demonstration, if cogent, is very far reaching. For the crux of the demonstration is to show that according to theory certain macroscopic observables, corresponding to the end product of a measurement (e.g., pointer positions), can take on no value whatsoever. The implication of this is that no laboratory observations can be cited in support of the quantum theory; e.g., the fact that an interference pattern emerges from a typical diffraction grating experiment is in *contradiction* with the theory. Surely, no one can take this seriously. The occurrence of an interference pattern, for example, is universally taken as supporting the theory. Thus in practice one treats the final superposed state of the joint object-apparatus system as having the same observational significance as some corresponding mixed state. The enormous difference between the two, which is the difference between the pointer aiming at some position or other in the mixed state but at no position at all in the superposed state, is treated as though it were no difference. This strategy of ignoring the difference is what I have referred to as an "approximate solution" to the measurement problem. It is the strategy adopted by all practitioners of the quantum theory, for it is the one that makes experimental support for the theory possible. There are, nevertheless, problems concerning the implementation of this strategy. These problems center around a precise formulation of alternative versions of the strategy, the scope and consistency of the interpretive rules embodied in these versions, and the type of support available for choosing one alternative over another. In Sec. V I have tried to lay the groundwork for discussing some of these issues. A clear and general statement of the proposed interpretive rules remains to be given.

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Examination of the "Leakage-Lifetime" Approximation in Cosmic-Ray Diffusion

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The diffusion of cosmic-ray particles in a finite volume of space with a simultaneous diffusion and/or transport in energy is considered. The solution of the appropriate differential equation may be expressed as an expansion in eigenfunctions of the differential operator. If one approximates the solution by keeping only the lowest or fundamental eigenfunction, one obtains the common "leakage-lifetime" approximation. In some situations this approximation can be justified, but in others (e.g., synchrotron or inverse-Compton losses) it cannot. The reason for the failure in this case can be seen from the point of view of the expansion. The solution of the general case of Fermi acceleration, synchrotron losses, and energy fluctuation acting together is also obtained by this method.

I. INTRODUCTION

THERE has been recent discussion in the literature as to the correct method of treating the loss of particles from a region of space where spatial diffusion and energy transport and/or diffusion are occurring simultaneously. A common method of treating this situation when it has arisen in the field of cosmic physics has been to describe it by an inhomogeneous, partial differential equation

$$\frac{\partial\rho(E,t)}{\partial t}+\mathcal{L}_E\rho(E,t)+\frac{\rho(E,t)}{\tau}=q(E,t)\,. \qquad (1)$$

In Eq. (1), $\mathcal{L}_E$ is a differential operator in energy that describes the various energy-changing processes at work within the region, τ is the average lifetime of a particle against a variety of loss mechanisms, including leakage from the boundary, and $q(E,t)$ describes the energy distribution of the particles when they are introduced into the region; the inhomogeneous term q is often referred to as the injection spectrum.

Solutions of this equation are usually sought for the steady-state case $\partial\rho/\partial t=0$ for a variety of energy-transport mechanisms and injection spectra. In his now classic papers, Fermi[1,2] in essence solved this equation for the case $\mathcal{L}_E\rho=\partial(aE\rho)/\partial E$. In his first paper,[1] he considered $\tau=\tau_c$, the lifetime against nuclear collisions of the cosmic-ray particles. In his second paper,[2] he had come to the opinion that diffusive leakage from the galaxy was the most significant loss mechanism and hence considered $\tau=\tau_e$ or the "leakage lifetime." At present it is not believed to be very likely that Fermi's mechanism offers the correct explanation of cosmic rays; however, it is generally believed that he presented a correct treatment of a plausible process that should, in fact, occur even though it might not produce cosmic rays.

The time-independent form of Eq. (1) has been used extensively to calculate equilibrium spectra for a wide variety of problems in cosmic physics. Energy loss as

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well as energy gain has been included in $\mathcal{L}_E$ and many different injection spectra have been considered. The use of this approach has been too widespread to give any references that would be complete. However, recently a particular application has been made to the case of cosmic-ray electrons,[3–12] where it is assumed that the injection spectrum is of the form $q(E)=kE^{-p}$ and that synchrotron radiation and inverse Compton scattering are the main contributers to the energy transport term, i.e., $\mathcal{L}_E\rho=\partial(-bE^2\rho)/\partial E$. This analysis predicts a solution of the form $\rho(E)\propto E^{-p}$ for $E\ll E_c$ and $\rho(E)\propto E^{-(p+1)}$ for $E\gg E_c$, where $E_c=(b\tau_e)^{-1}$. The significance of this analysis lies in the fact that an observation of the "break" in the spectrum at $E=E_c$ (although the term "break" should be used with great caution as I shall show in Sec. II) will yield the product $b\tau_e$, with the result that an assumption about the value of b (the strength of the magnetic field or the energy density of radiation) will give the properties of the diffusing region through the relation $\tau_e\approx R^2/D$, where R is the characteristic dimension of the diffusing region and D is the diffusion coefficient.

In all of the treatments of cosmic-ray electrons, the inhomogeneous term is included, but in some treatments of cosmic radio sources[13,14] and x-ray sources[15] the homogeneous equation is used. It is not appropriate for cosmic-ray electrons for, as Kardashev[13] points out, one cannot obtain steady flat spectra in the presence of inverse-Compton or synchrotron losses. Melrose[14] asserts the contrary and obtains solutions of the homogeneous equation that resemble the cosmic-ray-electron spectrum. There appear to be serious errors[16] in Melrose's paper, however, which invalidate his conclusion, so I shall consider only equations with the injection spectrum included hereafter.

In spite of its widespread use, Eq. (1) has been recently questioned by Jokipii and Meyer,[17] who quite correctly point out that the analogy between a collision lifetime and leakage lifetime is not a valid one. The probability of loss by a catastrophic event such as a nuclear collision may properly be characterized by a uniform probability per unit time but the probability that a particle will be lost by leakage at the boundary depends on the position of observation and even then cannot be characterized by a uniform probability per unit time.

At any point in the diffusing region the "ages" (time since injection) of the particles will be distributed exponentially with a mean age τ_c if collision is the dominant loss mechanism. If diffusive loss is dominant, on the other hand, the age distribution cannot, in general, be considered an exponential even with a variable mean age $\tau_e=\tau_e(\mathbf{r})$. Jokipii and Meyer point out that the correct equation to solve is not (1) but rather

$$\mathcal{L}_E\rho(E,\mathbf{r})-\nabla\cdot[D\nabla\rho(E,\mathbf{r})]+\rho(E,\mathbf{r})/\tau_c=q(E,\mathbf{r})\,, \quad (2)$$

where D is the diffusion coefficient and once again we consider the equilibrium case, $\partial\rho/\partial t=0$. The solution of this equation with the appropriate boundary conditions will yield the correct energy spectrum at any point.

Earlier, Shen[11] had pointed out that the diffusion term was important at high energies since here the leakage lifetime could be much longer than the lifetime against synchrotron and inverse-Compton losses and the spectrum could therefore depend on the spatial distribution of the sources. Nevertheless, Shen continued to treat the boundary conditions by means of the leakage-lifetime approximation.

Jokipii and Meyer consider a flat disk source of electrons embedded in a diffusing medium of infinite extent. They find that rather than one break of one power in the exponent, they obtain two breaks of one-half power each at energies $E_1=D/bR_1^2$, $E_2=D/bR_2^2$, where R_1 and R_2 are the diameter and thickness of the disk, respectively. A similar calculation had been performed earlier by Syrovatskii,[18] but his results were not presented in such a way as to make comparison with the leakage-lifetime approximation very easy.

More recently, Dogel and Syrovatskii[19] have considered a similarly shaped source distribution, but instead of an infinite diffusing medium, they consider a spherical region with free escape at the boundary of radius R_0, where $R_0\approx R_1$. They obtain results essentially identical to those of Jokipii and Meyer. Further departures from the leakage-lifetime approximation are indicated by Longair and Sunyaev.[20] Furthermore, Berkey and Shen[21] claim and amply demonstrate that in general the lifetime of a cosmic-ray electron in a general diffusing region has little or nothing to do with the equilibrium spectrum when sychrotron and in-

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verse-Compton losses are important. Rather the structure of the sources is the determining factor.

Clearly the leakage-lifetime approach does not work well for cosmic-ray electrons. This brings to mind the question of whether the theory of Fermi acceleration might also be in error. It turns out that this and other questions may be investigated from a general point of view. If Eq. (2) can be solved by means of an eigenfunction expansion of the Green's function, the leakage-lifetime approximation may be seen in its natural mathematical setting, and the question of its validity may be more easily investigated.

In Sec. II the method of eigenfunction expansion will be employed in this investigation. The approach will be entirely heuristic and nonrigorous; equations and expansions will be written down without justification and solutions to differential equations will be given with no derivation. The mathematical justification and unification of this material will be left to Sec. III and the Appendix. It is hoped that with this arrangement those readers who are interested only in the physical results as applied to cosmic physics may obtain these by reading only Sec. II (and perhaps Sec. IV) while those of a more mathematical bent can have their need for more rigor satisfied by reading on through Sec. III.

In Sec. II we discuss three particular physical processes: (a) Fermi acceleration, (b) synchrotron and inverse-Compton losses with an inverse-power-law injection spectrum, and (c) the full Fokker-Planck energy operator of the form

$$\mathcal{L}_E\rho=\frac{\partial}{\partial E}(a_1E-a_2E^2)\rho-\frac{1}{2}\frac{\partial^2}{\partial E^2}(a_3E^2)\rho\,.$$

In the above expression the term proportional to a_1 describes both Fermi acceleration and bremsstrahlung losses, and the term proportional to a_2 describes synchrotron and inverse-Compton losses. The second-order term proportional to a_3 describes a statistical spreading or diffusion in energy space. This term was first considered in a cosmic-ray setting by Terletski and Luganov[22] and later in a more general treatment by Davis.[23] Further discussion of the importance of this term may be found in work by Morrison[24] and it has been employed in many later works.[13–15,25]

Although all of the processes discussed above have been treated by many authors, to the knowledge of the present author the combination of the above mentioned Fokker-Planck energy operator with spatial diffusion, particle losses, and injection has never been treated before. It is therefore believed that the results of this part of Sec. II are essentially new.

