the integral becomes

$$P_3(k^2)=\tfrac{1}{4}G^2\pi^2\frac{\mu^2}{E_\mu}\int_{E_\mu+E_{2\mu}}^{\infty}\frac{EdE}{(E^2-s)[(E-E_\mu)^2-k^2]},\qquad \text{(B3)}$$

where $s=4E^2$. This choice of $\rho(w)$ enables us to evaluate the integral and obtain

$$P_3(k^2)=\frac{G^2\pi^2\mu^2}{8E_\mu}\bigg[\frac{E_\mu-k}{(E_\mu-k)^2-s}\frac{1}{k}\ln(E_{2\mu}+k)-\frac{E_\mu+k}{(E_\mu+k)^2-s}\frac{1}{k}\ln(E_{2\mu}-k)-\frac{1}{(E_\mu-\sqrt{s})^2-k^2}$$
$$\times\ln(|E_\mu+E_{2\mu}-\sqrt{s}|)-\frac{1}{(E_\mu+\sqrt{s})^2-k^2}\ln(E_\mu+E_{2\mu}+\sqrt{s})+\frac{4i\pi\theta[s-(E_\mu+E_{2\mu})^2]}{(E_\mu-\sqrt{s})^2-k^2}\bigg].\qquad \text{(B4)}$$

To obtain Regge trajectories in this approximation, we replace the factor $\pi G^2/[2(k^2+\mu^2)^{1/2}(k^2-q^2)]$ by $P_3(k^2)$ in (16).

PHYSICAL REVIEW D VOLUME 2, NUMBER 10 15 NOVEMBER 1970

Diagonalization of the Natural Parity in the Multiperipheral Equations*

MARCELLO CIAFALONI†
Department of Physics, University of California, Berkeley, California 94720

AND

HARRY J. YESIAN
Department of Physics, University of California at San Diego, La Jolla, California 92037

(Received 13 July 1970)

The crossed partial-wave analysis of the multiperipheral equations is extended to include the natural-parity quantum number. At nonvanishing momentum transfer, a constraint imposed by reflection symmetry on the two-Reggeon–particle vertex function is shown to diagonalize a discrete index κ, which therefore assumes the meaning of a parity index. At vanishing momentum transfer, the result is a selection rule for the vertex function between input Regge poles and an output Regge pole of Toller quantum number $M=0$. It turns out that, to leading order in the asymptotic expansion in the subenergies, the product of the three natural parities at the vertex has to be positive. The $t=0$ limit and some possible implications of this selection rule are also discussed.

I. INTRODUCTION

THE crossed partial-wave analysis[1–4] of the multiperipheral equations[5–8] has provided a mathematical framework for the Regge bootstrap.[9–13] Though the resulting equations are rather complicated in the multi-Regge-pole case, and their physical basis is being questioned,[13] they still provide a bootstrap model for arbitrary spin configurations and a reasonable motivation for factorized production amplitudes. If the Regge bootstrap is to be extended to lower-ranking trajectories, the use of multi-Regge-pole models will be useful to describe such nonvacuum quantum numbers as unnatural parity or Toller's quantum number $M\neq 0$.

The purpose of this paper is to fill a gap left by the previous group-theoretical analysis, by presenting a diagonalization of the crossed-channel natural parity

* Supported in part by the U. S. Atomic Energy Commission and in part by the Air Force Office of Scientific Research, Office of Aerospace Research, U. S. Air Force, under Contract No. F44620-70-C-0028.

† On leave of absence from Scuola Normale Superiore, Pisa, Italy and INFN, Sezione di Pisa, Italy. Present address: Scuola Normale Superiore, Pisa, Italy.

