Unitarization of the Veneziano Amplitude: A Simple Potential-Theory Model*

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In order to see what we would expect to happen to the poles in the Veneziano amplitude if we were able to unitarize it, we consider a potential-theory model with an energy-dependent potential with poles on the real axis. We show how in the T matrix the poles are removed from the real axis since their residues vanish. In the unitarized single-pole approximation, we see explicitly how corresponding poles appear on the second sheet.

I. INTRODUCTION

MUCH attention has been focused recently on the model proposed by Veneziano.[1] This model is crossing symmetric and displays Regge behavior but does not satisfy unitarity. There have been many attempts to remedy this difficulty. Many authors have treated the Veneziano amplitude as a K matrix, and others have approached the problem by means of dual diagrams. Since the Veneziano amplitude can be thought of as an infinite sum of single-particle exchange graphs, another approach would be to treat it as a Born approximation. We then obtain the full amplitude by summing the Born series. If we wish to do this while maintaining the crossing symmetry, the only possible approach is to use perturbation theory.

This Born approximation, however, has the peculiar property of having poles on the real axis above threshold. Since this violates unitarity, the poles must be removed from the real axis and must be made to reappear on the second sheet. We wish to investigate the mechanism by which this takes place in a potential-theory model. We shall take a model with an energy-dependent potential with poles on the real axis, and we shall show that in the T matrix the renormalized residues vanish, and hence these poles are no longer present on the real axis. We shall see how in the simple, soluble, unitarized single-pole model, the corresponding poles appear on the second sheet.

II. MODEL

In potential theory, the off-shell T matrix satisfies the Lippmann-Schwinger equation[2]

$$T(\mathbf{q},\mathbf{q}',E)=V(\mathbf{q},\mathbf{q}')+\int\frac{d^3q''}{(2\pi)^3}V(\mathbf{q},\mathbf{q}'')\times\frac{1}{E-q''^2+i\epsilon}T(\mathbf{q}'',\mathbf{q}',E),$$

or

$$T=V+VGT. \tag{1}$$

The physical, on-shell amplitude at a real energy E is obtained by setting $q^2=q'^2=E+i\epsilon$. If this equation is iterated, we obtain the Born series $T=V+VGV+VGVGV+\cdots$. The potential is usually assumed to be real, symmetric, and independent of the parameter E. In this case, it is easy to derive the elastic unitarity relation[3]

$$\operatorname{Im}T(\mathbf{q},\mathbf{q}',E)=-\pi\int\frac{d^3q''}{(2\pi)^3}T(\mathbf{q},\mathbf{q}'',E)\times\delta(q''^2-E)T^\dagger(q'',q',E). \tag{2}$$

If we consider the Veneziano amplitude to be a Born approximation, it has the peculiar properties of having singularities (poles) on the real axis. We wish to see what would happen to such poles in a potential-theory model. We cannot have such a situation with a real, energy-independent potential. We shall, therefore, assume that the potential is a function of the energy[4] which can have singularities in the physical region:

$$\begin{aligned}\Delta V(\mathbf{q},\mathbf{q}',E)&=V(\mathbf{q},\mathbf{q}',E)-V^\dagger(\mathbf{q},\mathbf{q}',E),\\ \Delta T(\mathbf{q},\mathbf{q}',E)&=T(\mathbf{q},\mathbf{q}',E)-T^\dagger(\mathbf{q},\mathbf{q}',E),\\ \Delta G(\mathbf{q}'',E)&=G(\mathbf{q}'',E)-G^\dagger(\mathbf{q}'',E)=-2\pi i\delta(q''^2-E).\end{aligned} \tag{3}$$

We can then write the adjoint of Eq. (1) as

$$T^\dagger=V^\dagger+T^\dagger G^\dagger V^\dagger=V^\dagger+V^\dagger G^\dagger T^\dagger. \tag{4}$$

Subtracting Eq. (4) from Eq. (1), we obtain

$$\Delta T=\Delta V+VGT-V^\dagger G^\dagger T^\dagger. \tag{5}$$

Using Eq. (3) and rearranging terms, we get

$$[1-VG]\Delta T=V\Delta GT^\dagger+\Delta V[1+G^\dagger T^\dagger]. \tag{6}$$

Since it can easily be seen[5] that $[1+TG][1-VG]=1$ and $[1+TG]V=T$, we can multiply Eq. (6) by

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[1] G. Veneziano, Nuovo Cimento **57A**, 190 (1968).