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II. APPROXIMATION, ITS SETTING AND VALIDITY

Let us assume that the solution to Eq. (2) may be written as a series

$$\rho(E,\mathbf{r})=\sum_{\mathbf{k}}R_{\mathbf{k}}(\mathbf{r})f_{\mathbf{k}}(E)\,,\tag{3}$$

where the $R_{\mathbf{k}}(\mathbf{r})$ are a complete set of orthonormal eigensolutions of the diffusion operator and the spatial boundary conditions. This means

$$\nabla\cdot[D\nabla R_{\mathbf{k}}(\mathbf{r})]=-k^2R_{\mathbf{k}}(\mathbf{r})\,,\quad k^2=\mathbf{k}\cdot\mathbf{k}\tag{4}$$

and

$$\int R_{\mathbf{k}'}(\mathbf{r})R_{\mathbf{k}}(\mathbf{r})d^3r=\delta_{\mathbf{k},\mathbf{k}'}\,,\tag{5}$$

where $\delta_{\mathbf{k}',\mathbf{k}}=1$ if $\mathbf{k}=\mathbf{k}'$ and $=0$ otherwise and the integration is taken over the entire diffusing volume.

If we now insert expression (3) for $\rho(E,\mathbf{r})$ into Eq. (2), multiply Eq. (2) by $R_{\mathbf{k}'}(\mathbf{r})$ and integrate over all space, we obtain the following equation for $f_{\mathbf{k}}(E)$ by making use of the relations (4) and (5):

$$\mathcal{L}_Ef_{\mathbf{k}}(E)+(k^2+1/\tau_c)f_{\mathbf{k}}(E)=q_{\mathbf{k}}(E)\,,\tag{6}$$

where

$$q_{\mathbf{k}}(E)=\int R_{\mathbf{k}}(\mathbf{r})q(E,\mathbf{r})d^3r\,.\tag{7}$$

We may now enquire into the physical meaning of the eigenvalue $\mathbf{k}$. We note that in the time-dependent case the eigenmode with a particular value of $\mathbf{k}$ has the time dependence e^{-k^2t}. Thus k^{-2} is the decay time of that mode. In particular, for the fundamental mode, the mode with the smallest value of k^2, we have the approximate relation

$$k_0^2\approx D/L^2\approx\bar{v}\lambda/L^2\,,\tag{8}$$

where $\bar{v}$ is the average particle speed, λ is the mean free path, and L is a typical linear dimension of the diffusing region. Expression (8) applies to the case where the boundary may be rather easily penetrated. If α is the ratio of particles penetrating the wall to those striking the wall, (8) applies to the case $\alpha\gtrsim\lambda/L$. If, however, $\alpha\ll\lambda/L$, we have

$$k_0^2\approx\alpha\bar{v}/L\,.\tag{9}$$

In either case we note that k_0^{-2} is the average time it takes a particle to leave the diffusing region either by a random walk of distance L to an easily penetrated boundary or by repeatedly striking a highly reflecting wall. If we then identify k_0^{-2} with τ_e, we may write (6) as

$$\mathcal{L}_Ef_0(E)+f_0(E)/\tau=q_0(E)\,,\tag{6$'$}$$

where $\tau^{-1}=\tau_e^{-1}+\tau_c^{-1}$ and $f_0(E)\equiv f_{\mathbf{k}_0}(E)$.

It is now immediately obvious that we have recovered the leakage-lifetime approximation if we neglect all modes in (3) but the lowest or fundamental mode. It should be equally obvious that the validity of approxi-

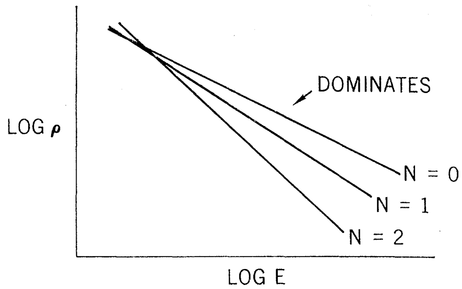

FIG. 1. Plot of $\log\rho(E)$ versus $\log E$ for the first three modes of Fermi acceleration showing how the fundamental mode $N=0$ dominates at high energy.

mation can depend in a complex way on the particular form of the higher modes. In the following sections we shall investigate this matter for several cases of interest in cosmic physics.

A. Fermi Acceleration

In this and the subsequent sections we shall assume that the injection or source function is separable, i.e., $q(E,\mathbf{r})+S(\mathbf{r})q(E)$. There is no particular loss of generality due to this choice and it does appear to be in accord with accepted ideas in the field of cosmic physics.

For Fermi acceleration[1,2] we have $\mathfrak{L}_E f=\partial(aEf)/\partial E$ and the usual choice of injection spectrum is a δ function at some low energy E_0. We have then

$$\frac{\partial}{\partial E}(aEf_{\mathbf{k}})+\frac{f_{\mathbf{k}}}{\tau_{\mathbf{k}}}=S(\mathbf{k})\delta(E-E_0)\,, \qquad (10)$$

where

$$\tau_{\mathbf{k}}^{-1}=k^2+1/\tau_c$$

and the corresponding form of the solution is

$$\rho(E,\mathbf{r})=\sum_{\mathbf{k}} C_{\mathbf{k}}R_{\mathbf{k}}(\mathbf{r})E^{-(1+1/a\tau_k)} \qquad (11)$$

and

$$C_{\mathbf{k}}=\frac{E_0{}^{\tau k/a}}{a}\int S(\mathbf{r})R_{\mathbf{k}}(\mathbf{r})d^3r\,. \qquad (12)$$

Since as k increases so does τ_k^{-1}, we see that $1+1/a\tau_k$ is an increasing function of k. For this reason the higher the mode, the steeper the spectrum, and for some energy $E\gg E_0$, one can neglect all but the fundamental mode. This is illustrated in Fig. 1, where $\log\rho(E)$ versus $\log E$ is sketched for the first three modes. In the case of Fermi's origin theory, the next mode to be considered (the first harmonic) would be of the form $E^{-\Gamma}$ where $\Gamma\approx 4$, so one may say that the leakage-lifetime approach was quite justified in this case.

B. Synchrotron and Inverse-Compton Losses

The process of synchrotron radiation and inverse-Compton effect cause high-energy electrons moving through a magnetic field or a radiation field to lose energy at a rate proportional to the square of their energy. For this case $\mathfrak{L}_E f=\partial(-bE^2 f)/\partial E$. The situation to which the leakage-lifetime approximation has been most often applied is that of an inverse-power-law spectrum $q(E)\propto E^{-p}$. For this case

$$\frac{\partial}{\partial E}[-bE^2 f_{\mathbf{k}}(E)]+\frac{f_{\mathbf{k}}(E)}{\tau_{\mathbf{k}}}=C_{\mathbf{k}}E^{-p}\,.$$

Once again we have

$$\rho(E,\mathbf{r})=\sum_{\mathbf{k}} C_{\mathbf{k}}R_{\mathbf{k}}(\mathbf{r})f_{\mathbf{k}}(E)\,, \qquad (11')$$

where

$$C_{\mathbf{k}}=\int S(r)R_{\mathbf{k}}(\mathbf{r})d^3r \qquad (12')$$

and

$$f_{\mathbf{k}}(E)=\frac{\exp[-(bE\tau_k)^{-1}]}{bE^2}\times\int_E^\infty E'^{-p}\exp[(bE'\tau_k)^{-1}]dE'\,. \qquad (13)$$

This expression is well known; it has been pointed out many times that $f_k(E)$ has two distinct asymptotic forms. For $E\ll E_c\equiv(\tau_k b)^{-1}$, $f_{\mathbf{k}}(E)\approx\tau_k E^{-p}$ and for $E\gg E_c$, $f_{\mathbf{k}}(E)\approx E^{-(p+1)}/b(p-1)$.

This is the spectrum (with $\tau_k=\tau_e$) that appears in the leakage-lifetime approximation, and it has led many authors[3,5-10] to refer to a break in the spectrum at the critical energy E_c. This is, in fact, rather unrealistic. In Fig. 2 we see plotted the value of the effective spectral-index increment

$$\delta=-d\log_{10}f/d\log_{10}x-p \quad \text{versus} \quad \log_{10}(p-1)x\,,$$

where

$$f=x^{-2}e^{-1/x}\int_x^\infty (x')^{-p}e^{1/x'}dx'$$

and $p=2.5$. We can see that the spectral index does not sharply break, but changes from 2.5 to 3.5 very smoothly over about *two decades* of parameter $x=E/E_c$. For comparison one should check the spectrum curves in the figures of Refs. 3, 5, and 8–10. At this point it should also be pointed out that the spectrum of photons produced by these particles in the inverse-Compton or synchrotron process will change its slope over about four decades, a fact that should be kept in mind when interpreting breaks in x-ray or γ-ray spectra.

In this situation, the effect of including higher modes is somewhat more complicated. At first glance one might think that there would always be a steepening at the critical energy of the fundamental mode $E_c=(b\tau_0)^{-1}\approx D\pi^2/bR^2$, where R is the linear dimension of the diffusing volume and for the moment we have neglected τ_c, assuming it to be very long compared to diffusion times. As a matter of fact, just such an argument was made[6] in attempting to explain the shape of the galactic cosmic-ray-electron spectrum. Such a conclusion would seem to be borne out by the calculations of Dogel and Syrovatskii,[19] who consider a spherical diffusing region of radius R_0 and an ellipsoidal source region of axes R_1 and R_2 where $R_1 \approx R_0$ and $R_1/R_2 = 10^2$. They obtain a spectrum that steepens by $\frac{1}{2}$ in the exponent at an energy of $D\pi^2/bR_0^2$ and again by a factor $\frac{1}{2}$ at an energy $D\pi^2/bR_2^2$. This would appear to indicate that R_0 is a dimension that determines the first break in the spectrum. This is misleading; it is, in fact, the dimension R_1, which in this problem is equal to R_0, that determines the position of the first break. The size of the diffusing region has very little effect on the shape of the equilibrium spectrum. This is almost entirely determined by the size and shape of the source region.

This point has been made by Berkey and Shen[21] and has been illustrated by them in a model calculation where a spherical source of radius B in a concentric diffusing volume of radius R is considered to inject electrons with a spectrum $q(E) \propto E^{-2}$. For this injection spectrum $f_{\rm k}(E)$ simplifies and may be expressed in closed form, i.e.,

$$f_{\rm k}(E) = \tau_k E^{-2}[1 - \exp(-1/b\tau_k E)].$$

For completeness we shall consider essentially the same case as Berkey and Shen but instead of limiting our consideration to an injection spectrum proportional to E^{-2} we shall consider a general power law and make use of a convenient approximate form of $f_{\rm k}(E)$. We shall also see that to an observer inside the source region, the outer diffusing volume serves only as a "quantization volume" for the eigenfunctions and for this reason has relatively little effect on the solution.

Consider a spherical diffusing region of radius R and a concentric source region of radius B with $B < R$. Due to the spherical symmetry of the problem, the eigenfunctions are $j_0(k_m r)$, where $k_m = m\pi/R$ and $j_0(k_m r) = (k_m r)^{-1} \sin(k_m r)$. If the source strength is η particles per unit volume, the resulting density will be

$$\rho(E,r) = \sum_{m=1}^{\infty} C_m j_0(k_m r) f_m(E), \tag{14}$$

where

$$C_m = \frac{\eta B}{R}\left[\frac{\sin(k_m B)}{k_m B} - \cos(k_m B)\right] \tag{15}$$

and $f_m(E)$ is given by Eq. (13).

