Phenomenological Lagrangian Model of Chiral $SU(3)\times SU(3)$ for Mesons and Baryons with Symmetry Breaking*

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Using the techniques of nonlinear realization of $SU(3)\times SU(3)$, we write a Lagrangian for the pseudoscalar nonet and $\frac{1}{2}^+$-baryon octet with a symmetry-breaking term transforming like a vector in a $(3,3^*)\oplus(3^*,3)$ representation of $SU(3)\times SU(3)$. This Lagrangian yields a mass formula (different from the Gell-Mann–Okubo mass formula) for the pseudoscalar octet, and, if we neglect η-η' mixing, then this formula fits the experimental masses to an accuracy of about 1 MeV. From this Lagrangian, explicit expressions for the axial-vector currents in terms of particle fields are obtained. The tree-diagram prescription then yields a modified Goldberger-Treiman relation consistent with experimental measurements.

I. INTRODUCTION

Ever since Nambu's work[1] on the idea of chiral $SU(3)\times SU(3)$ as realized on eight massless pseudoscalar mesons, the development and application of current algebra[2] and partial conservation of axial-vector currents (PCAC) have tended to support this idea, supplementing it with the hypothesis that the breaking of this symmetry contributes to the masses of these eight pseudoscalar mesons. Then the phenomenological Lagrangian method,[3] which is based on this idea explicitly and yields all current-algebra calculations, was introduced, and, later on, a uniquely and sound mathematical technique for this approach was laid down by Callan, Coleman, Wess, and Zumino.[4] With this technique, the $SU(3)\times SU(3)$-breaking term of Gell-Mann, Oakes, and Renner[5] can be explicitly realized in terms of particle fields.

In a phenomenological Lagrangian theory of a symmetry group (say, G), it is well known that there is a subgroup H of G under which all particle fields transform linearly. The rest of the symmetry group G, vaguely speaking, is then realized by having Goldstone bosons. Since $SU(2)$ symmetry is much more exactly followed than $SU(3)$, one may try to let the above-mentioned subgroup H be just $SU(2)$, and the Goldstone bosons would then consist of the κ meson and the ordinary pseudoscalar octet. However, the experimental status of the κ meson is highly uncertain, and even if it does exist, letting H be $SU(2)$ is not economical, since a smaller symmetry always leads to a larger number of parameters. These considerations lead us to use $SU(3)$ for H. Another compelling reason is the success of $SU(3)$ classification of particles.

For convenience, the generators V_i, A_i ($i=1,\ldots,8$) of $SU(3)\times SU(3)$ will be so defined that their commutators do not involve the imaginary number i:

$$[V_i,V_j]=[A_i,A_j]=f_{ijk}V_k\,,\qquad [V_i,A_j]=f_{ijk}A_k\,. \tag{1.1}$$

(In a unitary representation, these generators will correspond to non-Hermitian operators.) Any group element is of the form $\exp(\epsilon\cdot V+\delta\cdot A)$ and in a neighborhood of the identity, the group elements can be expressed uniquely in the form $\exp(\delta\cdot A)\times\exp(\epsilon\cdot V)$. The relation between the group generators and parity is well known:

$$PV_i=V_iP\,,\qquad PA_i=-A_iP\,. \tag{1.2}$$

In Sec. II, we give a brief description of the techniques of Ref. 4. The reader is referred to the original literature for details. A symmetry-conserving Lagrangian is constructed from the pseudoscalar and baryon fields in Sec. III, and then, in Sec. IV, functions of the particle fields are constructed such that these functions transform linearly under the entire group $SU(3)\times SU(3)$. These linearly transforming functions are then used to construct the symmetry-breaking terms u_0+cu_8 of the $(3,3^*)\oplus(3^*,3)$ representation, and their significance is discussed in the rest of the paper.

II. CHIRAL $SU(3)\times SU(3)$ TRANSFORMATIONS

We will now begin with a brief summary of the results of Ref. 4.

Let ψ be a vector in an N-dimensional representation of $SU(3)$. Then to each element $e^{u\cdot V}$ of $SU(3)$ there corresponds a transformation on ψ:

$$\psi\to\mathfrak{D}(e^{u\cdot V})\psi\,. \tag{2.1}$$

A nonlinear realization of $SU(3)\times SU(3)$ consists of the ordered pair of vectors (ξ,ψ), where ξ^i is an eight-dimensional vector. For each vector ξ^i and g in $SU(3)$

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[1] Y. Nambu, Phys. Rev. Letters **4**, 380 (1960); Y. Nambu and D. Lurié, Phys. Rev. **125**, 1429 (1962), and references therein.
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[3] S. Weinberg, in *Proceedings of the Fourteenth International Conference on High-Energy Physics, Vienna, 1968*, edited by J. Prentki and J. Steinberger (CERN, Geneva, 1968).
[4] S. Coleman, J. Wess, and B. Zumino, Phys. Rev. **177**, 2239 (1969); C. G. Callan, S. Coleman, J. Wess, and B. Zumino, *ibid.* **177**, 2247 (1969).
[5] M. Gell-Mann, R. J. Oakes, and B. Renner, Phys. Rev. **175**, 2193 (1968).