[2] B. A. Lippmann and J. Schwinger, Phys. Rev. **79**, 469 (1950).

[3] See, e.g., V. A. Alessandrini and R. L. Omnes, Phys. Rev. **139**, B167 (1965).

[4] It is interesting to note that while we cannot have infinitely rising trajectories in a potential-theory model with an energy-independent potential, such a situation becomes possible when the potential is energy dependent [G. Tiktopoulos, Phys. Letters **29B**, 185 (1969); U. Trivedi, Phys. Rev. **188**, 2241 (1969)].

[5] R. J. Yaes, Phys. Rev. **170**, 1236 (1968).

$[1+TG]$ and obtain[6]

$$\Delta T=T\Delta GT^{\dagger}+[1+TG]\Delta V[1+G^{\dagger}T^{\dagger}]. \quad (7)$$

If V were real and $\Delta V=0$, we would have only the first term, and Eq. (7) would reduce to Eq. (2) (since $\mathrm{Im}T=2i\Delta T$), which is the elastic unitarity relation.

We shall now assume that V is real except for a sum of at most a denumerably infinite number of simple poles with factorizable residues on the real axis above threshold,

$$V(\mathbf{q},\mathbf{q}',E)=\tilde{V}(\mathbf{q},\mathbf{q}',E)+\sum_i \frac{\lambda_i f_i(\mathbf{q})f_i(\mathbf{q}')}{E-E_i}. \quad (8)$$

$\tilde{V}$ has no singularities above threshold. Each of the E_i are assumed to be real and greater than zero. The unrenormalized form factors $f_i(\mathbf{q})$ have the normalization

$$\int \frac{d^3q}{(2\pi)^3}\frac{[f_i(\mathbf{q})]^2}{(E_i-q^2)^2}=1. \quad (9)$$

The coupling constants λ_i then determine the strength of the coupling of the amplitude to the pole at E_i and are all assumed to be nonzero. Since $\tilde{V}$ is real, ΔV is just a sum of δ functions,

$$\Delta V(\mathbf{q},\mathbf{q}',E)=-2\pi i\sum_i \lambda_i f_i(\mathbf{q})\delta(E-E_i)f_i(\mathbf{q}'). \quad (10)$$

Defining the renormalized form factors by

$$F_i=[1-TG]f_i \quad (11)$$

and using Eq. (7), ΔT becomes

$$\frac{\Delta T(\mathbf{q},\mathbf{q}',E)}{-2\pi i}=\int\frac{d^3q''}{(2\pi)^3}T(\mathbf{q},\mathbf{q}'',E)\delta(E-q''^2)T^{\dagger}(\mathbf{q}'',\mathbf{q}',E)+\sum_i \lambda_i F_i(\mathbf{q})\delta(E-E_i)F_i^{\dagger}(\mathbf{q}'). \quad (12)$$

The first term is the contribution from the elastic unitarity cut. The second term appears to be the contribution from a sum of poles.

We can write the solution of Eq. (1) formally as

$$T=V[1-VG]^{-1}. \quad (13)$$

As we approach one of the poles, $E\to E_i$, V is dominated by the pole at E_i, and we have

$$V\simeq[r_i/(E-E_i)] \quad r_i(\mathbf{q},\mathbf{q}')=\lambda_i f_i(\mathbf{q})f_i(\mathbf{q}'). \quad (14)$$

So Eq. (13) becomes

$$T\simeq\frac{r_i}{E-E_i}\left[1-\left(\tilde{V}+\frac{r_i}{E-E_i}\right)G\right]^{-1}$$
$$=r_i[(1-\tilde{V}G)(E-E_i)-r_iG]^{-1}. \quad (15)$$

Because of the r_iG term in the square brackets, T does not behave like $[E-E_i]^{-1}$ in the neighborhood of E_i, and hence there is no pole at this point.