A good approximation to (13) is given by

$$\tilde{f}_m(E) = \frac{E^{-p}}{(\tau_m)^{-1} + (p-1)bE} = \frac{E^{-p}}{(Dm^2\pi^2/R^2) + (p-1)bE}. \tag{16}$$

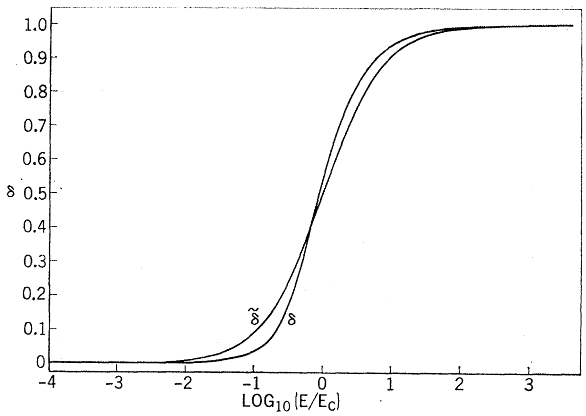

FIG. 2. Plot of the spectral index increment $\delta = -d \log_{10} f / d \log_{10} x - p$ versus $\log_{10} x$ $(x = E/E_c)$ for the functions

$$f = x^{-2} \exp[-(p-1)/x] \int_{x/(p-1)}^{\infty} (x')^{-p} \exp[(p-1)/x'] dx'$$

and $\tilde{\delta}$ for the approximate function $\tilde{f} = x^{-p}/(1+x)$, where $E_c = [(p-1)\tau_n b]^{-1}$ and $p = 2.5$.

It is easy to see that $\tilde{f}_m$ has the same asymptotic properties as f_m, and we can see in Fig. 2 that the spectral index is a similar function of E. The major difference between $\tilde{f}_m$ and f_m is that $\tilde{f}_m$ is overwhelmingly simpler.

Now using $\tilde{f}_m$ instead of f_m in (14), we may write

$$\rho(E,r) = \frac{\eta B}{RD} \sum_{m=1}^{\infty} \left[\frac{\sin(k_m B)}{k_m B} - \cos(k_m B)\right] \times \frac{\sin(k_m r)}{k_m r} \frac{E^{-p}}{(k_m^2 + k_E^2)}, \tag{17}$$

where $k_E = [(p-1)bE/D]^{1/2}$. Since $\Delta m = 1$, $\pi/R = \Delta m\pi/R = \Delta k_m$, we may write (17) as

$$\rho(E,r) = \frac{\eta B E^{-p}}{\pi D} \sum_{m=1}^{\infty} \Delta k_m \left[\frac{\sin(k_m B)}{k_m B} - \cos(k_m B)\right] \times \frac{\sin(k_m r)}{k_m r (k_m^2 + k_E^2)}. \tag{17'}$$

The only role that R plays in the above expression is to determine the closeness of the level spacing Δk_m. We can therefore see that the total diffusing region acts only as a quantization volume as in quantum mechanics and does not have a profound effect on the system (of size $\approx B$) provided $B \ll R$. In fact, if $B \ll R$, there will be many levels contributing to a small variation of $k_m B$ or $k_m r$, and we may approximate the sum in (17') by an integral

$$\rho(E,r) \approx \frac{\eta B E^{-p}}{\pi D} \int_0^{\infty} dk \left[\frac{\sin(kB)}{kB} - \cos(kB)\right] \times \frac{\sin(kr)}{kr(k^2 + k_E^2)}, \tag{18}$$

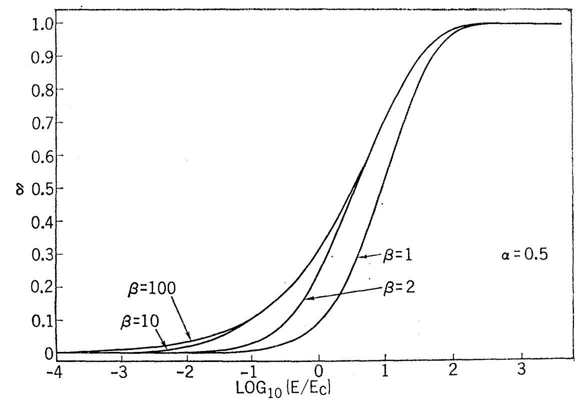

FIG. 3. Plot of the spectral index increment $\delta=-d\log_{10}\rho/d\log_{10}x-p$ versus $\log_{10}x$; $x=E/E_c$. ρ is given by Eq. (21); $\alpha\equiv r/B=0.5$; $\beta\equiv R/B=1$, 2, 10, and 100; $E_c=D[(p-1)bB^2]^{-1}$; and $p=2.5$.

and R no longer appears in the expression at all. This holds true for $f_m(E)$ as well as for $\tilde{f}_m(E)$. In fact, it would hold for any spectrum that is a function of k such that the integral or sum is not dominated by the lowest values of k. If it is so dominated, the approximation $k_{\min}=0$ is not valid and the result will be a function of $k_{\min}$ and hence R, since $k_{\min}=k_1=\pi/R$. The Fermi acceleration spectrum $f_k(E)\propto E^{-1-k^2/a}$ can be seen to be of the kind that is dominated by the smallest values of k.

If we note that $C_k\approx\frac{1}{3}k^2B^2$ for $kB\ll1$ and oscillates for $kB\gg1$, we may approximate the integral by simply cutting it off at $k=1/B$ and writing $(kr)^{-1}\sin(kr)\approx1$ (assuming $r<B$). We obtain

$$\rho(E,r)\approx\frac{\eta B^3E^{-p}}{3\pi D}\int_0^{1/B}\frac{k^2dk}{k^2+k_E^2}=\frac{\eta B^2E^{-p}}{3\pi D}[1-k_EB\tan^{-1}(1/k_EB)].\quad(19)$$

There are two asymptotic forms of (19), depending on the magnitude of k_EB:

$$\rho(E)\approx\frac{\eta B^2E^{-p}}{3\pi D},\quad k_EB\ll1\quad(20)$$

$$\rho(E)\approx\frac{\eta E^{-(p+1)}}{9\pi(p-1)b},\quad k_EB\gg1.\quad(20')$$

The critical energy is therefore seen to be determined by

$$k_EB\approx1\ \text{ or }\ E_c=D/(p-1)bB^2.$$

Expression (17) may in fact be summed exactly, giving

$$\rho(E,r)=\frac{\eta E^{-(p+1)}}{(p-1)2b}\left\{1+\left[(k_EB)\sinh(k_EB)-\cosh(k_EB)+\frac{\sinh(k_EB)-(k_EB)\cosh(k_EB)}{\tanh(k_ER)}\right]\times\frac{\sinh(k_Er)}{k_Er}\right\},\quad r<B\quad(21)$$

$$=\frac{\eta E^{-(p+1)}}{(p-1)2b}\left\{[(k_EB)\cosh(k_EB)-\sinh(k_EB)]\times\left[\frac{\cosh(k_Er)}{k_Er}-\frac{\sinh(k_Er)}{(k_Er)\tanh(k_ER)}\right]\right\},\quad r>B.\quad(21')$$

If $k_EB\ll1$, we may write (21) and (21′)

$$\rho(E,r)\approx\frac{\eta B^2E^{-p}}{4D}\left[1-\frac{1}{3}\left(\frac{r}{B}\right)^2-\frac{2}{3}\frac{k_EB}{\tanh(k_ER)}\right],\quad r<B,\quad(22)$$

$$\approx\frac{\eta B^3E^{-p}}{6Dr}\left[\cosh(k_Er)-\frac{\sinh(k_Er)}{\tanh(k_ER)}\right],\quad r>B.\quad(22')$$

For $k_EB\gg1$, we have

$$\rho(E,r)\approx\frac{\eta E^{-(p+1)}}{(p-1)2b}\left[1-(1+k_EB)\frac{\sinh(k_Er)}{k_Er}e^{-k_EB}\right],\quad r<B,\quad(23)$$

$$\approx\frac{\eta E^{-(p+1)}}{(p-1)2b}\frac{(k_EB-1)}{2k_Er}e^{-k_E(r-B)},\quad r>B.\quad(23')$$

We can see at once that for $r<B$ the leading terms are just those, apart from a numerical factor, obtained by the simple calculation of expressions (20) and (20′). The correction terms are small and do not affect the spectrum appreciably. If $R\approx B$ the third term in (22) is of the order unity; but in this case the two critical energies, $D/(p-1)bR^2$ and $D/(p-1)bB^2$, are approximately equal and one still sees only one break at a critical energy that is determined by the size of the source region rather than the whole diffusing volume.

In Fig. 3 the spectral index increment δ has been plotted as a function of $x=E/E_c$, where $E_c=D/(p-1)bB^2$. ρ is given by Eq. (21), $r=\frac{1}{2}B$, and $R/B=1$, 2, 10, and 100.

The primary effect of increasing R from B to $100B$ is to broaden the transition region to some degree. However, the transition remains in the vicinity of $E_c(B)$, and no partial break appears at a different characteristic energy $E_c(R)$.

From this we can see why the results of Dogel and Syrovatskii[19] agree so well with those of Jokipii and Meyer[17] even though the latter considered the case $R_0=\infty$.

C. Solution for Fokker-Planck Energy Operator

The Fokker-Planck energy operator is

$$\mathfrak{L}_Ef=\frac{\partial}{\partial E}(a_1E-a_2E^2)f-\frac{1}{2}\frac{\partial^2}{\partial E^2}(a_3E^2)f.\quad(24)$$

Inserting this operator into Eq. (6), we are led to a solution for the spectrum

$$f_k(E)=\int_0^\infty g_k(E,E')q_k(E')dE', \qquad (25)$$

where q_k is defined by (7) and

$$g_k(E,E')=\frac{E^\beta e^{-\delta E}\Gamma(-1-\beta+s_k)}{2a_2s_k(E')^{1+\beta}\Gamma(2s_k)}\left(\frac{E_<}{E_>}\right)^{s_k} \times {}_1F_1(-1-\beta+s_k;1+2s_k;\delta E_<)U(s_k,\delta E_>). \qquad (26)$$

In the above expression, ${}_1F_1(A;B;x)$ is the confluent hypergeometric function defined by

$${}_1F_1(A;B;x)\equiv\sum_{n=0}^\infty \frac{A(A+1)\cdots(A+n-1)}{B(B+1)\cdots(B+n-1)}\frac{x^n}{n!}, \qquad (27)$$

and

$$U(s_k,\delta E_>) \equiv \frac{\Gamma(-2s_k)}{\Gamma(-1-\beta-s_k)}{}_1F_1(-1-\beta+s_k;1+2s_k;\delta E_>) +(\delta E_>)^{-2s_k}\frac{\Gamma(2s_k)}{\Gamma(-1-\beta+s_k)} \times {}_1F_1(-1-\beta-s_k;1-2s_k;\delta E_>). \qquad (28)$$

Also,

$$\delta=a_2/a_3,\quad \beta=\tfrac{1}{2}(a_1/a_3)-\tfrac{3}{2},$$
$$s_k=[(1+\beta)^2+(k^2+\tau_c^{-1})/a_3]^{1/2},$$

and $E_>$ ($E_<$) is the larger (smaller) of E and E'.