1 A. H. Mueller and I. J. Muzinich, Ann. Phys. (N. Y.) **57**, 20 (1970).
2 A. H. Mueller and I. J. Muzinich, Ann. Phys. (N. Y.) **57**, 500 (1970).
3 M. Ciafaloni, C. DeTar, and M. Misheloff, Phys. Rev. **188**, 2522 (1969).
4 M. Ciafaloni and C. DeTar, Phys. Rev. D **1**, 2917 (1970).
5 D. Amati, A. Stanghellini, and S. Fubini, Nuovo Cimento **26**, 896 (1962), and references therein.
6 G. F. Chew, M. L. Goldberger, and F. Low, Phys. Rev. Letters **22**, 208 (1969).
7 I. G. Halliday and L. M. Saunders, Nuovo Cimento **60A**, 115 (1969); **60A**, 494 (1969); I. G. Halliday, *ibid.* **60A**, 177 (1969).
8 G. F. Chew and C. DeTar, Phys. Rev. **180**, 1577 (1969).
9 N. Bali, G. F. Chew, and A. Pignotti, Phys. Rev. **163**, 1572 (1967).
10 G. F. Chew and A. Pignotti, Phys. Rev. **176**, 2112 (1968).
11 L. Caneschi and A. Pignotti, Phys. Rev. **180**, 1525 (1969); **184**, 1915 (1969).
12 J. S. Ball and G. Marchesini, Phys. Rev. **188**, 2209 (1969); **188**, 2508 (1969).
13 G. F. Chew, T. W. Rogers, and D. R. Snider, Phys. Rev. D **2**, 765 (1970).

at $t \leq 0$. At $t<0$ it is introduced, following Toller,[14,15] as a discrete symmetry of the little group of spacelike momentum transfers connected with y inversion. It turns out that a discrete index κ,[16] needed for the treatment of the Toller angle at $t \neq 0$,[2-4] has the meaning of a parity index[17] and is diagonalized by the natural-parity quantum number.

At $t=0$, the parity under $\kappa \to -\kappa$ turns out to be positive for $M=0$ channels, and the result is a nontrivial selection rule, which can be stated as follows. At $t=0$, $M=0$ partial-wave amplitudes of definite natural parity σ receive leading-order contributions only from input Regge trajectories having product of upper and lower natural parities equal to σ. It turns out that nonleading orders are not forbidden, on account of the alternating parity under helicity inversion of the Clebsch-Gordan series of upper and lower Regge poles. Therefore, the physical meaning of this selection rule comes from the dynamical assumption that higher-ranking input cuts are more important, provided the cut discontinuities are of the same order of magnitude. Approximate extension of this result to $t \neq 0$ is possible only to the extent that $M=0$ and $M \neq 0$ states do not appreciably mix. This in turn requires a detailed dynamical discussion based on the Toller-angle dependence of the two-Reggeon–particle vertex function, and no conclusion can be reached in general.

In Sec. II we present the diagonalization of the natural parity at $t \neq 0$, as discussed before. In Sec. III we treat the $t=0$ case and give the corresponding selection rule. In Sec. II only the leading order model of Ref. 3 is considered in detail. The generalization to the Bali-Chew-Pignotti[9] (BCP) model of the production amplitudes and the $t=0$ limit are given in Sec. IV. While Secs. II and III are more or less self-contained, Sec. IV and the Appendices rely heavily on previous literature, especially on the lower-orders formalism of Ref. 4. Finally, in Sec. V we discuss the possible uses of the $t=0$ selection rule. As an example, we give an argument in favor of natural-parity trajectories having higher intercept than unnatural-parity ones.

II. NATURAL PARITY AND TOLLER-ANGLE DEPENDENCE

Absorptive-part integral equations have been diagonalized for the BCP model,[2,4] for the leading $O(1,1)$ pole model,[3] and for the other leading power models with various approximations to the phase space.[11,18,19]

[14] M. Toller, Nuovo Cimento **53A**, 671 (1968).

[15] M. Toller, Nuovo Cimento **54A**, 295 (1968).

[16] This index is called τ by Mueller and Muzinich (Ref. 2), who first introduced it.

[17] This fact has first been conjectured by C. DeTar, Lawrence Radiation Laboratory (private communication).

[18] M. L. Goldberger, C. I-Tan, and J. M. Wang, Phys. Rev. **184**, 1920 (1969); see also D. Silverman and C. I-Tan, Phys. Rev. D **1**, 3479 (1970).

[19] S. Pinsky and W. Weisberger. Phys. Rev. D **2**, 1640 (1970); **2**, 2365 (1970).

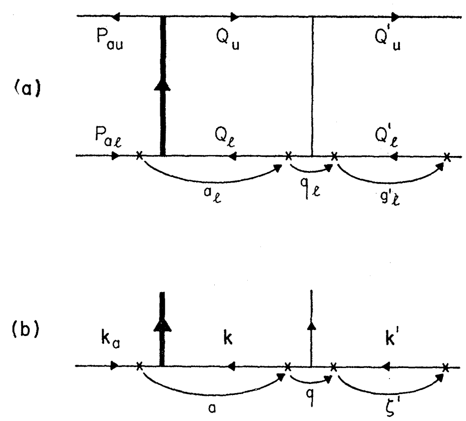

FIG. 1. Schematic picture of (a) four-dimensional versus (b) three-dimensional BCP variables. P_{al} and $\mathbf{k}_a$ refer to the initial on-shell particles. The crosses represent properly defined (see Refs. 3 and 8) c.m., or Breit, frames of the occurring momenta.