To see that this does not contradict Eq. (12), we write the resolvent identity[5]

$$[1-TG][1-VG]=1. \quad (16)$$

Since there is no pole on the right-hand side of Eq. (16), the coefficient of the pole on the left-hand side must vanish, giving us

$$[1+TG]f=F=0. \quad (17)$$

Hence the second term in the unitarity relation [Eq. (12)] vanishes identically, and the T matrix satisfies the exact elastic unitarity relation, Eq. (2). Since the coefficients of the poles vanish identically, the T matrix has no poles on the real axis.

However, if we refer back to Eq. (15), we see that if $\lambda_i\ll1$ and the r_iG term is therefore small, the square bracket will have a zero at a value E_i' in the neighborhood of E_i. We write E_i' in terms of its real and imaginary parts, $E_i'=E_R+iE_I$. Then in the neighborhood of E_i', if we assume $E_I\ll E_R$, we have

$$T\simeq\bar{f}(\mathbf{q})\bar{f}(\mathbf{q}')/(E-E_R-iE_i),$$
$$T^{\dagger}\simeq\bar{f}(\mathbf{q})\bar{f}(\mathbf{q}')/(E-E_R+iE_I), \quad (18)$$
$$\Delta T\simeq2iE_I\bar{f}(\mathbf{q})\bar{f}(\mathbf{q}')/(E-E_R)^2.$$

Substituting Eq. (14) into Eq. (2), we have

$$E_I=-\pi\int\frac{d^3q[\bar{f}(q)]^2}{(2\pi)^3}. \quad (19)$$

Since the integral is positive definite, $E_I<0$, and E_i must be on the second sheet below the unitarity cut.

We can see explicitly how the pole appears on the second sheet in a simple model which is exactly soluble. We treat the unitarized single-pole model, the properties of which have been discussed in detail by Amado.[7]

We take $\tilde{V}$ and all the λ_i except one to vanish. The potential then consists of a single pole,

$$V(\mathbf{q},\mathbf{q}',E)=\lambda f(\mathbf{q})f(\mathbf{q}')/(E-E_0). \quad (20)$$

Since the potential is now separable, the Lippmann-Schwinger equation becomes trivially soluble. The solution is[7]

$$T(\mathbf{q},\mathbf{q}',E)=\lambda f(\mathbf{q})P(E)f(\mathbf{q}'), \quad (21)$$

$$P(E)=\left[E-E_0-\lambda\int\frac{d^3q''}{(2\pi)^3}\frac{f^2(\mathbf{q}'')}{E+i\epsilon-q''^2}\right]^{-1}. \quad (22)$$

It should be clear that a pole appears in T only through P, and a pole appears in P when the expression in the square brackets vanishes. T then has a pole at E_0' which

[6] This relation was obtained for the quasipotential equation via a more circuitous route by P. N. Bogoliubov, Dubna Report (unpublished).

[7] R. D. Amado, Phys. Rev. **132**, 485 (1963).

is the solution of

$$E_0'=E_0+\lambda\int\frac{d^3q''}{(2\pi)^3}\frac{f^2(\mathbf{q}'')}{E_0'+i\epsilon-q''^2}\,. \tag{23}$$

Here we can see explicitly that in the case of weak coupling, $\lambda\ll1$, E_0' is in the neighborhood of E_0.

If both E_0 and E_0' are real and negative, we have Amado's renormalized bound-state model. It should be clear that there is no solution with E_0' real and positive since the pole would then appear in the range of integration and the integral term would pick up an imaginary part.