The Green's function of Eq. (26) is quite complicated, and one would not expect simple spectra to result in general; however, in the case of an inverse-power-law injection $q(E)\propto E^{-p}$, one simple result can be obtained. If we examine the case of large E, we may investigate the asymptotic form of the Green's function employing results from Sec. III. If $E>E'$, we have

$$g_k(E,E')\approx\frac{\Gamma(-1-\beta-s_k)}{2a_2s_k\Gamma(2s_k)\delta}\frac{E^{1+2\beta}}{(E')^{3+2\beta}}e^{-\delta(E-E')}. \qquad (29)$$

The exponential quickly damps any contribution from values of E' much removed from E, so little contribution is obtained from energies $E'<E$. On the other hand, if $E'>E$ we have

$$g_k(E,E')\approx\frac{\Gamma(-1-\beta-s_k)}{2a_2s_k\Gamma(2s_k)\delta}E^{-2}. \qquad (30)$$

The resulting spectrum is approximately

$$f_k(E)\approx\int_E^\infty g_k(E,E')E'^{-p}dE' \approx\frac{\Gamma(-1-\beta+s_k)}{2a_2s_k\Gamma(2s_k)\delta}E^{-2}\int_E^\infty E'^{-p}dE' =\text{const}\times E^{-(p+1)}. \qquad (31)$$

This result is the same as for the case of synchrotron and inverse-Compton losses alone. This is indeed reasonable since if one assumes that $f(E)$ is a power law, the original differential equation [Eq. (24)] shows that as $E\to\infty$ the synchrotron or Compton-loss term proportional to a_2 dominates the entire process.

A more detailed examination of the asymptotic forms of ${}_1F_1$ and U shows that the asymptotic forms are valid if we have

$$\frac{s_k^2-(1+\beta)^2}{\delta E}\ll 1.$$

Inserting the expression for s_k, β, and δ, we obtain the condition

$$E\gg\frac{k^2+\tau_c^{-1}}{a_2}=(a_2\tau_k)^{-1},$$

where τ_k is the total lifetime for the kth mode. This is the same condition as for synchrotron and inverse-Compton losses alone, so we may conclude that inclusion of energy diffusion and Fermi acceleration does not substantially alter the form of cosmic-ray-electron spectra at high energies.

III. MATHEMATICAL METHOD

The equation we wish to solve is

$$\mathcal{L}_E\rho(E,\mathbf{r})-\nabla\cdot[D\nabla\rho(E,\mathbf{r})]+\rho(E,\mathbf{r})/\tau_c=q(E,\mathbf{r}). \qquad (2)$$

If we define the total operator $\mathcal{L}$ by

$$\mathcal{L}=\mathcal{L}_E-\nabla\cdot D\nabla+\tau_c^{-1},$$

we may write (2) as

$$\mathcal{L}\rho(E,\mathbf{r})=q(E,\mathbf{r}). \qquad (2')$$

The solution may be written in terms of the Green's function

$$\rho(E,\mathbf{r})=\int d^3r'dE'G(E,\mathbf{r};E',\mathbf{r}')q(E',\mathbf{r}'), \qquad (32)$$

where

$$\mathcal{L}G(E,\mathbf{r};E',\mathbf{r}')=\delta(E-E')\delta(\mathbf{r}-\mathbf{r}') \qquad (33)$$

and the Green's function is constructed to fit the proper boundary conditions. If the operator $\mathcal{L}$ and its adjoint $\mathcal{L}^\dagger$ have eigensolutions

$$\mathcal{L}\rho_n=\lambda_n\rho_n,\quad \mathcal{L}^\dagger\rho_n{}^\dagger=\lambda_n\rho_n{}^\dagger, \qquad (34)$$

where n represents all of the parameters required to specify the solutions, and if the δ function may be expanded in these functions (i.e., if they form a complete basis set), then

$$\delta(E-E')\delta(\mathbf{r}-\mathbf{r}')=\mathbf{S}_n\,\rho_n(E,\mathbf{r})\rho_n{}^\dagger(E',\mathbf{r}'), \qquad (35)$$

where $\mathbf{S}_n$ represents summation over discrete parameters and integration over continuous ones; then the

Green's function may be also expanded by[26,27]

$$G(E,\mathbf{r};E',\mathbf{r}')=\mathop{\mathbf{S}}_{n}\frac{\rho_n(E,\mathbf{r})\rho_n^{\dagger}(E',\mathbf{r}')}{\lambda_n}. \tag{36}$$

In many cases the notion of eigensolution must be taken with a grain of salt since the solutions ρ_n will not fit the boundary conditions. Nevertheless, much of the time (and in particular in the cases we shall consider) an expansion of the form (35) can be defined in terms of transform theory, and the analysis follows through.

At this point we should note that (36) follows almost trivially from (35); for, applying $\mathcal{L}$ to (31), we have

$$\mathcal{L}G=\mathop{\mathbf{S}}_{n}\frac{\mathcal{L}\rho_n(E,\mathbf{r})\rho_n^{\dagger}(E',\mathbf{r}')}{\lambda_n}=\mathop{\mathbf{S}}_{n}\frac{\lambda_n\rho_n(E,\mathbf{r})\rho_n^{\dagger}(E',\mathbf{r}')}{\lambda_n}$$
$$=\mathop{\mathbf{S}}_{n}\rho_n(E,\mathbf{r})\rho_n^{\dagger}(E',\mathbf{r}')=\delta(E-E')\delta(\mathbf{r}-\mathbf{r}'),$$

making use of (35). It is, however, (35) that is not obviously true and must be established for the cases we wish to consider.

At this point we shall make the assumption that the problem may be separated into a spatial part and an energy part. That is to say, that the diffusion coefficient D is independent of energy, and the energy operator has coefficients that are independent of position. If this is the case, the solutions are separable:

$$\rho_{\mathbf{k},\nu}=R_{\mathbf{k}}(\mathbf{r})f_\nu(E),$$

and (34) becomes

$$\mathcal{L}\rho_{\mathbf{k},\nu}=R_{\mathbf{k}}(\mathbf{r})\mathcal{L}_E f_\nu(E)-f_\nu(E)\nabla\cdot D\nabla R_{\mathbf{k}}(\mathbf{r})$$
$$+R_{\mathbf{k}}(\mathbf{r})f_\nu(E)\tau_c^{-1}=\lambda_{\mathbf{k},\nu}R_{\mathbf{k}}(\mathbf{r})f_\nu(E).$$

Dividing by $\rho_{\mathbf{k},\nu}$ gives

$$\frac{\nabla\cdot[D\nabla R_{\mathbf{k}}(\mathbf{r})]}{R_{\mathbf{k}}(\mathbf{r})}-\frac{\mathcal{L}_E f_\nu(E)}{f_\nu(E)}=\tau_c^{-1}-\lambda_{\mathbf{k},\nu}. \tag{37}$$

Since $\mathbf{r}$ and E are independent variables, the first and second terms are functions of $\mathbf{r}$ only and E only, respectively, and hence must be equal to constants; thus we have

$$\nabla\cdot[D\nabla R_{\mathbf{k}}(\mathbf{r})]+k^2R_{\mathbf{k}}(\mathbf{r}), \tag{38}$$

$$\mathcal{L}_E f_\nu(E)-\nu f_\nu(E)=0, \tag{39}$$

and

$$\lambda_{\mathbf{k},\nu}=k^2+\nu+\tau_c^{-1}. \tag{40}$$

We shall not consider solutions to Eq. (38) in detail. It will be sufficient to know that since we are dealing with a self-adjoint operator in a finite domain with homogeneous boundary conditions, it is possible[28] to find a complete denumerable set of orthornal eigenfunctions as long as we confine our solutions to square-integrable functions on the domain of the diffusing volume.

It can be shown[29] that if the separated parts of a partial differential equation allow of eigenfunction expansions then the appropriate expansion for the complete solution is just the product of the separate expansions. Therefore,

$$\delta(E-E')\delta(\mathbf{r}-\mathbf{r}')=\sum_{\mathbf{k}}R_{\mathbf{k}}(\mathbf{r})R_{\mathbf{k}}(r')\int_\nu f_\nu(E)f_\nu^{\dagger}(E')d\nu,$$

provided the expansion in $f_\nu(E)$ exists in a meaningful sense.

In the case where the functions $f_\nu(E)$ are square-integrable and the differential equation is self-adjoint, the problem has been well developed both from a straightforward point of view[27–29] and from the theory of linear operators in Hilbert space.[27,30,31] In this case an expansion of the form

$$\delta(E-E')=\int f_\nu(E)f_\nu(E')d\nu \tag{41}$$

is known to exist with the "eigenvalues" ν distributed along the real axis.

However, unfortunately, in our case the differential operator $\mathcal{L}_E$ is not, in general, self-adjoint and, what is worse, we are not dealing with functions that form a Hilbert space. For functions that describe the distribution of particles in energy, the property of square-integrability has no physical meaning. What we require instead is that the density of particles be finite or that the spectra be absolutely integrable:

$$\int_0^\infty|f(E)|dE<\infty. \tag{42}$$

Unfortunately, property (42) does not allow us to establish an inner product; rather we are dealing with a Banach space with the norm defined by expression (42). The theory of general operators on a Banach space is not nearly as well developed as the corresponding theory of Hilbert space. About all one can say is that an expansion of the form (41) does exist,[32,33] but the distribution of the "eigenvalues" in the complex plane is not known in general and must be investigated for each particular case.

