf{\Delta})=\frac{1}{2\pi i}\int_{-\infty}^{\infty}dk_0\Psi(k;P\pm\Delta).$$

If ϵ and the inverse range of the binding force, μ, are much smaller than M, we may write

$$\begin{aligned}\Psi_{\rm NR}(\mathbf{k};\pm\mathbf{\Delta})&\\&=\frac{1}{2\pi i}\int dk_0G_1G_2\chi(k,P\pm\Delta)\\&\cong\chi_{\rm NR}(\mathbf{k})[\epsilon+(\mathbf{k}+\tfrac{1}{2}\mathbf{\Delta})^2/2m+(\mathbf{k}-\tfrac{1}{2}\mathbf{\Delta})^2/2m]^{-1}\\&\equiv\Psi_{\rm NR}(\mathbf{k}).\end{aligned}$$

In other words, in performing the k_0 integration it is only necessary to pick up the poles in the free Green's functions. Those in χ can be ignored to lowest order in ϵ/m and μ/m. Using the same reasoning in the integrals of Eqs. (21) and (25) allows the k_0 and q_0 integrations to be performed. We find

$$H_1=4p\cdot PM\int\frac{d^3k}{(2\pi)^3}\Psi_{\rm NR}^*(\mathbf{k})\Psi_{\rm NR}(\mathbf{k}-\mathbf{\Delta})\times[E_{13}(2\mathbf{\Delta})+E_{23}(2\mathbf{\Delta})]$$

and

$$H_2=4p\cdot PM\int\frac{d^3k}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\Psi_{\rm NR}^*(\mathbf{k})\Psi_{\rm NR}(\mathbf{k}+\mathbf{q})\times iE_{13}(\mathbf{q}+\mathbf{\Delta})E_{23}(\mathbf{q}-\mathbf{\Delta}).$$

Combining the above equations and going to coordinate space gives

$$H=-iM(s-u)\int d^3rd^2b|\Psi_{\rm NR}(r)|^2e^{2i\Delta\cdot\mathbf{b}}\times[e^{i[\delta_{13}(\mathbf{b}+\frac{1}{2}\mathbf{r})+\delta_{23}(\mathbf{b}-\frac{1}{2}\mathbf{r})]}-1],\quad(27)$$

where $\Psi_{\rm NR}(\mathbf{r})$ is the Fourier transform of $\Psi_{\rm NR}(\mathbf{k})$. Equation (27) is, of course, just the Glauber formula with relativistic kinematics for the projectile.

B. Short-Range Binding Forces

The second case in which our results simplify is when the binding forces are of very short range. Here it is most convenient to work in the center-of-mass system. We take the z axis along $\mathbf{P}$ and introduce the variables

$$k_\pm=k_0\pm k_z.$$

It is also useful to introduce the three-dimensional wave function

$$\begin{aligned}\Psi_+(\boldsymbol{\kappa},k_+,P\pm\Delta)&=\frac{P_+}{2\pi i}\int dk_-\Psi(k,P\pm\Delta)\\&=\frac{P_+}{2\pi i}\int dk_-G_1G_2\chi(k,P\pm\Delta).\end{aligned}$$

If the binding forces are of very short range, then one can do the k_- integration by picking up the poles in the free Green's functions and ignoring the singularities in χ. We then find

$$\begin{aligned}\Psi_+(\boldsymbol{\kappa},x;P\pm\Delta)=\chi\theta(1-x^2)[m^2+\tfrac{1}{2}(1-x)(\boldsymbol{\kappa}+\tfrac{1}{2}\mathbf{\Delta})^2\\+\tfrac{1}{2}(1+x)(\boldsymbol{\kappa}-\tfrac{1}{2}\mathbf{\Delta})^2-\tfrac{1}{4}(1-x^2)P^2]^{-1}\\=\Psi_+(\boldsymbol{\kappa},x),\end{aligned}$$

where we have written $k_+=\frac{1}{2}xP_+$. Under the present assumptions the k_- and q_- integrals in H_1 and H_2 can be evaluated and give

$$H_1=4p\cdot P\int\frac{d^2\kappa}{(2\pi)^3}\int_{-1}^{1}\frac{dx}{8}(1-x^2)\Psi_+^*(\boldsymbol{\kappa},x)\Psi_+(\boldsymbol{\kappa}-\mathbf{\Delta},x)\times[E_{13}(2\mathbf{\Delta})+E_{23}(2\mathbf{\Delta})],\quad(28)$$

$$H_2=4p\cdot P\int\frac{d^2\kappa}{(2\pi)^3}\int_{-1}^{1}\frac{dx}{8}(1-x^2)\times\int\frac{d^2q}{(2\pi)^2}\Psi_+^*(\boldsymbol{\kappa},x)\Psi_+(\boldsymbol{\kappa}+\mathbf{q},x)\times iE_{13}(\mathbf{q}+\mathbf{\Delta})E_{23}(\mathbf{q}-\mathbf{\Delta}).\quad(29)$$