$\times SU(3)$, define ξ'^i and μ^i by

$$ge^{\xi\cdot A}=e^{\xi'\cdot A}e^{\mu\cdot V}\,. \tag{2.2}$$

Then the transformation of the pair (ξ,ψ) under g is defined to be

$$(\xi,\psi)\xrightarrow{g}(\xi',\psi')\,, \tag{2.3}$$

where

$$\psi'=\mathfrak{D}(e^{\mu\cdot V})\psi\,. \tag{2.4}$$

Because everything transforms linearly under $SU(3)$, all particles are classified into $SU(3)$ multiplets.

To include parity, we easily see that property (1.2) requires that

$$\xi\xrightarrow{P}-\xi\,,\quad \psi\xrightarrow{P}\pm\eta\psi\,, \tag{2.5}$$

where η is the appropriate parity operator consistent with the representation of the Lorentz group.

Because of the nonlinear nature of transformations under $SU(3)\times SU(3)$, the simple derivatives $\partial_\mu\xi$ and $\partial_\mu\psi$ do not have simple transformations, but instead we must use the covariant derivatives, defined by

$$\begin{aligned} D_\mu\xi(x)&=p_\mu(x)\,,\\ D_\mu\psi(x)&=\partial_\mu\psi(x)+[q_\mu(x)\cdot T]\psi(x)\,, \end{aligned} \tag{2.6}$$

where $p_\mu{}^i(x)$ and $q_\mu{}^i(x)$ $(i=1,\ldots,8)$ are defined to be functions of $\xi(x)$ and $\partial_\mu\xi(x)$ by

$$e^{-\xi(x)\cdot A}\partial_\mu e^{\xi(x)\cdot A}=p_\mu(x)\cdot A+q_\mu(x)\cdot V\,, \tag{2.7}$$

and T^i $(i=1,\ldots,8)$ are the generators of $SU(3)$ in the space $\{\psi\}$. Their transformation is similar to that of ψ:

$$\begin{aligned} D_\mu\xi(x)&\xrightarrow{g}(e^{\mu\cdot t})D_\mu\xi(x)\,,\\ D_\mu\psi(x)&\xrightarrow{g}\mathfrak{D}(e^{\mu\cdot V})D_\mu\psi(x)\,, \end{aligned} \tag{2.8}$$

where $\mu=\mu(x)$ is defined by (2.2) and the matrix elements of the generators t^i $(i=1,\ldots,8)$ are simply related to the structure constants f_{ijk} by

$$(t^i)_{jk}=-f_{ijk}\,. \tag{2.9}$$

p_μ and q_μ can be expanded as a series in ξ:

$$\begin{aligned} D_\mu\xi=p_\mu&=\frac{\sinh(\xi\cdot t)}{\xi\cdot t}\partial_\mu\xi\,,\\ q_\mu&=\frac{1-\cosh(\xi\cdot t)}{\xi\cdot t}\partial_\mu\xi\,. \end{aligned} \tag{2.10}$$

Though ξ and ψ do not transform linearly, functions of them which transform linearly can be constructed. Throughout this paper, the construction of such functions will be referred to as *linearization*. Let $D(g)$ be any $M\times M$ matrix representation of $SU(3)\times SU(3)$ with $M>N$, such that, when g is restricted to $SU(3)$, they contain $\mathfrak{D}(g)$ in the form

$$D(g)=\begin{pmatrix}\mathfrak{D}(g) & 0\\ 0 & R(g)\end{pmatrix}. \tag{2.11}$$

Then the M functions

$$\phi^\alpha(\xi,\psi)\equiv\sum_{i=1}^{N}D(e^{\xi\cdot A})^\alpha{}_i\psi^i \tag{2.12}$$

transform linearly under $SU(3)\times SU(3)$ according to the matrices $D(g)$. We will find this linearization used repeatedly in this paper.

III. $SU(3)\times SU(3)$-INVARIANT LAGRANGIAN

As is evident from the previous section on transformations, the fields $\xi(x)$ must be dimensionless quantities. While it is well known that the pseudoscalar meson fields $P^i(x)$ carry the dimension of mass, the symmetry-invariant term $-\frac{1}{2}D_\mu\xi^iD^\mu\xi^i$ must be multiplied by a constant F^2 of dimension mass squared before it can be placed in a symmetric Lagrangian as its meson kinetic term. On the other hand, the group transformations do not place any restriction on the dimension of ψ (do not confuse dimension here with that of a vector space). So let us take ψ to be the usual $\frac{1}{2}^+$-baryon fields B^i and the pseudoscalar singlet field $F'\xi^0(x)$ and write the following $SU(3)\times SU(3)$-invariant Lagrangian:

$$\begin{aligned} \mathfrak{L}_0(x)=-\tfrac{1}{2}F^2D_\mu\xi\cdot D^\mu\xi+\bar{B}\cdot(\tfrac{1}{2}i\gamma^\mu\overleftrightarrow{D}_\mu-M)B\\ -\tfrac{1}{2}F'^2\partial_\mu\xi^0\partial^\mu\xi^0-\tfrac{1}{2}\mu_0'^2(\xi^0)^2\\ +(iG_Ff_{ijk}+G_Dd_{ijk})D_\mu\xi^i\bar{B}^j\gamma^\mu\gamma_5B^k\,, \end{aligned} \tag{3.1}$$

where the pseudoscalar octet fields are

$$P^i(x)=F\xi^i(x)\,,\quad i=1,\ldots,8\,. \tag{3.2}$$

Here $F'\xi^0(x)$ is the pseudoscalar singlet field, and $B^i(x)$ $(i=1,\ldots,8)$ are the baryon $\frac{1}{2}^+$-octet fields. Note the following features.