In dealing with expansions of the form of (41) we have placed the word "eigenvalues" in quotes because for

[26] G. Goertzel and N. Tralli, *Some Mathematical Methods of Physics* (McGraw-Hill, New York, 1960).
[27] I. Stakgold, *Boundary Value Problems of Mathematical Physics* (MacMillan, New York, 1967), Vol. I.
[28] R. Courant and D. Hilbert, *Methods of Mathematical Physics* (Interscience, New York, 1953), Vol. I.
[29] E. C. Titchmarsh, *Eigenfunction Expansions* (Oxford U. P., London, 1958, 1962), Vols. I and II.
[30] W. Schmeidler, *Linear Operators in Hilbert Space* (Academic, New York, 1965).
[31] P. R. Halmos, *Introduction to Hilbert Space* (Chelsea, New York, 1957).
[32] E. R. Lorch, *Spectral Theory* (Oxford U. P., New York, 1962).
[33] F. Riesz and B. Sz.-Nagy, *Functional Analysis* (Ungar, New York, 1955).

these values of ν, there are no solutions of Eq. (39) that have the property (42). These values of ν are, therefore, *not* eigenvalues properly speaking. However, for values of ν in the continuous spectrum it can be shown[31] that one can find functions with the property (42) such that, for any $\epsilon>0$,

$$\int_0^\infty |\mathfrak{L}_E f_{\nu,\epsilon}(E)-\nu f_{\nu,\epsilon}(E)|dE<\epsilon. \tag{43}$$

It is for this reason that the continuous spectrum is sometimes called the "approximate point spectrum."[31] From this a theory of integral transforms may be developed, and it is in the light of such transforms that expression (41) must be understood.

If we have an integral transform such that

$$F(x)=\int_C \mathfrak{F}(\nu)f_\nu(x)d\nu, \tag{44}$$

where

$$\mathfrak{F}(\nu)=\int_0^\infty f_\nu{}^\dagger(x)F(x)dx \tag{45}$$

and C is some contour in the complex plane, then we may write

$$F(x)=\int_0^\infty dx'F(x')\int_C d\nu\, f_\nu{}^\dagger(x')f_\nu(x). \tag{46}$$

Hence

$$\delta(x-x')=\int_C d\nu\, f_\nu{}^\dagger(x')f_\nu(x). \tag{47}$$

Since we require (45) to be finite, $f_\nu{}^\dagger(x')$ must be an element of the Banach space that is "dual" or "adjoint"[32,33] to our original space of integrable functions. This adjoint space is in this case the space of all essentially bounded functions, or functions which are bounded "almost everywhere."[33]

We can see the relationship between $f_\nu(E)$ and $f_\nu{}^\dagger(E)$ from the following consideration. If we assume that both $F(E)$ and $\mathfrak{L}_E F(E)$ are integrable, we have from (46)

$$\begin{aligned}\mathfrak{L}_x F(x)&=\mathfrak{L}_x\int_0^\infty dx'F(x')\int_C d\nu\, f_\nu{}^\dagger(x')f_\nu(x)\\ &=\int_0^\infty dx'F(x')\int_C d\nu\, f_\nu{}^\dagger(x')\mathfrak{L}_x f_\nu(x)\\ &=\int_0^\infty dx'F(x')\int_C d\nu\, f_\nu{}^\dagger(x')\nu f_\nu(x).\end{aligned} \tag{48}$$

But also

$$\mathfrak{L}_x F(x)=\int_C d\nu\, f_\nu(x)\int_0^\infty dx' f_\nu{}^\dagger(x')\mathfrak{L}_{x'}F(x').$$

But

$$\int_0^\infty dx' f_\nu{}^\dagger(x')\mathfrak{L}_{x'}F(x')=\int_0^\infty dx'F(x')\mathfrak{L}_{x'}{}^\dagger f_\nu{}^\dagger(x'),$$

where $\mathfrak{L}_{x'}{}^\dagger$ is the formal adjoint[28] of the operator $\mathfrak{L}_{x'}$; therefore,

$$\begin{aligned}\mathfrak{L}_x F(x)&=\int_0^\infty dx'F(x')\int_C d\nu\, f_\nu(x)\mathfrak{L}_{x'}{}^\dagger f_\nu{}^\dagger(x')\\ &=\int_0^\infty dx'F(x')\int_C d\nu\, f_\nu(x)\nu f_\nu{}^\dagger(x').\end{aligned} \tag{49}$$

We see therefore that we must have

$$\mathfrak{L}_E{}^\dagger f_\nu{}^\dagger(E)=\nu f_\nu{}^\dagger(E), \tag{50}$$

and $f_\nu{}^\dagger(E)$ is the essentially bounded solution of the differential equation formally adjoint to (39) with the *same* eigenvalue ν.

In the following the relevant transform will be apparant for the case of Fermi acceleration and for inverse-Compton and synchrotron losses. However, in treating the Fokker-Planck equation, the transform does not correspond to any of the well-known ones and will therefore be developed in this paper.

A. Fermi Acceleration

We have the equation

$$\mathfrak{L}_E f_\nu(E)=\frac{\partial}{\partial E}(aEf_\nu)=\nu f_\nu$$

or

$$\frac{\partial f_\nu}{\partial E}=\frac{(\nu/a-1)}{E}f_\nu. \tag{51}$$

The solution to (51) is

$$f_\nu=\text{const}\, E^{(\nu/a-1)}, \tag{52}$$

$$\mathfrak{L}_E{}^\dagger f_\nu{}^\dagger(E)=aE(\partial f_\nu{}^\dagger/\partial E)=\nu f_\nu{}^\dagger, \tag{53}$$

$$f_\nu{}^\dagger=\text{const}\, E^{-\nu/a}. \tag{54}$$

It is easy to see that there is no value of ν for which f_ν is integrable from 0 to ∞. However, it is also easy to see that the relevant transform theory is that of the Mellin transform;

$$\int_0^\infty f_\nu{}^\dagger q(E')dE'=\int_0^\infty E'^{-\nu/a}q(E')dE'<\infty$$

for $\mathrm{Re}\,\nu/a=0$ and an integrable $q(E')$. We may therefore immediately write

$$\delta(E-E')=\frac{1}{2\pi iE}\int_{-i\infty}^{i\infty}\left(\frac{E}{E'}\right)^{\nu/a}d(\nu/a). \tag{55}$$

Remembering Eq. (40), we have from Eq. (36)

$$G(E,\mathbf{r};E',\mathbf{r}')=\sum_{\mathbf{k}}R_{\mathbf{k}}(\mathbf{r})R_{\mathbf{k}}(\mathbf{r}')g_{\mathbf{k}}(E,E'), \tag{56}$$

where

$$g_k(E,E')=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{E^{\nu/a-1}E'^{-\nu/a}}{\nu+k^2+\tau_c^{-1}}d(\nu/a). \quad (57)$$

For $E<E'$ we may close the contour on the right-hand infinite semicircle without adding anything to the integral. There are no singularities in the right-hand plane, so

$$g_k(E,E')=0 \quad \text{for} \quad E<E'.$$

For $E>E'$ we may similarly close the contour to the left and pick up the pole at $\nu=-\tau_k^{-1}\equiv-(k^2+\tau_c^{-1})$ to obtain

$$g_k(E,E')=\frac{E'^{1/a\tau_k}}{aE^{1+1/a\tau_k}} \quad \text{for} \quad E>E'.$$

If all of the particles are injected at a single, low energy E_0, i.e., $q(E,\mathbf{r})=S(\mathbf{r})\delta(E-E_0)$, inserting the appropriate expression into (32) then yields (11) and (12).

B. Inverse-Compton or Synchrotron Losses

If we consider the case of electrons injected into the region with a power-law spectrum $q(E,\mathbf{r})=S(\mathbf{r})E^{-p}$ and subsequently undergoing energy loss by synchrotron radiation[25,34] or inverse-Compton scattering,[35] we have

$$\mathfrak{L}_E f_\nu=\frac{\partial}{\partial E}(-bE^2f_\nu)=\nu f_\nu, \quad (58)$$

where the solution is

$$f_\nu(E)=\text{const } E^{-2}e^{\nu/bE}.$$

The adjoint solution is

$$f_\nu{}^\dagger=\text{const } e^{-\nu/bE}.$$

If we consider the variable $y=E^{-1}$, we see that the transform in this case is equivalent to the Laplace transform, and we have

$$\delta(E-E')=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}E^{-2}\exp[\nu/bE-\nu/bE']d(\nu/b).$$

Using the same analysis as before, we obtain for $g_k(E,E')$

$$g_k(E,E')=0 \quad \text{for} \quad E'<E$$

and

$$g_k(E,E')=(bE^2)^{-1}\exp[(\tau_k bE')^{-1}-(\tau_k bE)^{-1}] \quad \text{for} \quad E'>E. \quad (59)$$

Continuing, we obtain for $\rho(E,\mathbf{r})$ expressions (11′), (12′), and (13).

[34] V. L. Ginzburg and S. I. Syrovatskii, *Annual Review of Astronomy and Astrophysics* (Annual Reviews, Palo Alto, Calif., 1969), Vol. 7.

[35] F. C. Jones, Phys. Rev. **137**, B1306 (1965).

C. Fokker-Planck Energy Operator

We now want to consider the equation

$$\mathfrak{L}_E f_\nu(E)=\frac{\partial}{\partial E}(a_1E-a_2E^2)f_\nu-\frac{1}{2}\frac{\partial^2}{\partial E^2}(a_3E^2)f_\nu=\nu f_\nu. \quad (60)$$

This equation may be written in the form

$$E^2f''+(bE+\delta E^2)f'+(c+2\delta E)f=0, \quad (60')$$

where

$$b=4-(a_1/a_3), \quad c=2+(\nu-a_1)/a_3,$$

and

$$\delta=a_2/a_3.$$

If we make the substitution

$$f(E)=E^\alpha u(\alpha,E),$$

then we obtain an equation for $u(\alpha,E)$:

$$E^2u''+E[2\alpha+b+\delta E]u'+[\alpha(\alpha-1)+b\alpha+c+(\alpha+2)\delta E]u=0. \quad (61)$$

If we choose α such that

$$\alpha(\alpha-1)+b\alpha+c=0,$$

i.e.,

$$\alpha_\pm=\tfrac{1}{2}\{1-b\pm[(1-b)^2-4c]^{1/2}\}$$
$$=\tfrac{1}{2}(a_1/a_3-1)-1\pm[\tfrac{1}{4}(a_1/a_3-1)^2-\nu/a_3]^{1/2},$$

then we may make the transformation

$$z=\delta E, \quad A=\alpha+2, \quad B=2\alpha+b$$

and obtain

$$zu''+(B-z)u'-Au=0. \quad (62)$$

This is Kummer's equation, whose solutions are the confluent hypergeometric functions[36]

$$u_1={}_1F_1(A;B;z)={}_1F_1(\alpha+2;2\alpha+b;-\delta E). \quad (63)$$

These functions are defined by the series

$${}_1F_1(A;B;z)\equiv\sum_{n=0}^{\infty}\frac{(A)_nz^n}{(B)_nn!}, \quad (64)$$

which is convergent for all finite z if $B\neq-1,-2,-3$, etc., where

$$(A)_n\equiv A(A+1)(A+2)\cdots(A+n-1)$$

and

$$(A)_0\equiv1.$$

An independent solution is

$$u_2=z^{1-B}\,{}_1F_1(1+A-B;2-B;z).$$

It may be seen by straightforward substitution that although there are two independent solutions to Eq. (62) and two values of α (namely, α_+ and α_-), the solu-

[36] L. J. Slater, *Confluent Hypergeometric Functions* (Cambridge U. P., Cambridge, England, 1960).

tions of Eq. (60) have the property

$$z^{\alpha_+}u_1(\alpha_+,z)=z^{\alpha_-}u_2(\alpha_-,z)\,,$$
$$z^{\alpha_-}u_1(\alpha_-,z)=z^{\alpha_+}u_2(\alpha_+,z)\,, \tag{65}$$

so there are still only two independent solutions of Eq. (60).