We refer in this section to the model of Ref. 3, and we show in Sec. IV how the results are modified for the more general case of Refs. 2 and 4. By using three-dimensional BCP variables, the $t \neq 0$ equation reads

$$B_{\kappa'}{}^{\gamma'}(a',\mathcal{T}') = {}_{(0)}B_{\kappa'}{}^{\gamma'} + \sum_{\gamma,\kappa} \int_{-\infty}^{+\infty} d\zeta' d\mathcal{T} \times e^{\alpha_{\gamma'}\kappa'\zeta'}\theta(\kappa'\zeta') B_{\kappa}{}^{\gamma}(a'\zeta'^{-1}q^{-1},\mathcal{T}) R_{\kappa,\kappa'}{}^{\gamma\gamma'}(\mathcal{T},\mathcal{T}'), \quad (2.1)$$

where $\mathcal{T} \equiv \{k,w\}$, $d\mathcal{T} \equiv dk\, dw$, and $\gamma \equiv \{\gamma_u,\gamma_l\}$ is the label of upper and lower Regge trajectories, and the other variables are defined as follows.[2,3]

In a given Breit frame of the over-all momentum transfer Q ($Q^2=t$), we define (Fig. 1)

$$Q_{u,l} = [\mathbf{k},\, w \pm \tfrac{1}{2}(-t)^{1/2}], \quad Q_{u,l}{}^2 \equiv t_{u,l} \equiv -u_{u,l} \quad (2.2)$$

for any momentum transfer Q_{iu} or Q_{il} of the chain. The three-vector $\mathbf{k}$ is timelike for the on-shell left-hand particles ($\mathbf{k}_a{}^2 = m_a{}^2 - \frac{1}{4}t$ for equal masses) and may be spacelike or timelike for the exchanged particles. The spacelike case is the only one which can propagate indefinitely along the chain, and also spans the most important portion of the phase space.[3] Referring to this case we define $k=(-\mathbf{k}^2)^{1/2}$ and a as the $O(2,1)$ transformation connecting a rest frame of $\mathbf{k}_a$ with a properly specified Breit frame of $\mathbf{k}$. The production amplitudes are functions of the invariant variables k_i, w_i, and of the parameters ζ_i of the little groups of $\mathbf{k}_i$ [essentially $O(1,1)$ groups] which are asymptotically connected with the subenergies

$$s_i \sim k_{i-1}k_{i+1} \sinh q_{i-1} \sinh q_i\, e^{|\zeta_i|}, \quad \cosh q_i \equiv [k_i{}^2 + k_{i+1}{}^2 + (w_i - w_{i+1})^2 + m^2]/2k_ik_{i+1}. \quad (2.3)$$

The kinematical basis for Eq. (2.1) is the recursive relation[2,3,8]

$$a_{i+1} = a_i q_i \zeta_{i+1}, \quad (2.4)$$

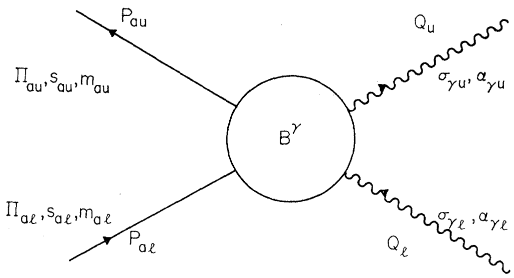

FIG. 2. Helicity and Regge-trajectory labels for the incomplete absorptive part.

where, in our convention, q_i is a (positive) x boost as defined in (2.3) and ζ_i is a y boost.