We must then search for a solution of the form $E_0'=E_R'+iE_I'$. To further simplify matters, we take the pole to be an s-state pole. Hence, the form factor is a function only of the square of the momentum, $f(\mathbf{q}'')=f(q''^2)$. Taking $q''^2=E''$,

$$\int\frac{d^3q''}{(2\pi)^3}\to\int_0^\infty\frac{E''^{1/2}dE''}{(2\pi)^2}\,,$$

and we can rewrite Eqs. (22) and (23):

$$P^{-1}(E)=\left[E-E_0+\int_0^\infty\frac{E''^{1/2}dE''}{(2\pi)^2}\frac{\lambda f^2(E'')}{E''-E+i\epsilon}\right]^{-1}, \tag{24}$$

$$E_0'=E_0-\int_0^\infty\frac{E''^{1/2}dE''}{(2\pi)^2}\frac{\lambda f^2(E'')}{E''-E_0'+i\epsilon}\,. \tag{25}$$

Taking $E_0'=E_R'+iE_I'$ in Eq. (21) and identifying the imaginary parts, we get

$$E_I\left[1+\frac{1}{(2\pi)^2}\int_0^\infty\frac{E''^{1/2}\lambda f^2(E'')dE''}{(E''-E_R)^2+E_I^2}\right]=0\,. \tag{26}$$

Since the integral is positive definite, the only solution is $E_I=0$. Hence there are no poles above threshold on the first sheet.

To see if there are poles on the second sheet, we continue $P^{-1}(E)$ onto the second sheet below the cut by adding the discontinuity $(-i/2\pi)E^{1/2}\lambda f^2(E)$. So that instead of Eq. (25) we have

$$E_0'=E_0-\int\frac{E''^{1/2}dE''}{(2\pi)^2}\frac{\lambda f^2(E'')}{E''-E_0'+i\epsilon}-\frac{i}{2\pi}E_0'^{1/2}\lambda f^2(E_0')\,, \tag{27}$$

which will have a solution above threshold. If we assume that $E_I'\ll E_R'$, we obtain

$$\begin{aligned}E_R'&=E_0-\frac{1}{(2\pi)^2}\int_0^\infty\frac{E''^{1/2}dE''\lambda f^2(E'')}{(E''-E_R)}\,,\\ E_I'&=\frac{E_R'^{1/2}\lambda f^2(E_R')}{2\pi}\Big/\left[1+\frac{1}{(2\pi)^2}\int_0^\infty\frac{f^2(E'')E''^{1/2}dE''}{(E''-E_R)^2+E_I^2}\right].\end{aligned} \tag{28}$$

It is also easy to see that if we continue onto the second sheet above the cut, a corresponding pole will appear.

III. DISCUSSION

We have seen how, in a simple potential-theory model, poles which appear on the real axis in the potential become poles on the second sheet in the T matrix. We would expect the same mechanism to be operative if we had a means for unitarizing the Veneziano amplitude. However, if we wish to do so, we must solve, not the Lippmann-Schwinger equation, but the crossing-symmetric Bethe-Salpeter equations.[8] As we have shown elsewhere,[9] these equations are essentially impossible to solve, not only because they are nonlinear, but also because the operators which appear therein are not mutually associative. We are thus forced to use perturbation theory. However, on the basis of what we have seen in the potential-theory model, we do not expect perturbation theory to give much useful information. We were able to remove the poles from the real axis and obtain the positions of the renormalized poles on the second sheet only by explicitly solving the equations, i.e., by summing the *whole* Born series. If we cut off the Born series at any point, the singularities would still be on the real axis, not on the second sheet. (This is similar to the way in which bound-state poles appear in potential theory, as the energies where the Born series diverges, not as poles in the individual terms of the series.) It thus seems that if the Veneziano amplitude is actually a Born approximation, we cannot obtain much information about the full amplitude by conventional techniques.

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[9] R. J. Yaes, Phys. Rev. D **2**, 2457 (1970).