The adjoint equation is

$$E^2f^{\dagger\prime\prime}+(b^\dagger E+\delta^\dagger E^2)f^{\dagger\prime}+c^\dagger f^\dagger=0\,, \tag{66}$$

where

$$b^\dagger=a_1/a_3\,,\quad c^\dagger=\nu/a_3\,,\quad \text{and}\quad \delta^\dagger=-a_2/a_3=-\delta\,.$$

By the same procedure as before, we obtain solutions

$$f^\dagger(E)=E^{-1-\alpha_\pm}\,{}_1F_1(-1-\alpha_\pm;\,2+2\alpha_\pm-b;\,\delta E)\,. \tag{67}$$

The solutions depend on the eigenvalue ν through $\alpha_\pm=\alpha_\pm(\nu)$, The functional relationship of α on ν is rather complicated, so instead of ν, we will define a new variable, s, with which we will label our solutions. Define

$$\alpha_\pm=\beta\mp s\,,$$
$$\beta=\tfrac{1}{2}(a_1/a_3)-\tfrac{3}{2}\,, \tag{68}$$
$$s=-[\tfrac{1}{4}(a_1/a_3-1)^2-\nu/a_3]^{1/2}\,.$$

We now have

$$f_{\pm s}(E)=E^{\beta\pm s}\,{}_1F_1(2+\beta\pm s;\,1\pm 2s;\,-\delta E) \tag{69}$$

and

$$f_{\pm s}^\dagger(E)=E^{-1-\beta\pm s}\,{}_1F_1(-1-\beta\pm s;\,1\pm 2s;\,\delta E)\,. \tag{70}$$

We must now determine whether or not the solutions we have chosen form a complete set (and if so, in what sense), and if not, we must find the proper combination of solutions that will have this property. Making use of the asymptotic forms of the confluent hypergeometric function[36]

$${}_1F_1(A;B;z)\to 1\,,\quad z\to 0$$

and

$${}_1F_1(A;B;z)\to\frac{\Gamma(B)}{\Gamma(B-A)}z^{-A}\,,\quad |z|\to\infty\,,\ \mathrm{Re}z<0$$
$$\to\frac{\Gamma(B)}{\Gamma(A)}z^{A-B}e^z\,,\quad |z|\to\infty\,,\ \mathrm{Re}z>0$$

we see at once that the adjoint solution has the property

$$f^\dagger(E)\to e^{\delta E}\quad\text{as}\quad E\to+\infty\,.$$

This means that

$$\int_0^\infty f^\dagger(E)q(E)dE$$

will be defined only for a very restricted class of injection spectra.

It turns out that the functions we want are the linear combinations

$$f_s(E)=(\delta E)^{\beta+s}e^{-\delta E}U(s,\delta E)\,, \tag{71}$$

$$f_s^\dagger(E)=(\delta E)^{-1-\beta-s}U(-s,\,\delta E)\,, \tag{72}$$

where

$$U(s,z)\equiv\frac{\Gamma(-2_s)}{\Gamma(-1-\beta-s)}\,{}_1F_1(-1-\beta+s;\,1+2s;\,z)$$
$$+z^{-2s}\frac{\Gamma(2s)}{\Gamma(-1-\beta+s)}\,{}_1F_1(-1-\beta-s;\,1-2s;\,z)\,. \tag{73}$$

Using the Kummer transformation,[36]

$$e^{-z}\,{}_1F_1(A;B;z)={}_1F_1(B-A;B;-z)\,,$$

it is straightforward to verify that (71) and (72) are indeed solutions of Eqs. (60) and (66), respectively. One may also verify from Eq. (73) that these solutions are symmetric in s, i.e., $z^sU(s,z)=z^{-s}U(-s,\,z)$.

The asymptotic forms of $U(s,z)$ are[36]

$$U(s,z)\to z^{1+\beta-s}\,,\quad |z|\to\infty$$

so we see that

$$f_s^\dagger(E)\to\text{const}\,,\qquad |E|\to\infty$$
$$\to E^{-1-\beta\pm s}\qquad |E|\to 0\,.$$

Since $-1-\beta=\frac{1}{2}(1-a_1/a_3)$, if $a_1<a_3$ and $|\mathrm{Re}s|<\frac{1}{2}(1-a_1/a_3)$, then we have

$$\int_0^\infty f_s^\dagger(E)q(E)dE<\infty\quad\text{if}\quad\int_0^\infty|q(E)|dE<\infty\,,$$

and we have a well-defined transform. It is not clear that there is any physical reason to require $a_1<a_3$; however, if we require this to be formally true, we will see that our final Green's function may be quite trivially continued from $a_1<a_3$ to $a_1\geq a_3$, so no real restriction is imposed.

It is shown in the Appendix that the proper form of the unit operator appropriate to this transform is

$$\delta(x-x')=\frac{x^\beta e^{-x}}{4\pi i(x')^{1+\beta}}\times\int_{-i\infty}^{i\infty}\frac{\Gamma(-1-\beta+s)\Gamma(-1-\beta-s)}{\Gamma(2s)\Gamma(-2s)}\left(\frac{x}{x'}\right)^s\times U(s,x)U(-s,\,x')ds\,, \tag{74}$$

where $x=\delta E$.

To obtain the Green's function, we note that $\nu=a_3[(1+\beta)^2-s^2]$; since $\lambda_{k,\nu}=k^2+\nu+\tau_c^{-1}$, we have

$$\lambda=-a_3(s-s_k)(s+s_k)\,,$$

where

$$s_k=[(1+\beta)^2+(k^2+\tau_c^{-1})/a_3]^{1/2}\,.$$

Therefore,

$$g_k(x,x')=-\frac{x^\beta e^{-x}}{4\pi i a_3(x')^{1+\beta}}\times\int_{-i\infty}^{i\infty}\frac{\Gamma(-1-\beta+s)\Gamma(-1-\beta-s)}{\Gamma(2s)\Gamma(-2s)}\times\frac{x^sU(s,x)(x')^{-s}U(-s,x')}{(s-s_k)(s+s_k)}ds. \quad (75)$$

To evaluate the integral in Eq. (75), we use the definition of the function $U(s,x)$ [Eq. (73)] to expand the integrand as

$$g_k(x,x')=\frac{-x^\beta e^{-x}}{4\pi i a_3(x')^{1+\beta}}\int_{-i\infty}^{i\infty}(\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV})ds,$$

where

$$\mathrm{I}=\left(\frac{x}{x'}\right)^s\frac{F(s,x)F(-s,x')}{(s-s_k)(s+s_k)},$$

$$\mathrm{II}=\left(\frac{x}{x'}\right)^{-s}\frac{F(-s,x)F(s,x')}{(s-s_k)(s+s_k)},$$

$$\mathrm{III}=\frac{\Gamma(-2s)\Gamma(-1-\beta+s)}{\Gamma(2s)\Gamma(-1-\beta-s)}\frac{(xx')^sF(s,x)F(s,x')}{(s-s_k)(s+s_k)}, \quad (76)$$

$$\mathrm{IV}=\frac{\Gamma(2s)\Gamma(-1-\beta-s)}{\Gamma(-2s)\Gamma(-1-\beta+s)}\times\frac{(xx')^{-s}F(-s,x)F(-s,x')}{(s-s_k)(s+s_k)},$$

where we have used the notation

$$F(s,x)\equiv {}_1F_1(-1-\beta+s;1+2s;x).$$

We may now determine the behavior of each of the four terms as $s\to\infty$ from the fact[36] that as $s\to\infty$, $F(s,x)\to$ const. We see immediately that the first two terms are dominated by the value of the ratio x/x' and that we may close the contour of integration in I to the right or left as x/x' is less than or greater than one, respectively. The opposite is true for II. The third and fourth terms, however, are dominated as $s\to\infty$ by the Γ functions since

$$\left|\frac{\Gamma(2s)\Gamma(-1-\beta-s)}{\Gamma(-2s)\Gamma(-1-\beta+s)}\right|\to\left(\frac{4s}{e}\right)^{2s}$$

as $s\to\infty$; so we see that III may always be closed to the right and IV may always be closed to the left regardless of the value of x/x'.

If we designate by C_R and C_L the contours that run up the imaginary axis and are closed on the right and left infinite semicircles, respectively, we may write, if $x<x'$,

$$g_k(x,x')=-\frac{x^\beta e^{-x}}{4\pi i a_3(x')^{1+\beta}}\times\left[\int_{C_R}\frac{\Gamma(-1-\beta+s)}{\Gamma(2s)}\left(\frac{x}{x'}\right)^s\frac{F(s,x)U(-s,x')}{(s-s_k)(s+s_k)}ds+\int_{C_L}\frac{\Gamma(-1-\beta-s)}{\Gamma(-2s)}\left(\frac{x}{x'}\right)^{-s}\frac{F(-s,x)U(s,x')}{(s-s_k)(s+s_k)}ds\right]. \quad (77)$$

If $x>x'$,

$$g_k(x,x')=-\frac{x^\beta e^{-x}}{4\pi i a_3(x')^{1+\beta}}\times\left[\int_{C_R}\frac{\Gamma(-1-\beta+s)}{\Gamma(2s)}\left(\frac{x'}{x}\right)^s\frac{F(s,x')U(-s,x)}{(s-s_k)(s+s_k)}ds+\int_{C_L}\frac{\Gamma(-1-\beta-s)}{\Gamma(-2s)}\left(\frac{x'}{x}\right)^{-s}\frac{F(-s,x')U(s,x)}{(s-s_k)(s+s_k)}ds\right]. \quad (78)$$

We must now investigate the positions of the various singularities of the integrands in the s plane. We first note that the Γ functions will contribute no singularities, within their respective contours, as long as $-1-\beta>0$ ($a_1<a_3$). Furthermore, $U(s,x)$ is an entire function of s,[36] and $F(s,x)$ has only simple poles at $s=-\frac{1}{2}(n+1)$.[36] We see, therefore, that in each of the integrals of Eqs. (77) and (78) the only poles that contribute to the integrals are those of the denominator $(s-s_k)(s+s_k)$. Evaluation of the residues is straightforward from this point on. If we adopt the notation $x_>$ ($x_<$) is the larger (smaller) of the two variables x and x', we may write the result in the simple form

$$g_k(x,x')=\frac{x^\beta e^{-x}}{2a_3s_k(x')^{1+\beta}}\frac{\Gamma(-1-\beta+s_k)}{\Gamma(2s_k)}\left(\frac{x_<}{x_>}\right)^{s_k}\times F(s_k,x_<)U(-s_k,x_>). \quad (79)$$

In terms of E, we have

$$g_k(E,E')=\frac{E^\beta e^{-\delta E}}{2a_2s_k(E')^{1+\beta}}\frac{\Gamma(-1-\beta+s_k)}{\Gamma(2s_k)}\left(\frac{E_<}{E_>}\right)^{s_k}\times F(s_k,\delta E_<)U(-s_k,\delta E_>). \quad (80)$$

At this point we may note that since $s_k=[(1+\beta)^2+(k^2+\tau_c^{-1})/a_3]^{1/2}>|1+\beta|$, $-1-\beta+s_k>0$ for any value of a_1/a_3. So the restriction $a_1<a_3$ may be relaxed, and our Green's function remains perfectly regular.