The index $\kappa=\pm$ in the incomplete absorptive part $B_\kappa{}^\gamma(a)$ (see Fig. 2) is defined by the asymptotic covariance condition[4]

$$B_\kappa{}^\gamma(a\xi) \underset{|\xi|\to\infty}{\sim} B_\kappa{}^\gamma(a)e^{\kappa\alpha_\gamma\xi}\theta(\kappa\xi), \qquad (2.5)$$

where ξ is a y boost and $\alpha_\gamma=\alpha_{\gamma u}(t_u)+\alpha_{\gamma l}(t_l)$. $B_\kappa{}^\gamma$ is actually κ dependent when the left-hand particles are not spinless and/or the two-Reggeon–particle vertex function depends on the Toller angle. In fact, the upper and lower Toller angles ω_{iu} and ω_{il} are, for large subenergies, functions of $\mathcal{T}_i$, $\mathcal{T}_{i+1}$ and also of $\kappa_i\equiv\operatorname{sgn}\zeta_i$ and κ_{i+1}. By asymptotically evaluating $W_i\equiv s_is_{i+1}/s_{i-1,i+1}$ in terms of the four-dimensional and three-dimensional BCP variables, we find the relation[20]

$$\left[\frac{2(t_it_{i+1})^{1/2}(\sinh q_i)^2}{\cosh q_i+\cos\omega_i}\right]_{u,l} =(\kappa_ik_i-\kappa_{i+1}k_{i+1})^2+(w_i-w_{i+1})^2+m^2. \qquad (2.6)$$

By more sophisticated methods,[21] we can also prove that

$$\omega(\mathcal{T},\ \mathcal{T}';\ -\kappa,\ -\kappa')=-\omega(\mathcal{T},\mathcal{T}';\ \kappa,\kappa'). \qquad (2.7)$$

In the standard Breit frame[2,3] in which $\mathbf{k}_i$ has only the x component and $\mathbf{k}_{i+1}$ only x and t components, a y inversion both changes ζ_i, ζ_{i+1} into $-\zeta_i$, $-\zeta_{i+1}$ and inverts the sign of the z rotations. This explains why (2.7) holds and also why the whole interval $-\infty<\zeta_i<+\infty$ is needed to span the phase space.

The main point in diagonalizing the κ index is that reflection symmetry implies the relation

$$R_{-\kappa,-\kappa'}{}^{\gamma\gamma'}(\mathcal{T},\mathcal{T}')=\sigma_\gamma\sigma_{\gamma'}R_{\kappa,\kappa'}{}^{\gamma\gamma'}(\mathcal{T},\mathcal{T}'), \qquad (2.8)$$

where $\sigma_\gamma\equiv\sigma_{\gamma u}\sigma_{\gamma l}$ is the product of the natural parities of the upper and lower Regge trajectories in a given channel. To prove (2.8), we recall that for spinless produced particles,[3]

$$R_{\kappa,\kappa'}{}^{\gamma\gamma'}(\mathcal{T},\mathcal{T}')=\tilde{\beta}^{\gamma l\gamma l'}(t_l,\omega_l,t_l')[\tilde{\beta}^{\gamma u\gamma u'}(t_u,\omega_u,t_u')]^* \times(\sinh q)^{\alpha_\gamma+\alpha_{\gamma'}}k^{\alpha_{\gamma'}}k'^{\alpha_\gamma}, \qquad (2.9)$$

where ω_l and ω_u are given in terms of $\mathcal{T}$, $\mathcal{T}'$ and κ, κ' by Eqs. (2.6) and (2.7).[21] Due to (2.7) we have to know the parity of the two Reggeon-particle vertex function[22] $\tilde{\beta}(\omega)\equiv\sum_m\tilde{\beta}_me^{-im\omega}$ under the transformation $\omega\to-\omega$. For the lower amplitude this is given by[23,24]

$$\tilde{\beta}_{-m}{}^{\gamma l\gamma l'}(t_l,t'_l)=\Pi\sigma_{\gamma l}\sigma_{\gamma' l}\tilde{\beta}_m{}^{\gamma l\gamma l'}(t_l,t'_l), \qquad (2.10)$$

where Π is the intrinsic parity of the spinless produced particle. Equation (2.10), which proves (2.8), has been given in a generalized version by Toller,[23] and can be understood by the following argument. In the c.m. frame of the produced spinless particle, a y inversion $\mathbf{s}'$ multiplies its wave function by Π. On the other hand, $\mathbf{s}'$ can be carried through to the Breit frames of Q_l, Q_l', because it commutes with the relevant z boosts. In those frames, $\mathbf{s}'$ inverts the helicities (which by helicity conservation are the same for the two Reggeons) and multiplies the wave function of the two Reggeons by their natural parities, by definition. Then conservation of parity implies Eq. (2.10).