(i) Covariant derivatives replace the ordinary derivative because they have *suitable* transformation properties. The only exception is the case of the pseudoscalar singlet ξ^0, where the covariant derivative is by definition simply $\partial_\mu\xi^0$.

(ii) The mass term of the pseudoscalar octet is prohibited by the symmetry group, while the pseudoscalar singlet may acquire any mass.

(iii) The only $SU(3)\times SU(3)$-symmetric pseudoscalar octet coupling with the baryon octet is a derivative coupling. Even with linearization technique, one cannot construct a symmetric nonderivative coupling. This, of course, agrees with soft-pion calculations.

IV. LINEARIZATION OF PARTICLE FIELDS AND THEIR PRODUCTS

Our sole aim is to find the symmetry-breaking term of the form proposed by Gell-Mann, Oakes, and

Renner.[5] So we will look for representations $(3,3^*)\oplus(3^*,3)$ of $SU(3)\times SU(3)$. Since representations transform linearly, we naturally make use of the linearization procedure of Ref. 4, as described in Sec. II. Notice that a convenient basis of the $(3,3^*)\oplus(3^*,3)$ is such that a vector in the representation consists of the pair (u,v) where both u_i and v_i $(i=1,\ldots,8,0)$ are nine-dimensional and the generators of the group take the form

$$V^i \to \begin{pmatrix} -f_{ijk} & 0 \\ 0 & -f_{ist} \end{pmatrix}, \qquad A^i \to \begin{pmatrix} 0 & d_{ijk} \\ -d_{ist} & 0 \end{pmatrix}. \tag{4.1}$$

Then the parity relation (1.2) leaves us no choice for parity transformation except for an over-all sign. We shall choose, as in Ref. 5,

$$(u,v) \xrightarrow{P} (u,\,-v). \tag{4.2}$$

To construct functions of fields that transform linearly like a $(3,3^*)\oplus(3^*,3)$ representation, let us identify the generators of (4.1) with those of the representation matrices D of Sec. II, and note that when restricted to the subgroup $SU(3)$, the vector (u,v) decomposes into two octets and two singlets. Thus the linearization procedure says that if $(\mathcal{U},\mathcal{V})$ are 18 functions of the fields ξ, B, and ξ^0 such that under g of $SU(3)\times SU(3)$ they transform as

$$\begin{aligned} \mathcal{U}_i &\xrightarrow{g} \sum_{j=1}^{8} (e^{\mu\cdot t})_{ij}\mathcal{U}_j\,,\\ \mathcal{V}_i &\xrightarrow{g} \sum_{j=1}^{8} (e^{\mu\cdot t})_{ij}\mathcal{V}_j\,,\\ \mathcal{U}_0 &\xrightarrow{g} \mathcal{U}_0\,,\\ \mathcal{V}_0 &\xrightarrow{g} \mathcal{V}_0\,, \end{aligned} \tag{4.3}$$

with μ defined by (2.2), then the 18 functions $(u(\xi,\mathcal{U},\mathcal{V}),\ v(\xi,\mathcal{U},\mathcal{V}))$, defined by

$$\begin{pmatrix} u(\xi,\mathcal{U},\mathcal{V}) \\ v(\xi,\mathcal{U},\mathcal{V}) \end{pmatrix} \equiv \exp\begin{pmatrix} 0 & \xi\cdot d \\ -\xi\cdot d & 0 \end{pmatrix}\begin{pmatrix} \mathcal{U} \\ \mathcal{V} \end{pmatrix} = \begin{pmatrix} \cos\xi\cdot d & \sin\xi\cdot d \\ -\sin\xi\cdot d & \cos\xi\cdot d \end{pmatrix}\begin{pmatrix} \mathcal{U} \\ \mathcal{V} \end{pmatrix}, \tag{4.4}$$

where

$$(\xi\cdot d)_{ij} \equiv \sum_{s=1}^{8} \xi^s d_{sij}\,, \tag{4.5}$$

transform linearly under $SU(3)\times SU(3)$ like a $(3,3^*)\oplus(3^*,3)$ representation. Now the symmetry-breaking term is nothing but (according to Ref. 5) u_0+cu_8, where c is a constant. If (u,v) are expanded as power series in ξ, successive terms in the series differ by two powers of ξ.

Noting that the transformations of $D_\mu\xi$, $D_\mu B$, B, etc., behave *like* linear representations of $SU(3)$, construction of $(\mathcal{U},\mathcal{V})$ becomes an easy task. Here we will list contributions from various combinations.