IV. SUMMARY

We have investigated the validity of the leakage-lifetime approximation that has had wide use in cosmic physics. We have seen that this approximation has its natural setting in the eigenfunction expansion of the differential equation describing spatial diffusion and energy transport. The spectra that are derived in the leakage-lifetime approximation are the energy spectra of the various eigenmodes of the operator plus spatial boundary conditions. Each mode has its own leakage lifetime with that of the nth mode being approximately $\tau_k \approx \tau_0/n^2$, where τ_0 is the random-walk time across the diffusing region of linear dimension L, $\tau_0 \approx L^2/\pi D$, D being the diffusion coefficient.

We saw that for Fermi acceleration the lowest mode dominates, and hence the leakage-lifetime approximation is quite good. On the other hand, for the case of inverse-power-law injection of electrons with subsequent synchrotron or inverse-Compton losses, all higher modes contribute significantly, with the fundamental lifetime τ_0 being of no particular significance. The resulting spectrum in this case is determined almost wholly by the spatial distribution of the sources.

Finally we were able to obtain a solution to the more general Fokker-Planck energy operator in terms of confluent hypergeometric functions. These solutions are, in general, quite complex. However, for inverse-power-law injection they were seen to have the familiar property of steepening the spectrum by one power at high energy.

All in all, the eigenfunction expansion of the Green's function is seen to be an effective method for putting the leakage-lifetime approximation in its natural mathematical setting, thereby making it possible to investigate the limits of its validity.

APPENDIX

We wish to consider the transform

$$\mathfrak{F}(s)=\frac{\Gamma(-1-\beta+s)}{\Gamma(2s)}\int_0^\infty (x')^{-1-\beta-s}U(-s,\,x')F(x')dx'$$

and its inverse

$$\tilde{F}(x)=\frac{e^{-x}}{4\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma(-1-\beta-s)}{\Gamma(-2s)}x^{\beta+s}U(s,x)\mathfrak{F}(s)ds$$

and discover to what extent we may say that $\tilde{F}(x)=F(x)$.

First we will rewrite the transform formula as a Lebesgue-Stieltjes (LS) integral:

$$\mathfrak{F}(s)=\frac{\Gamma(-1-\beta-s)}{\Gamma(2s)}\int_{Q(0)}^{Q(\infty)}F(x')dQ(s,x')\,,$$

where

$$Q(s,x')=\int^{x'}(x'')^{-1-\beta-s}U(-s,\,x'')dx''$$

$$=\frac{\Gamma(2s)}{\Gamma(-1-\beta+s)}(x')^{-\beta-s}\mathcal{G}(-s,\,x')+\frac{\Gamma(-2s)}{\Gamma(-1-\beta-s)}(x')^{-\beta+s}\mathcal{G}(s,x')\,,$$

$$\mathcal{G}(s,x')\equiv\sum_{n=0}^{\infty}\frac{(-1-\beta+s)_n x'^n}{(1+2s)_n(-\beta+s+n)n!}\,.$$

We note the following properties of $\mathcal{G}(s,x')$:

$$\mathcal{G}(s,x')\xrightarrow[s\to\beta-n]{}\frac{(-1-n)_n x'^n}{(1+2\beta-2n)_n n!}\left(\frac{1}{s-\beta+n}\right),$$

$$\mathcal{G}(\pm s,\,x')\xrightarrow[1\pm s\to -m]{}\Gamma(-m)(x')^{m+1}\times\frac{(-1-\beta-\frac{1}{2}(m+1))_{m+1}}{(m+1)!}\mathcal{G}(-\tfrac{1}{2}(m+1),\,x')\,.$$

If $F(x')$ is of unbounded variation and/or of unbounded range in x', the proper definition of the LS integral is

$$\int_{Q(0)}^{Q(\infty)}F(x')dQ=\lim_{N\to\infty;\,X\to\infty}\int_{Q(0)}^{Q(X)}F_N(x')dQ\,,$$

where $F_N(x')=F(x')$ if $|F(x')|\leq N$ and $F_N(x')=\pm N$ if $F(x')>N$ or $<-N$, respectively. It is assumed that the limits exist. We may now define

$$\mathfrak{F}_{N,X}(s)\equiv\frac{\Gamma(-1-\beta+s)}{\Gamma(2s)}\int_{Q(0)}^{Q(X)}F_N(x')dQ$$

and

$$\mathfrak{F}(s)=\lim_{N\to\infty;\,X\to\infty}\mathfrak{F}_{N,X}(s)\,.$$

Since $Q(-s,\,x')=Q(s,x')$, we see that for pure imaginary s, Q is real. We may therefore introduce two auxiliary functions of s,

$$\sigma_{N,X}(s)=\sum_{r=0}^{R}l_r(s)q_r(s)$$

and

$$\Sigma_{N,X}(s)=\sum_{r=0}^{R}L_r(s)q_r(s)\,,$$

where $q_r(s)$ is the variation of $Q(s,x')$ over the set E_r of points x',

$$E_r=\{x':\,f_r\leq F_N(x')<f_{r+1},\,x'\leq X\}$$

and $l_r=f_r$ if $q_r(s)\geq 0$ and $l_r=f_{r+1}$ if $q_r(s)<0$. Likewise $L_r=f_{r+1}$ if $q_r(s)\geq 0$ and $L_r=f_r$ if $q_r(s)<0$ [$q_r(s)$ is real], where we have divided the ordinate between $-N$ and $+N$ into R intervals. From the theory of the LS integral, we know that

$$\frac{\Gamma(-1-\beta+s)}{\Gamma(2s)}\sigma_{N,X}(s)\leq \mathfrak{F}_{N,X}(s)\leq \frac{\Gamma(-1-\beta+s)}{\Gamma(2s)}\Sigma_{N,X}(s).$$

We note the fact that since for s pure imaginary $(x')^{-1-\beta-s}U(-s,\,x')$ is bounded uniformly in x' and s, $|(x')^{-1-\beta-s}U(-s,\,x')|<M$, where M is a constant independent of x' and s. Therefore, we have

$$|Q(s,x_2)-Q(s,x_1)|=\left|\int_{x_1}^{x_2}(x')^{-1-\beta-s}U(-s,\,x')dx'\right|\leq\int_{x_1}^{x_2}|(x')^{-1-\beta-s}U(-s,x')|\,dx'<M(x_2-x_1).$$

Thus, $|q_r(s)|<MmE_r$, where mE_r is the measure of the set E_r. If we now let R be large enough so that the maximum separation of ordinate subdivisions $(f_{r+1}-f_r)_{\max}=\epsilon$, we can consider $\Sigma_{N,X}(s)-\sigma_{N,X}(s)$. We have

$$\Sigma_{N,X}(s)-\sigma_{N,X}(s)=\sum_r[L_r(s)-l_r(s)]q_r(s)=\sum_r|f_{r+1}-f_r|\,|q_r(s)|<\epsilon M\sum_r mE_r=\epsilon MX.$$

We see that as we let $R\to\infty$, so that $\epsilon\to 0$, the two functions $\Sigma_{N,X}(s)$ and $\sigma_{N,X}(s)$ converge *uniformly in s* to one another. We may define another auxiliary function

$$\mu_{N,X}(s)=\sum_r f_rq_r(s)$$

and readily see that

$$\sigma_{N,X}(s)\leq\mu_{N,X}(s)\leq\Sigma_{N,X}(s).$$

Since the functions $\sigma_{N,X}(s)$ and $\Sigma_{N,X}(s)$ bracket *both* the functions $\mu_{N,X}(s)$ and $\mathfrak{F}_{N,X}(s)\Gamma(2s)/\Gamma(-1-\beta+s)$, we see that as $R\to\infty$ and $\epsilon\to 0$, we have

$$\mu_{N,X}(s)\xrightarrow[\text{uniformly in } s]{}\mathfrak{F}_{N,X}(s)\frac{\Gamma(2s)}{\Gamma(-1-\beta+s)}.$$

Also from the uniform boundedness of $(x')^{-1-\beta-s}\times U(-s,\,x')$, we note that

$$\left|\frac{\Gamma(2s)}{\Gamma(-1-\beta+s)}(\mathfrak{F}(s)-\mathfrak{F}_{N,X}(s))\right|=\left|\int_0^\infty[F(x')-F_N(x')S(x;0,X)]\times(x')^{-1-\beta-s}U(-s,\,x')dx'\right|$$

(where $S(x;\,a,b)\equiv 1$ if $a\leq x\leq b\equiv 0$ otherwise)

$$\leq\int_0^\infty|F(x')-F_N(x')S(x';0,X)|\times|(x')^{-1-\beta-s}U(-s,\,x')|\,dx'$$

$$<M\int_0^\infty|F(x')-F_N(x')S(x';0X)|\,dx'=M\left[\int_0^X|F(x')-F_N(x')|\,dx'+\int_X^\infty|F(x')|\,dx'\right].$$

Since we have assumed that

$$\int_0^\infty|F(x')|\,dx'<\infty,$$

we have

$$\lim_{N\to\infty;\,X\to\infty}\left|\frac{\Gamma(2s)}{\Gamma(-1-\beta+s)}[\mathfrak{F}(s)-\mathfrak{F}_{N,X}(s)]\right|=0,$$

and the limit is uniform in s. We may now write

$$\tilde{F}(x)=\frac{e^{-x}}{4\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma(1-\beta-s)}{\Gamma(-2s)}x^{\beta+s}U(s,x)\frac{\Gamma(-1-\beta+s)}{\Gamma(2s)}\times[\lim_{N\to\infty;\,X\to\infty}\ \lim_{R\to\infty;\,\epsilon\to 0}\mu_{N,X}(s)]ds.$$

The limits in the brackets are uniform in s and the remainder of the integrand outside of the brackets is uniformly bounded in s; hence the entire integrand approaches its limiting value uniformly in s, so we may write

$$\tilde{F}(x)=\lim_{N\to\infty;\,X\to\infty}\ \lim_{R\to\infty;\,\epsilon\to 0}\times\left[\frac{e^{-x}}{4\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma(-1-\beta-s)\Gamma(-1-\beta+s)}{\Gamma(-2s)\Gamma(2s)}\times x^{\beta+s}U(s,x)\mu_{N,X}(s)ds\right].$$

Since

$$\mu_{N,X}(s)=\sum_{r=0}^R f_rq_r(s),$$

we may write

$$\tilde{F}_{N,X,R}(x)=\frac{e^{-x}}{4\pi i}\sum_{r=0}^R f_r\int_{-i\infty}^{i\infty}\frac{\Gamma(-1-\beta-s)\Gamma(-1-\beta+s)}{\Gamma(-2s)\Gamma(2s)}\times x^{\beta+s}U(s,x)q_r(s)ds$$

and

$$\tilde{F}(x)=\lim_{N\to\infty;\,X\to\infty}\ \lim_{R\to\infty;\,\epsilon\to 0}\tilde{F}_{N,X,R}(x).$$

Since the set E_r may be covered by a series of intevals

$$mE_r=\sum_i|I_{r,i}|,$$

where $|I_{r,i}|$ is the length of the interval $I_{r,i}$, $|I_{r,i}|=x_{2,r,i}-x_{1,r,i}$, where $x_{2,r,i}$ and $x_{1,r,i}$ are the upper and lower endpoints, respectively, of the (r,i)th interval. We have then $q_r=\sum_i q_{r,i}$, where $q_{r,i}=Q(s_{2,r,i})-Q(x_{1,r,i})$.