Let us now consider the diagonalized equation for the leading-pole approximation[3] of Eq. (2.1):

$$b_{\kappa'}{}^{l\gamma'}(\mathcal{T}')={}_{(0)}b_{\kappa'}{}^{l\gamma'}+\pi\sum_{\gamma,\kappa}\int d\mathcal{T} \times b_\kappa{}^{l\gamma}(\mathcal{T})R_{\kappa,\kappa'}{}^{\gamma\gamma'}(\mathcal{T},\mathcal{T}')\hat{d}_{\alpha_\gamma,\kappa\kappa'\alpha_{\gamma'}}{}^l(q^{-1}). \qquad (2.11)$$

Since the $\hat{d}$ function (defined in Ref. 3) is function of the product $(\kappa\kappa')$ only, it is clear that the kernel of Eq. (2.11) satisfies the reflection symmetry (2.8). Therefore, we can define amplitudes of definite κ symmetry ($\sigma=\pm$)

$$b_\kappa{}^{l\sigma,\gamma}(\mathcal{T})=\tfrac{1}{2}[b_\kappa{}^{l\gamma}(\mathcal{T})+\sigma\sigma_\gamma b_{-\kappa}{}^{l\gamma}(\mathcal{T})] \qquad (2.12)$$

which diagonalize Eq. (2.11) with the kernel

$$R_{++}{}^{\gamma\gamma'}(\mathcal{T},\mathcal{T}')\hat{d}_{\alpha_\gamma,\alpha_{\gamma'}}{}^l(q^{-1}) +\sigma\sigma_\gamma R_{-+}{}^{\gamma\gamma'}(\mathcal{T},\mathcal{T}')\hat{d}_{\alpha_\gamma,-\alpha_{\gamma'}}{}^l(q^{-1}). \qquad (2.13)$$

Equations (2.12) and (2.13) are the final results of this section, if we interpret $\sigma=\pm$ as the natural parity of the output trajectory (or channel). To prove this, we define σ as follows.[14,15] If $\mathbf{s}$ is the parity operation, consider the y inversion

$$\mathbf{s}'=\mathbf{s}r_y(\pi), \qquad (2.14)$$

an element of the little group of the over-all momentum transfer Q. The definition of $B_\kappa{}^\gamma(a)$ can be extended to

[20] See Ref. 3, footnote 18; see also Silverman and C. I-Tan (Ref. 18) and Ref. 19.

[21] See Ref. 4, Eq. (2.8a). This equation determines the sign of ω in a way consistent with Eq. (2.7).

[22] This is the vertex function used in Ref. 3, and in phenomenological applications; see, e.g., H. M. Chan, K. Kajantie, and G. Ranft, Nuovo Cimento 49, 157 (1964).

[23] M. Toller, Rivista Nuovo Cimento 1, 403 (1969).

[24] J. M. Kosterlitz, Nucl. Phys. B9, 273 (1969); see also I. T. Drummond, P. V. Landshoff, and W. J. Zakrzewski, Phys. Letters **28B**, 676 (1969).

group elements of the form $a\mathbf{s}'$ by means of the covariance condition

$$B_\kappa{}^\gamma(a\mathbf{s}')=\sigma_\gamma B_{-\kappa}{}^\gamma(a)\,. \qquad (2.15)$$

Due to the fact that $\mathbf{s}'$ anticommutes with the y boost generator and that $R_{\kappa\kappa'}{}^{\gamma\gamma'}$ satisfies condition (2.8), Eq. (2.15) is consistent with Eqs. (2.1) and (2.5).[25]

We now use the crossed partial-wave analysis to construct states of definite natural parity. For a given representation $\{l,\sigma\}$,[14] the element $\mathbf{s}'$ is given, in the mixed basis[2,3] (see Appendix A), by

$$\begin{gathered}D_{m,\mu r}{}^{l\sigma\epsilon}(a\mathbf{s}')=\sigma_r D_{m,-\mu r}{}^{l\sigma\epsilon}(a)\,,\\ \sigma_+=\sigma=(-)^{2\epsilon}\sigma_-,\quad \sigma^2=1\,,\end{gathered} \qquad (2.16)$$