A. Linearization of $(\xi)^2$

$$\begin{aligned} &\mathcal{U}_i \sim 0\,, \quad i=1,\ldots,8\\ &\mathcal{U}_0 \sim a_\xi\,,\\ &\mathcal{V}_i \sim 0\,, \quad i=1,\ldots,8,0 \end{aligned} \tag{4.6}$$

where a_ξ is a constant. $\mathcal{V}_0$ cannot be a constant because of parity. The leading terms from the expansion of (u,v) as obtained from this $(\mathcal{U},\mathcal{V})$ will contain the meson mass term and meson-meson scattering. The u's are

$$u_i(\xi) \sim a_\xi[\delta_{0i} - \tfrac{1}{2}(\xi\cdot d)^2{}_{i0} + (1/4!)(\xi\cdot d)^4{}_{i0} + O(\xi^6)]\,. \tag{4.7}$$

B. Linearization of $D_\mu\xi^i D^\mu\xi^j$

$$\begin{aligned} &\mathcal{U}_i \sim a_\xi' \sum_{j,k=1}^{8} d_{ijk} D_\mu\xi^j D^\mu\xi^k + b_\xi'\delta_{i0} D_\mu\xi\cdot D^\mu\xi\,,\\ &\mathcal{V}_i \sim 0\,, \quad i=1,\ldots,8,0 \end{aligned} \tag{4.8}$$

where a_ξ' and b_ξ' are arbitrary real constants. In addition to the meson-meson scattering-term contribution to (u,v), this will also yield a *breaking* to the kinetic term $(\partial_\mu P^i)^2$ of the pseudoscalar octet. Its consequences relevant to weak interaction will be described in Sec. VI.

C. Linearization of ξ^0, $(\xi^0)^2$, and $(\partial_\mu\xi^0)^2$

Although $\xi^0(x)$ is already a linearly transforming field (being a singlet), a further combination with ξ^i can be made that contributes an additional meson-meson interaction. The relevant parts are

$$\begin{aligned} &\mathcal{U}_i \sim \mathcal{V}_i \sim 0\,, \quad i=1,\ldots,8\\ &\mathcal{U}_0 \sim a_{\xi^0}(\xi^0)^2 + b_{\xi^0}(\partial_\mu\xi^0)^2\,,\\ &\mathcal{V}_0 \sim \bar{a}_{\xi^0}\xi^0\,, \end{aligned} \tag{4.9}$$

where a_{ξ^0}, $\bar{a}_{\xi^0}$, and b_{ξ^0} are arbitrary constants. $\mathcal{V}_0$ will eventually lead to an η-η' mixing system.

D. Linearization of $\bar{B}^i B^j$

$$\begin{aligned} &\mathcal{U}_i \sim [\delta_{i0}(\sqrt{\tfrac{3}{2}})S\delta_{jk} + iFf_{ijk} + Dd_{ijk}]\bar{B}^j B^k\,,\\ &\mathcal{V}_i \sim (\delta_{i0}a_B\delta_{jk} + ib_B f_{ijk} + c_B d_{ijk})i\bar{B}^j\gamma_5 B^k\,,\\ &\qquad\qquad i=1,\ldots,8,0 \end{aligned} \tag{4.10}$$

where S, F, D, a_B, b_B, and c_B are, in principle, arbitrary constants. As we have reported earlier,[6] if we take the

[6] Y. M. P. Lam and Y. Y. Lee, Phys. Rev. Letters **23**, 734 (1969).

quarks and pseudoscalar octet as basic entities and assign to the quarks a transformation under $SU(3)\times SU(3)$ exactly like the vector ψ as described in Sec. II, then a simple assumption on this quark model will yield the relation

$$S+F+D=0\,, \tag{4.11}$$

which agrees well with the result obtained from an analysis of threshold amplitudes by von Hippel and Kim.[7] Their result essentially says that the nucleon mass remains practically unchanged as $SU(3)\times SU(3)$ is broken. The v's will contribute a broken baryon-meson trilinear *nonderivative* coupling to the interaction Lagrangian:

$$i(\sqrt{\tfrac{2}{3}})(ib_Bf_{ijk}+c_Bd_{ijk})\xi^i\bar{B}^j\gamma_5B^k+ci[(\sqrt{\tfrac{2}{3}})a_B\xi^8\bar{B}\cdot\gamma_5B+d_{8ij}(ib_Bf_{jst}+c_Bd_{jst})\xi^i\bar{B}^s\gamma_5B^t]\,. \tag{4.12}$$

More and more terms can be constructed for $(\mathfrak{U},\mathfrak{V})$ as we go up with the power of the fields involved. This will then lead to an infinite number of parameters, and this we wish to avoid. Thus we will stop here and hope that the symmetry-breaking terms so far included are adequate to describe our limited system. Thus we take the total Lagrangian to be

$$\mathfrak{L}=\mathfrak{L}_0+u_0+cu_8\,, \tag{4.13}$$

where u_i is obtained by summing up all contributions (4.6), (4.8), (4.9), and (4.10) to $(\mathfrak{U},\mathfrak{V})$ and substituting the sum into (4.4).

V. PSEUDOSCALARS

Let us look at the free Lagrangian for the pseudoscalars. Of course, the mixing term between ξ^8 and ξ^0 must be included. So the relevant contribution to (u,v) is from the linearization of $(\xi)^2$, $D_\mu\xi^iD^\mu\xi^j$, ξ^0, and $(\xi^0)^2$, and summing them from (4.6), (4.8), and (4.9) we have

$$\mathfrak{U}_i\sim a_\xi'\sum_{j,k=1}^{8}d_{ijk}D_\mu\xi^jD^\mu\xi^k+\delta_{i0}[a_\xi+a_{\xi^0}(\xi^0)^2+b_\xi'D_\mu\xi\cdot D^\mu\xi+b_{\xi^0}\partial_\mu\xi^0\partial^\mu\xi^0]\,, \tag{5.1}$$

$$\mathfrak{V}_i\sim\delta_{i0}\bar{a}_{\xi^0}\xi^0\,,\quad i=1,\,\ldots,\,8,\,0\,.$$