Since the sum over i may be at times an infinite series, we must question whether its convergence (which is assumed) is also uniform in s. We have

$$|q_r-\sum_{i=1}^{n} q_{r,i}|=|\sum_{i=n+1}^{\infty} q_{r,i}|\le\sum_{i=n+1}^{\infty}|q_{r,i}|<M\sum_{i=n+1}^{\infty}|I_{r,i}|.$$

This may be made as small as desired by making n sufficiently large independently of s, so we may write

$$\tilde{F}_{N,X,R}(x)=\frac{e^{-x}}{4\pi i}\sum_{r,i} f_r\int_{-i\infty}^{i\infty}\frac{\Gamma(-1-\beta-s)\Gamma(-1-\beta+s)}{\Gamma(-2s)\Gamma(2s)}\times x^{\beta+s}U(s,x)q_{r,i}(s)ds.$$

We must now evaluate integrals of the type

$$\int_{-i\infty}^{i\infty}\frac{\Gamma(-1-\beta-s)\Gamma(-1-\beta+s)}{\Gamma(-2s)\Gamma(2s)}x^{\beta+s}U(s,x)Q(s,x_r)ds$$
$$=\int_{-i\infty}^{i\infty}ds\bigg[\left(\frac{x}{x_r}\right)^{\beta+s}F(s,x)\mathcal{G}(-s,x_r)+\left(\frac{x}{x_r}\right)^{\beta-s}F(-s,x)\mathcal{G}(s,x_r)$$
$$+\frac{\Gamma(-2s)\Gamma(-1-\beta+s)}{\Gamma(2s)\Gamma(-1-\beta-s)}\left(\frac{x}{x_r}\right)^{\beta}(xx_r)^{s}F(s,x)\mathcal{G}(s,x_r)$$
$$+\frac{\Gamma(2s)\Gamma(-1-\beta-s)}{\Gamma(-2s)\Gamma(-1-\beta+s)}\left(\frac{x}{x_r}\right)^{\beta}(xx_r)^{-s}\times F(-s,x)\mathcal{G}(-s,x_r)\bigg].$$

It can readily be seen that the third (and fourth) term makes no contribution to the integral for any value of x and x_r. This is because all of the poles of this term are in the left- (right-) hand plane and the combination of Γ functions always ensures that the contour may be closed in the right- (left-) hand infinite semicircle, thereby enclosing no poles. We may therefore drop these terms and consider simply

$$\left(\frac{x}{x_r}\right)^{\beta}\int_{-i\infty}^{i\infty}ds\bigg[\left(\frac{x}{x_r}\right)^{s}F(s,x)\mathcal{G}(-s,x_r)+\left(\frac{x}{x_r}\right)^{s}F(-s,x)\mathcal{G}(s,x_r)\bigg].$$

We note that as $s\to\infty$, $F(\pm s,x)\to e^{x/2}$ and $\mathcal{G}(\pm s,x_r)\to e^{x_r/2}s^{-1}$. Therefore, the behavior on the infinite circle is dominated by the factors $(x/x_r)^{\pm s}$. If $x<x_r$, we may close the first (second) term to the left (right) and vice versa if $x>x_r$. If $x=x_r$, we may close the contours either way but the contribution of the infinite semicircle is finite.

Next we note that the poles at $1\pm 2s=-m$ make no contribution as long as the first and second terms are closed in opposite directions. To see this, observe that when $1\pm 2s\to -m$, the first term approaches

$$(xx_r)^{(m+1)/2}\frac{\Gamma(-m)(-1-\beta-\frac{1}{2}(m+1))_{m+1}}{(m+1)!}\times F(\mp\tfrac{1}{2}(m+1),x)\mathcal{G}(\pm\tfrac{1}{2}(m+1),x_r).$$

The second term approaches the same value for $1\mp s\to -m$ and since the two sets of poles would be encircled in opposite directions the contributions would cancel, term for term. In order to ignore these poles, we will always close the two terms in opposite directions even when $x=x_r$.

We next turn to the poles at $s=\pm(m-\beta)$. The first term has the poles along the positive axis at $s=m-\beta$, and the second term has the poles along the negative axis at $s=-m+\beta$. We see at once that when $x>x_r$, the integral is zero since in this case each term will be closed to the side where it has no poles.

If $x<x_r$, we close the first term to the right and the second to the left. Each term will give a contribution when $s=\pm(m-\beta)$ equal to

$$2\pi i\left(\frac{x}{x_r}\right)^{-\beta}\frac{x^m(-1-m)_m}{(1+2\beta-2m)_m m!}F(m-\beta,x).$$

Adding these together, we obtain for the integral

$$4\pi i\sum_{m=0}^{\infty}\frac{(-1-m)_m}{(1+2\beta-2m)_m m!}x^m F(m-\beta,x).$$

This series may be evaluated simply by evaluating the case $x=x_r$. If we close the contours in the same manner as for $x<x_r$, we will obtain the same series but with a correction to compensate for the contributions from the infinite circle. We obtain for the integral

$$4\pi i(\text{series})-2\pi ie^x.$$

However, we may also close the contours in the opposite manner as for the case $x>x_n$ and get no contributions from the poles. The contribution from the infinite circle is of opposite sign, so we obtain a value of $2\pi ie^x$ for the integral. The two methods of evaluation must yield equal results, so we have

$$4\pi i(\text{series})-2\pi ie^x=2\pi ie^x$$

or

$$\sum_{m=0}^{\infty}\frac{(-1-m)_m}{(1+2\beta-2m)_m m!}x^m F(m-\beta,x)=e^x.$$

Combining these results, we see that the integral

$$\int_{-i\infty}^{i\infty} \frac{\Gamma(-1-\beta-s)\Gamma(-1-\beta+s)}{\Gamma(-2s)\Gamma(2s)} x^{\beta+s} U(s,x) Q(s,x_r) ds$$

$$\begin{aligned} &= 0, \quad && x > x_r \\ &= 4\pi i e^x, \quad && x < x_r \\ &= 2\pi i e^x, \quad && x = x_r . \end{aligned}$$

If now instead of $Q(s,x_r)$ we insert $q(r,i)=Q(s,x_{2,r,i})-Q(s,x_{1,r,i})$ into the integral, we will obtain 0 if x is not in, or an endpoint of, the interval $I_{r,i}$, $4\pi i e^x$ if x is in the interval $I_{r,i}$, and $2\pi i e^x$ if x is an endpoint of the interval $I_{r,i}$.

We now note that the intervals $I_{r,i}$ do not overlap and must cover the entire interval 0 to X. Therefore, if we assume that x is in the interval 0 to X, the point x will fall in one of three categories: (i) It will lie in an interval $I_{r,i}$, (ii) It will be a boundary point of two adjacent intervals (with different values of r), or (iii) it will be a point of accumulation of an infinite sequence of intervals.

In case (iii) the point x cannot be unambiguously assigned to any interval and hence the value of the integral is indeterminate. It should be noted, however, that the only way an infinite sequence of intervals can arise is for the function $F(x)$ to have a point of infinite oscillation, and at this point the function itself is indeterminate, e.g., $\sin[1/(x-x_p)]$ at the point $x=x_p$.

For case (i) we have $\tilde{F}_{N,X,R}(x)=f_r$, and for case (ii) we have $\tilde{F}_{N,X,R}(x)=\frac{1}{2}(f_r+f_{r'})$, where r and r' are the values of r for the two intervals that have x as a common boundary point.

We must now inquire as to the effect of taking the limit $R\rightarrow\infty$, $\epsilon\rightarrow 0$. A little reflection will show that for both cases (i) and (ii) the limit

$$\lim_{R\rightarrow\infty;\ \epsilon\rightarrow 0} [f_r \text{ or } \tfrac{1}{2}(f_r+f_{r'})] = \lim_{\epsilon\rightarrow 0} \frac{1}{2\epsilon}\int_{x-\epsilon}^{x+\epsilon} F_N(x')dx' .$$

This may or may not be equal to $F_N(x)$ but it will be equal for "almost all" values of x [values taken by $F_N(x)$ on sets of measure zero are ignored]. It is also easy to see that case (i) corresponds to a point of continuity and case (ii) may (but not necessarily) correspond to a point of discontinuity of the function $F_N(x)$ (once again ignoring values taken on sets of measure zero). If x is a point of discontinuity, the transform will take on a value half-way between the values of either side.

Taking the limit $X\rightarrow\infty$ is now trivial since we have assumed that X was large enough so that $x<X$, and increasing it further has no effect; thus

$$\lim_{X\rightarrow\infty} \tilde{F}_{N,X}(x) \equiv \tilde{F}_N(x) = \lim_{\epsilon\rightarrow 0} \frac{1}{2\epsilon}\int_{x-\epsilon}^{x+\epsilon} F_N(x')dx' .$$

If $\lim_{N\rightarrow\infty} F_N(x)$ exists, so will $\lim_{N\rightarrow\infty} \tilde{F}_N(x)$ and

$$\lim_{N\rightarrow\infty} \tilde{F}_N(x) \equiv \tilde{F}(x) = \lim_{\epsilon\rightarrow 0} \frac{1}{2\epsilon}\int_{x-\epsilon}^{x+\epsilon} F(x')dx' .$$

We therefore have $\tilde{F}(x)=F(x)$ almost everywhere.