where $\rho\equiv(-i\mu)$ is the eigenvalue of the y-boost generator K_y in the $O(1,1)$ basis, $r=\pm$ labels the two $O(1,1)$ states occurring in the completeness relation, and $\epsilon=0,\ \frac{1}{2}$ labels boson and fermion representations, respectively. By using the asymptotic completeness relation of Ref. 4 we obtain

$$B_\kappa{}^\gamma(a)\sim\sum_{r,\sigma=\pm}\int d[l]b_{m_a,\kappa r}{}^{l\sigma,\gamma}e^{-im_a\phi}\times D_{m_a,\,\kappa\alpha,r}{}^{l\epsilon\sigma}(\eta)e^{\kappa\alpha_\gamma\xi}\theta(\kappa\xi)\,, \qquad (2.17)$$

where the helicity flip $m_a=m_{al}-m_{au}$ is now explicitly shown, we have parametrized the proper $O(2,1)$ transformations as

$$a=r_z(\phi)b_x(\eta)b_y(\xi)\,, \qquad (2.18)$$

and η stands for either $b_x(\eta)$ or $b_x(\eta)\mathbf{s}'$. Since $\mathbf{s}'$ and K_z anticommute, Eqs. (2.15) and (2.16) can be satisfied by (2.17), provided that

$$b_{m_a,-\kappa r}{}^{l\epsilon\sigma}=\sigma_r\sigma_\gamma b_{m_a,\kappa r}{}^{l\epsilon\sigma}\,. \qquad (2.19)$$

For the relevant case $r=+$, the partial-wave amplitudes (2.19) have the same κ parity as in Eq. (2.12), and this proves the identification of σ with the natural parity in Eq. (2.13).

III. NATURAL PARITY AT $t=0$

For the case of forward scattering and pairwise equal masses, the production amplitude, its complex conjugate, and therefore the unitarity integrand, can be expressed in terms of the same variables, which we take to be the four-dimensional BCP variables. The absorptive-part equation of Chew and DeTar[8] (CD) is

$$\bar{B}_{m_am'}{}^{\gamma'}(\tilde{a}',u')={}_{(0)}B_{m_am'}{}^{\gamma'}(\tilde{a}',u')+\sum_{m,\gamma}\int dg'du\,\sinh\tilde{q}\,(u,u')\times\bar{B}_{m_am}{}^{\gamma}(\tilde{a},u)G_{mm'}{}^{\gamma\gamma'}(u,u',g')\,, \qquad (3.1)$$

where $t=-u$ is the momentum transfer squared of the Reggeons, g' is an element of $O(2,1)$, and $\tilde{q}$ is a positive z boost. The m's are labels for helicities defined in the special frames of CD, and the arguments of the absorptive parts, $\tilde{a}'$ and $\tilde{a}$, are the relative $O(3,1)$ transformations between these special frames. The kernel $G^{\gamma\gamma'}$ is proportional to the second-kind representation functions of $O(2,1)$ (the little group of the internal momentum transfers) and represents a piece of the multiple decomposition of the subenergy, momentum transfer, and Toller-angle dependence of the production amplitude. The physical basis for this equation lies in the factorizability assumption inherent in the BCP expansion and in the recursion relation

$$\tilde{a}'=\tilde{a}\tilde{q}g'\,, \qquad (3.2)$$

which leads to a suitably factorizable phase space.

If we employ the BCP expansion of Ref. 4, the kernel of our integral equation can be written as

$$G_{mm'}{}^{\gamma\gamma'}(u,u',g')=R_m{}^{\gamma\gamma'}(u,u')\tilde{a}_{mm'}{}^{-\alpha_{\gamma'}-1}(g')\,, \qquad (3.3)$$

where

$$\delta_{pm}R_m{}^{\gamma\gamma'}(u,u')=\sum_{p_u,p_l,m_u,m_l,n}C(\alpha_u,\alpha_l,\nu;\,p_u,p_l,p)(G_{p_unm_u}{}^{\gamma_u\gamma_u'})^*\times G_{p_lnm_l}{}^{\gamma_l\gamma_l'}C'(\alpha_u',\alpha_l'\nu';\,m_u,m_l,m)\,. \qquad (3.4)$$

The index γ is short for $\{\gamma_u,\gamma_l;\,\nu\}$, $\alpha_\gamma\equiv\alpha_{\gamma u}+\alpha_{\gamma l}-\nu$, and $\nu=0,1,2,\ldots$ arises from the Clebsch-Gordan series of the product of the two representation functions of of $O(2,1)$ from the upper and lower BCP expansions.[4] The coefficients C and C' are more precisely defined in Appendix B. Due to helicity conservation, $C(C')$ are nonzero only when $p_l-p_u=p(m_l-m_u=m)$, and $G_{p_lnm_l}$ is nonzero only when $p_l-m_l=n=$ helicity of the produced particle. This explains the δ_{pm} factor in Eq. (3.4).