An expansion of u_i to second order in ξ is

$$u_i\sim a_{\xi'}\sum_{j,k=1}^{8}d_{ijk}\partial_\mu\xi^j\partial^\mu\xi^k+\delta_{i0}[a_\xi+b_{\xi^0}(\partial_\mu\xi^0)^2+a_{\xi^0}(\xi^0)^2+b_\xi'\partial_\mu\xi\cdot\partial^\mu\xi]-\tfrac{1}{2}a_\xi(\xi\cdot d)^2{}_{i0}+\bar{a}_{\xi^0}(\xi\cdot d)_{i0}\xi^0+O(\xi^4)\,. \tag{5.2}$$

Upon neglecting additive constants that have no physical meaning, we find

$$u_0+cu_8\sim\sum_{j,k=1}^{8}\{[(\sqrt{\tfrac{2}{3}})a_\xi'+b_\xi']\delta_{jk}+ca_\xi'd_{8jk}\}\partial_\mu\xi^j\partial^\mu\xi^k+b_{\xi^0}(\partial_\mu\xi^0)^2+a_{\xi^0}(\xi^0)^2-\frac{1}{\sqrt{6}}a_\xi\sum_{j,k=1}^{8}[(\sqrt{\tfrac{2}{3}})\delta_{jk}+cd_{8jk}]\xi^j\xi^k+(\sqrt{\tfrac{2}{3}})c\bar{a}_{\xi^0}\xi^8\xi^0\,. \tag{5.3}$$

Adding this to the meson part of the symmetric Lagrangian (3.1), we obtain the free-meson Lagrangian

$$\mathfrak{L}_{\text{free}}{}^{\text{meson}}(x)=-\tfrac{1}{2}\sum_{j,k=1}^{8}\{[F^2-2(\sqrt{\tfrac{2}{3}})a_\xi'-2b_\xi']\delta_{jk}-2ca_\xi'd_{8jk}\}\partial_\mu\xi^j\partial^\mu\xi^k-\tfrac{1}{2}(F'^2-2b_{\xi^0})(\partial_\mu\xi^0)^2-\frac{1}{\sqrt{6}}a_\xi\sum_{j,k=1}^{8}[(\sqrt{\tfrac{2}{3}})\delta_{jk}+cd_{8jk}]\xi^j\xi^k+(\sqrt{\tfrac{2}{3}})c\bar{a}_{\xi^0}\xi^8\xi^0-\tfrac{1}{2}(\mu_0'^2-2a_{\xi^0})(\xi^0)^2\,. \tag{5.4}$$

Let us define new pseudoscalar fields P^i and constants μ_{80}^2, F_i^2, μ_i^2 $(i=1,\,\ldots,\,8,\,0)$ by

$$\begin{aligned}
&F_i^2\equiv F_\pi^2\equiv F^2-2(\sqrt{\tfrac{2}{3}})a_\xi-2b_\xi'-(2/\sqrt{3})ca_\xi'\,, && i=1,2,3\\
&F_i^2\equiv F_K^2\equiv F^2-2(\sqrt{\tfrac{2}{3}})a_\xi-2b_\xi'+(1/\sqrt{3})ca_\xi'\,, && i=4,5,6,7
\end{aligned} \tag{5.5}$$

$$F_8^2\equiv F^2-2(\sqrt{\tfrac{2}{3}})a_\xi-2b_\xi'+(2/\sqrt{3})ca_\xi'\,,$$

$$F_0^2\equiv F'^2-2b_{\xi^0}\,,$$

$$\begin{aligned}
&\mu_i^2\equiv\mu_\pi^2\equiv(\sqrt{\tfrac{2}{3}})a_\xi[(\sqrt{\tfrac{2}{3}})+c/\sqrt{3}]\,, && i=1,2,3\\
&\mu_i^2\equiv\mu_K^2\equiv(\sqrt{\tfrac{2}{3}})a_\xi[(\sqrt{\tfrac{2}{3}})-c/2\sqrt{3}]\,, && i=4,5,6,7\\
&\mu_8^2\equiv(\sqrt{\tfrac{2}{3}})a_\xi[(\sqrt{\tfrac{2}{3}})-c/\sqrt{3}]\,,\\
&\mu_0^2\equiv\mu_0'^2-2a_{\xi^0}\,,\\
&\mu_{80}^2\equiv-2(\sqrt{\tfrac{2}{3}})c\bar{a}_{\xi^0}\,,
\end{aligned} \tag{5.5'}$$

$$P^i(x)\equiv F_i\xi^i(x)\,,\quad i=0,1,\,\ldots,\,8\,. \tag{5.6}$$

Then the free-meson Lagrangian simply becomes

$$\mathfrak{L}_{\text{free}}{}^{\text{meson}}(x)=-\tfrac{1}{2}\sum_{i=0}^{8}(\partial_\mu P^i)^2-\tfrac{1}{2}\sum_{i=0}^{8}\frac{\mu_i^2}{F_i^2}(P^i)^2-\frac{1}{2}\frac{\mu_{80}^2}{F_8F_0}P^8P^0\,. \tag{5.7}$$

To diagonalize the P^8-P^0 system, introduce the usual fields η, η' by

$$\begin{pmatrix}P^8\\P^0\end{pmatrix}=\begin{pmatrix}\cos\phi & \sin\phi\\ -\sin\phi & \cos\phi\end{pmatrix}\begin{pmatrix}\eta\\ \eta'\end{pmatrix}, \tag{5.8}$$

where

$$\tan2\phi=\frac{\mu_{80}^2}{F_8F_0}\Big/\left(\frac{\mu_0^2}{F_0^2}-\frac{\mu_8^2}{F_8^2}\right). \tag{5.9}$$

[7] F. von Hippel and J. K. Kim, Phys. Rev. Letters **22**, 740 (1969).