The total amplitude H becomes

$$H=-i(s-u)\int\frac{d^2q_1d^2q_2}{(2\pi)^2}\delta(\mathbf{q}_1+\mathbf{q}_2-2\mathbf{\Delta})\tfrac{1}{2}F(\mathbf{q}_1-\mathbf{q}_2,2\mathbf{\Delta})\times\{[iE_{13}(\mathbf{q}_1)+(2\pi)^2\delta(\mathbf{q}_1)][iE_{23}(\mathbf{q}_2)+(2\pi)^2\delta(\mathbf{q}_2)]-(2\pi)^4\delta(\mathbf{q}_1)\delta(\mathbf{q}_2)\},\quad(30)$$

where

$$F(\mathbf{q},2\mathbf{\Delta})=\int_{-1}^{1}\frac{dx}{4}(1-x^2)\int\frac{d^2k}{(2\pi)^3}\Psi_+^*(\boldsymbol{\kappa},x)\Psi_+(\boldsymbol{\kappa}+\tfrac{1}{2}\mathbf{q},x).$$

It should be noticed that the vector form factor of the bound state is given by $F(2\mathbf{\Delta},2\mathbf{\Delta})$. Thus, although Eq. (30) represents a considerable simplification of the general result, it is simply not possible to express the scattering amplitude in terms of the on-shell two-body amplitudes and the vector form factor.

In the limit of infinitely short-range forces, $\chi \to 1$. In that case the function F becomes

$$F(\mathbf{q}_1-\mathbf{q}_2, 2\boldsymbol{\Delta})=\frac{1}{(2\pi)^2}\int_0^1 dy\, y(1-y)\int_0^1 d\beta \times\{M^2-y(1-y)m^2+\beta(1-\beta)[y\mathbf{q}_1-(1-y)\mathbf{q}_2]^2\}^{-1},$$

which is just the impact factor of Cheng and Wu.[7]

Since it is probably easier to parametrize the coordinate-space wave functions, we shall rewrite our results in terms of them. We find

$$\Pi=-i(s-u)\int d^4x\, p\cdot P\delta(p\cdot x)\left[\frac{p\cdot(\frac{1}{2}P+i\partial)}{p\cdot P}\Psi_{p+\Delta}(x)\right] \times\left[\frac{p\cdot(\frac{1}{2}P-i\partial)}{p\cdot P}\Psi_{p-\Delta}(x)\right]\int d^2b \times e^{2i\Delta\cdot\mathbf{b}}[e^{i\delta_{13}(\mathbf{b}+\frac{1}{2}\mathbf{x}_\perp)+i\delta_{23}(\mathbf{b}-\frac{1}{2}\mathbf{x}_\perp)}-1].$$

The $\delta(p\cdot x)$ reduces the x integration to three dimensions and should greatly simplify the parametrization of the wave functions.

It is interesting that our results do not hold for the exchange of spinless quanta. This is not important in practice since these amplitudes go to the Born approximation at large energies, but it does illustrate the fact that one must use care in applying the functional-derivative techniques. It is always essential to ascertain whether the leading behavior of the amplitude really corresponds to straight-line propagation of the particles. One can see that this is not true in the case of spin-zero exchange by looking at the single-scattering diagrams in momentum space. It is clear that the leading behavior comes from the region $k\sim\frac{1}{2}P$, i.e., when the non-interacting bound particle carries all the large momentum. Then even though the momentum of the exchanged quanta is small, it cannot be neglected with respect to the momentum of the interacting bound particle. Thus, the entire expansion procedure breaks down. In the case of spin-one exchange it is profitable for the large momentum to be shared by the bound particles because of the momentum factors arising from spin-one coupling.

For exchange of spinless quanta, the only case in which we have been able to obtain a simple result is when the wave function can be treated nonrelativistically. Then the bound particles cannot get far off the mass shell without the wave functions falling off. As a result the previous techniques do go through and one obtains Eq. (27) with δ_{i3} replaced by $\delta_{i3}/2\sqrt{(-su)}$. Again one finds the Glauber formula with relativistic kinematics.

V. CONCLUSION

For two-body scattering, the relativistic and non-relativistic eikonal expansions are strikingly similar. Both rest on the fact that it is most favorable for a high-energy particle to propagate through the interaction region in a straight line. In the relativistic case the high-energy particles interact via the exchange of quanta whose momenta are small compared to their own. Since the propagators of the exchanged quanta fall off in momentum space, scattering in all but the forward direction is suppressed at high energies. For those rare events in which large-momentum-transfer scattering does occur, it is most favorable for the large momentum to be carried across by a single hard quantum. All this is in exact analogy with the nonrelativistic case.

The eikonal approximation for the scattering of a relativistic projectile from a bound state is virtually identical to the nonrelativistic Glauber formula provided the bound-state wave function can be treated nonrelativistically. Only when this cannot be done are there added complications.

The major difference between the nonrelativistic and relativistic eikonal expansions is that in the former case one has obtained an accurate approximation for the entire amplitude while in the latter one only has an approximation for a subset of Feynman diagrams. It is certainly not true that all Feynman graphs are susceptible to the eikonal approximation; the interesting question is whether those graphs that contribute to the leading asymptotic behavior are susceptible. In this respect it is encouraging that Feynman graphs corresponding to the exchange of more complicated objects such as Reggeons can be treated using the ideas discussed here.[11]

[11] H. Cheng and T. T. Wu, Phys. Rev. **186**, 1611 (1969); S. J. Chang and T. M. Yan (unpublished); G. Cicuta and R. L. Sugar (unpublished).