We recall that the vertex function G is defined[26] in Ref. 4 by using as the Regge-pole contribution the function

$$\begin{gathered}\tilde{a}_{mm'}{}^{-l-1}(g)=U_m{}^l a_{mm'}{}^{-l-1}(g)U_{m'}{}^{-l-1}\,,\\ U_m{}^l=\Gamma(l+1+m)/\Gamma(-l+m)\,,\end{gathered} \qquad (3.5)$$

where $a_{mm'}{}^{-l-1}(g)$ is the $O(2,1)$ representation function of the second kind, as defined in Sciarrino and Toller.[27] We further specify the model by defining the $\tilde{a}$ function for improper $O(2.1)$ transformations as [25]

$$\tilde{a}_{mm'}{}^{-\alpha-1,\sigma}(g\mathbf{s}')=\sigma\tilde{a}_{m,-m'}{}^{-\alpha-1}(g)\,. \qquad (3.6)$$

Consider now the constraint imposed by parity invariance on the residue function (3.4). When the produced particle has spin s and intrinsic parity Π, the

[25] A γ-independent phase factor could in general be added to the right-hand side of Eq. (2.15), depending on the phase conventions for the helicity states. It has been set equal to 1, by assuming that the $\tilde{a}$ function (see Ref. 4) representing the Regge-pole contribution in the BCP model is actually the $\tilde{a}^{-\alpha-1,\sigma}(g)$ which is defined in Eqs. (3.5) and (3.6). We keep the phase conventions of Refs. 14 and 15 throughout the paper. Note that those of Ref. 23 are different.

[26] The relation between $G^{\alpha\alpha'}$ and the previously defined $\beta^{\alpha\alpha'}$ is, for spinless produced particle, $G_{m0m}{}^{\alpha\alpha'}\Gamma(2\alpha+1)=\Gamma(\alpha+1+m)\times\Gamma(\alpha+1-m)\beta_m{}^{\alpha\alpha'}(2\sinh\tilde{q})^{\alpha+\alpha'}u'^{\alpha/2}u^{\alpha'/2}$.

[27] A. Sciarrino and M. Toller, J. Math. Phys. **8**, 1252 (1967).

vertex function has the reflection symmetry[28]

$$G_{-p_l,-n,-m_l}{}^{\gamma_l\gamma_l'}(u,u')=\Pi(-)^{s-n}\sigma_{\gamma l}\sigma_{\gamma' l}G_{p_l n m_l}{}^{\gamma_l\gamma_l'}(u,u'). \quad (3.7)$$

Furthermore, the asymptotic Clebsch-Gordan coefficient (see Appendix B) has the property

$$C(\alpha_u,\alpha_l,\nu;\,-m_u,\,-m_l-m)=(-)^\nu C(\alpha_u,\alpha_l,\nu\;;m_u,m_l,m), \quad (3.8)$$

and similarly for C'. It then follows that

$$R_{-m}{}^{\gamma\gamma'}(u,u')=\sigma_\gamma\sigma_{\gamma'}(-1)^{\nu+\nu'}R_m{}^{\gamma\gamma'}(u,u'), \quad (3.9)$$

where $\sigma_\gamma=\sigma_{\gamma u}\sigma_{\gamma l}$. In comparison with Eq. (2.8) we see that Eq. (3.9) has the additional phase factor $(-)^{\nu+\nu'}$ coming from the reflection symmetry of C and C'.

By using Eq. (3.6), we extend the definition of the incomplete absorptive part to include elements of the form $\tilde{a}\mathbf{s}'$ by the following covariance condition:

$$\tilde{B}_{m_a m}{}^\gamma(\tilde{a}\mathbf{s}')=(-)^\nu\sigma_\gamma\tilde{B}_{m_a,-m}{}^\gamma(\tilde{a}). \quad (3.10)$$

By virtue of Eq. (3.9) we see that Eq. (3.10) is consistent with the integral equation (3.1). In an analogous fashion we find the covariance condition on the left-hand side to be

$$\tilde{B}_{m_a m}{}^\gamma(\mathbf{s}\tilde{a})=\Pi_{al}(\Pi_{au})^*\tilde{B}_{m_a m}{}^\gamma(\tilde{a}), \quad (3.11)$$

where Π_{al}, Π_{au} refer to the intrinsic parities of the left-hand on-shell particles.