Then

$$\mathcal{L}_{\text{free}}{}^{\text{meson}} = -\tfrac{1}{2}\sum_{i=1}^{7}[(\partial_\mu P^i)^2 + m_i{}^2(P^i)^2]$$
$$-\tfrac{1}{2}[(\partial_\mu \eta)^2 + m_\eta{}^2\eta^2] - \tfrac{1}{2}[(\partial_\mu \eta')^2 + m_{\eta'}{}^2\eta'^2], \quad (5.10)$$

where the particles now have masses squared

$$m_i{}^2 \equiv \mu_i{}^2/F_i{}^2,$$
$$m_\pi{}^2 = m_i{}^2, \quad i=1,2,3$$
$$m_K{}^2 = m_i{}^2, \quad i=4,5,6,7 \quad (5.11)$$
$$m_\eta{}^2 = m_8{}^2\cos^2\phi + m_0{}^2\sin^2\phi + \tfrac{1}{2}(m_8{}^2 - m_0{}^2)\tan 2\phi \sin 2\phi,$$
$$m_{\eta'}{}^2 = m_8{}^2\sin^2\phi + m_0{}^2\cos^2\phi - \tfrac{1}{2}(m_8{}^2 - m_0{}^2)\tan 2\phi \sin 2\phi.$$

Here we do not have a Gell-Mann–Okubo mass formula for the mesons. The modification is made clear if we define a constant x for convenience by

$$2x = F^2 - 2(\sqrt{\tfrac{2}{3}})a_\xi - 2b_\xi'. \quad (5.12)$$

Then we see that

$$m_\pi{}^2 = \frac{a_\xi}{\sqrt{6}}\left(\frac{(\sqrt{\tfrac{2}{3}}) + c/\sqrt{3}}{x - (1/\sqrt{3})ca_\xi'}\right),$$
$$m_K{}^2 = \frac{a_\xi}{\sqrt{6}}\left(\frac{(\sqrt{\tfrac{2}{3}}) - c/2\sqrt{3}}{x + (1/2\sqrt{3})ca_\xi'}\right), \quad (5.13)$$
$$m_8{}^2 = \frac{a_\xi}{\sqrt{6}}\left(\frac{(\sqrt{\tfrac{2}{3}}) - c/\sqrt{3}}{x + (1/\sqrt{3})ca_\xi'}\right),$$

so that now the masses squared are fractions, both numerator and denominator of which satisfy the Gell-Mann–Okubo formula. Our mass formula (5.13) cannot be tested before finding what F_π, F_K, and F_8 are because, as it is, the mass formula involves too many parameters. So we will take a look at the axial-vector currents in Sec. VI.

VI. AXIAL-VECTOR CURRENTS, WEAK DECAY CONSTANTS, AND MASS FORMULA

Let us write the transformation of (ξ,ψ) under $\exp(\epsilon\cdot A)$ of $SU(3)\times SU(3)$ as

$$(\xi,\psi) \xrightarrow{e^{\epsilon\cdot A}} (f(\epsilon,\xi), \bar{f}(\epsilon,\xi,\psi)), \quad (6.1)$$

where, of course,

$$f(\epsilon,\xi) = \xi', \quad (6.2)$$
$$\bar{f}(\epsilon,\xi,\psi) = \mathfrak{D}(e^{\mu\cdot V})\psi, \quad (6.3)$$
$$e^{\epsilon\cdot A}e^{\xi\cdot A} = e^{\xi'\cdot A}e^{\mu\cdot V}. \quad (6.4)$$

Then define the axial infinitesimal transformations

$$\Lambda^{ij}(\xi) = \frac{\partial}{\partial\epsilon^i} f^j(\epsilon,\xi)\bigg|_{\epsilon=0}, \quad (6.5)$$
$$\bar{\Lambda}^{ij}(\xi,\psi) = \frac{\partial}{\partial\epsilon^i} \bar{f}^j(\epsilon,\xi,\psi)\bigg|_{\epsilon=0}, \quad (6.6)$$

and, by Noether's theorem, the axial-vector currents are

$$A_\mu{}^i(\xi,\psi) = -\sum_j \frac{\delta\mathcal{L}}{\delta(\partial^\mu\xi^j)}\Lambda^{ij}(\xi) - \sum_j \frac{\delta\mathcal{L}}{\delta(\partial^\mu\psi^j)}\bar{\Lambda}^{ij}(\xi,\psi). \quad (6.7)$$

Inspection of the definitions of $\Lambda^{ij}(\xi)$ and $\bar{\Lambda}^{ij}(\xi,\psi)$ shows that

$$\Lambda^{ij}(\xi) = \delta_{ij} + O(\xi^2),$$
$$\bar{\Lambda}^{ij}(\xi,\psi) = O(\xi^2). \quad (6.8)$$

Therefore, an expansion of $A_\mu{}^i(\xi,\psi)$ in lower-order terms in ξ^i gives

$$A_\mu{}^i(\xi,\psi) = -\frac{\delta\mathcal{L}}{\delta(\partial^\mu\xi^i)} + O(\xi^2)$$
$$= F_i{}^2\partial_\mu\xi^i - (iG_F f_{ijk} + G_D d_{ijk})\bar{B}^j\gamma_\mu\gamma_5 B^k + O(\xi^2). \quad (6.9)$$

Thus when sandwiched between the vacuum and the single jth meson state of momentum p_μ,

$$\langle 0|A_\mu{}^i(x)|j,p_\mu\rangle = \delta_{ij}F_i\partial_\mu\langle 0|P^i(x)|i,p_\mu\rangle$$
$$= -\delta_{ij}F_i ip_\mu(2\pi)^{-3/2}e^{-ip\cdot x}. \quad (6.10)$$

Therefore, F_i are the weak-decay constants of the pseudoscalars which are related to their leptonic weak-decay lifetime τ, universal weak-interaction coupling G, and the Cabibbo angle θ. The leptonic decay lifetime of the π and K are well known. To calculate F_π and F_K from them, let us make the assumption that the Cabbibbo angle for leptonic pseudoscalar decay is the same as the pseudovector Cabibbo angle for leptonic hyperon decay. The latter has been determined by Filthuth[8] to be

$$\theta = 0.238 \pm 0.018.$$

With the values of the pion and kaon muonic-decay mean life τ as given in the Particle Data Group tables,[9]

$$\tau_\pi = (2.604 \pm 0.007)\times 10^{-8}\ \text{sec},$$
$$\tau_K{}^{-1} = (51.54 \pm 0.30)\times 10^6\ \text{sec}^{-1},$$

we obtain for the π and K the decay constants

$$F_\pi = 131.5 \pm 1.1\ \text{MeV},$$
$$F_K = 149.4 \pm 11.5\ \text{MeV}, \quad (6.11)$$

[8] H. Filthuth, Proceedings of the Topical Conference on Weak Interactions, CERN, 1969 (unpublished).
[9] Particle Data Group, Rev. Mod. Phys. 41, 109 (1969).

which, when substituted into (5.5), determine the constants

$$2x = 2.065 \times 10^4 \text{ MeV}^2 , \quad ca_\xi' = -5.802 \times 10^3 \text{ MeV}^2 , \tag{6.12}$$

and predict

$$F_8 = 154.9 \text{ MeV} . \tag{6.13}$$

Now taking the masses of the π and K to be the average of their respective isospin multiplet, the first two equations of (5.13) allow one to determine the parameters

$$(1/\sqrt{6})a_\xi = 4.615 \times 10^9 \text{ MeV}^2 , \quad (1/\sqrt{6})a_\xi c = -5.955 \times 10^9 \text{ MeV}^2 , \tag{6.14}$$

from which one obtains

$$c = -1.29 , \tag{6.15}$$

$$m_8 = 548.0 \text{ MeV} . \tag{6.16}$$

When this value of m_8 is compared with the mass of the η ($m_\eta = 548.8 \pm 0.6$ MeV), one sees that they are surprisingly close, less than 2 standard deviations apart. Therefore, the mixing of the η-η' system must be very small. Thus if this mixing is assumed to be nonexistent, then one sees that our mass formula fits the meson masses exceedingly well.

Note that we obtain a value of c slightly greater in absolute magnitude than that obtained by Gell-Mann, Oakes, and Renner.[5]

VII. STRONG-INTERACTION TRILINEAR COUPLING AND GOLDBERGER-TREIMAN RELATION

If we look at expression (6.9) for axial-vector currents, we notice that only the *derivative* part of the strong baryon-meson coupling contributes to the axial-vector currents. Thus the Goldberger-Treiman relation obtained from this model will not involve the total nucleon-pion coupling, but only its derivative part. To see this let us extract the pion-nucleon part of the total Lagrangian (4.13):

$$\mathcal{L}^{\pi N} = \mathcal{L}(\text{free}) + G_0^{NN\pi} F_\pi^{-1} \bar{B} \gamma^\mu \gamma_5 \boldsymbol{\tau} B \cdot \partial_\mu \boldsymbol{\pi} + i G_1^{NN\pi} F_\pi^{-1} \bar{N} \gamma_5 \boldsymbol{\tau} N \cdot \boldsymbol{\pi} , \tag{7.1}$$

where

$$G_0^{NN\pi} = \tfrac{1}{2}(G_F + G_D) , \tag{7.2}$$

$$G_1^{NN\pi} = (b_B + c_B)(c + \sqrt{2})/2\sqrt{3} . \tag{7.3}$$

Then the unrenormalized axial-vector coupling g_A to the nucleons and the strong πNN coupling g are

$$g_A = -G_0^{NN\pi} , \tag{7.4}$$

$$g = -(2M_N G_0^{NN\pi} + G_1^{NN\pi})/\sqrt{2} F_\pi , \tag{7.5}$$

where M_N is the mass of the nucleon. If g is broken into two parts,

$$g = g_0 + g_1 , \tag{7.6}$$

$$g_0 \equiv -2M_N G_0^{NN\pi}/\sqrt{2} F_\pi , \tag{7.7}$$

$$g_1 = -G_1^{NN\pi}/\sqrt{2} F_\pi , \tag{7.8}$$

then elimination of $G_0^{NN\pi}$ from (7.4) and (7.7) yields the modified Goldberger-Treiman relation

$$F_\pi g_0 = \sqrt{2} M_N g_A . \tag{7.9}$$

Therefore, the usual 10% "discrepancy" between experimental data and the original Goldberger-Treiman relation now allows us to calculate in this model the nonderivative part g_1 of the $NN\pi$ vertex.

VIII. CONCLUSION

We have demonstrated how easy it is to construct phenomenological Lagrangians for the $SU(3) \times SU(3)$ dynamical group with symmetry-breaking terms. Of course, one of the advantages of a Lagrangian theory is the unambiguous expression of currents and hence charges as functions of interacting fields, and we have found that these currents play an important role even in the determination of pseudoscalar masses.

Our $u_0 + cu_8$ type of breaking not only introduces a non-$SU(3)$-symmetric mass term, but also a non-$SU(3)$-symmetric kinetic term, so that, with the redefinition of particle fields $P^i = F_i \xi^i$, it gives a natural breaking in the weak decay constants. The reason for this behavior is that it is ξ^i, not P^i, that transforms as a unitary representation of $SU(3)$, and that the particle fields P^i do not form a *unitary* representation. It is also the same argument that leads to a modified Gell-Mann–Okubo mass formula. As to this mass formula, we have used the average mass of the isospin multiplets of the pseudoscalars to predict that there is little, if any, mixing between P^0 and P^8, so that we are tempted to conjecture that the mixing is identically zero, and that $\eta = P^8$, $\eta' = P^0$. However, the discrepancy between the η mass and the calculated m_8 is smaller than the mass difference in an isospin multiplet. To make a more conclusive statement would require a detailed knowledge of the mechanism of mass breaking within an isospin multiplet.

The discrepancy between the Goldberger-Treiman relation and experiment has long been known. Some have tried to estimate hadronic contributions into the unitarity sum other than the single-pion states, and found that they in fact contribute too poorly to bridge the gap.[10] If one really takes our modified Goldberger-Treiman relation seriously, then one is no longer disturbed by this discrepancy.

We have only dealt with the $\frac{1}{2}^+$-baryon octet. The virtual of the nonlinear realizations of $SU(3) \times SU(3)$ is that any $SU(3)$ multiplet can be included into the

[10] H. Pagels, Phys. Rev. **179**, 1337 (1969).

Lagrangian. By adding more and more multiplets into the vector ψ, we get a more and more complicated system. However, in the construction of the breaking term u_0+cu_8, more and more parameters are involved, and we feel that some principle is badly needed to curb this population explosion.

Finally, we would like to mention that we have not treated the problem of scattering, as the results are just the well-known ones of current-algebra calculations.

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Studies of Multiperipheral Integral Equations*

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The Chew-Goldberger-Low type of multiperipheral integral equation has been derived in terms of invariant variables, without assuming subenergies to be large compared with momentum transfers and particle masses. Both forward direction and nonforward direction have been worked out in detail, and the multiperipheral model with Toller-angle-dependent vertex functions has been discussed. We have furthermore demonstrated that all the qualitative physical properties of the model proposed by Amati, Bertocchi, Fubini, Stanghellini, and Tonin in 1962 remain true in this generalized multiperipheral model.

I. INTRODUCTION

SINCE early 1963[1] the multi-Regge model, which is one kind of multiperipheral model, has been used with some success to describe production processes at high energies. Since 1968 two new developments have made this model more attractive. One is the "duality hypothesis" proposed by Dolen, Horn, and Schmid[2] and extended by Chew and Pignotti[3]; another is the phenomenological model for production processes proposed by Chan, Łoskiewicz, and Allison (CŁA).[4] The "duality hypothesis" asserts that if we extrapolate the smooth high-energy Regge representation down to low energy, the Regge representation gives a certain semilocal average over the resonance peaks. This duality simplifies multiperipheral calculations enormously. The phenomenological CŁA model supplements the multi-Regge model with the assumption that the structure of nonresonant low-mass clusters is governed only by phase space. The CŁA model has been used to analyze data for $\pi^{\pm}p \to n\pi+p$, $K^-p \to n\pi+\Lambda$, $pp \to pp+n\pi$, $p\bar{p} \to n\pi$, and some other reactions of multiplicities n ranging from 2 to 9, and at $p_{\rm lab}$ ranging from 2 to 28 GeV/c.[4-6] The qualitative agreement with experiment is very good. Subsequently, Białas, Michejda, and Turnau[6] used this model, incorporating the usual Regge phase factor, to calculate the absorptive part of the two-particle elastic scattering amplitude. Assuming the amplitude to be purely imaginary at high energy for small momentum transfer, their result gives a sharp momentum-transfer dependence of the elastic differential cross section. This result had not previously been achieved from a dynamical model.

Recently Chew, Goldberger, and Low (CGL)[7] and Halliday[8] have proposed a multiperipheral model which contains the multi-Regge or CŁA models as a special case, and also leads to a unitarity integral equation for the elastic amplitude. The method of deriving this equation is very similar to the method used by Amati, Bertocchi, Fubini, Stanghellini, and Tonin[9] in their 1962 papers (the ABFST model).

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[1] See footnote 1 of Ref. 14 of this paper.

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