Consider now the asymptotic $O(3,1)$ expansion of Ref. 4 for a definite "channel spin" (j_a,m_a) [14] for the on-shell particles

$$\tilde{B}_{m_a m}{}^\gamma(\tilde{a})\simeq\sum_{M,s}\int d[\lambda]b_{\alpha_\gamma s}{}^{\lambda M}\times\sum_{m'}D_{j_a m_a,\,\alpha_\gamma s,m'}{}^{\lambda M}(\tilde{r}\tilde{\eta})\tilde{a}_{m'm}{}^{-\alpha_\gamma-1}(\tilde{g}), \quad (3.12)$$

where $d[\lambda]$ is the $O(3.1)$ measure, $s=\pm$ is the index of the two $O(2,1)$ basis states, M runs over both positive and negative allowed values, and we have parametrized

$$\tilde{a}=\tilde{r}\tilde{\eta}\tilde{g},\quad \tilde{r}\in O(3),\quad \tilde{\eta}=b_z(\tilde{\eta}),\quad \tilde{g}\in O(2,1). \quad (3.13)$$

Note the important fact that the residue amplitudes $b_{\alpha_\gamma s}{}^{\lambda M}$ are m independent. This is due to the asymptotic covariance condition

$$\tilde{B}_{m_a m}{}^\gamma(\tilde{a}g)\sim\sum_{m'}\tilde{B}_{m_a m'}{}^\gamma(\tilde{a})\tilde{a}_{m'm}{}^{-\alpha_\gamma-1,\sigma_\gamma(-)^\nu}(g), \quad (3.14)$$

valid when the z-boost of $g\in O(2,1)$ is large. This condition can be read off from Eq. (3.1) or from the definition of $\tilde{B}$ itself in CD.

Now, in general, Toller poles do not have a definite parity, but for $M=0$ representations there is an additional valid quantum number $\sigma=\pm1$, the Lorentz natural parity, which is the same for the parent trajectory and all its daughters. The $M=0$ term in the completeness relation represents a sum of two terms

$$\sum_{s=\pm}\sum_{\sigma=\pm}\int_0^{i\infty}d[\lambda]b_{\alpha_\gamma s}{}^{\lambda0\sigma}\times\sum_{m'}D_{j_a m_a,\,\alpha_\gamma s,m'}{}^{\lambda0\sigma}(\tilde{r}\tilde{\eta})\tilde{a}_{m'm}{}^{-\alpha_\gamma-1,\sigma}(\tilde{g}), \quad (3.15)$$

corresponding to natural and unnatural parity, respectively, and where our representations have been enlarged to include elements of the form $\tilde{a}'=\tilde{a}\mathbf{s}'$ and $\tilde{a}''=\mathbf{s}\tilde{a}$, $\tilde{a}\in O(3.1)$. The representation functions satisfy the relations (Ref. 14 and Appendix A)[29]

$$D_{j_a m_a;\,\alpha_\gamma s,m}{}^{\lambda0\sigma}(\tilde{a}\mathbf{s}')=\sigma D_{j_a m_a;\,\alpha_\gamma s,-m}{}^{\lambda0\sigma}(\tilde{a}), \quad (3.16a)$$

$$D_{j_a m_a;\,\alpha_\gamma s,m}{}^{\lambda0\sigma}(\mathbf{s}\tilde{a})=\sigma(-)^{j_a}D_{j_a m_a;\,\alpha_\gamma s,m}{}^{\lambda0\sigma}(\tilde{a}). \quad (3.16b)$$

From the covariance conditions [Eqs. (3.10) and (3.11)] and Eq. (3.15), we see that the partial-wave amplitudes must vanish unless

$$\sigma=\sigma_\gamma(-)^\nu=\Pi_{al}(\Pi_{au})^*(-)^{j_a}. \quad (3.17)$$

As stated above, the first relation represents a selection rule which says that to leading order ($\nu=0$), $M=0$ absorptive parts can receive contributions from Reggeon channels having product of natural parities equal to the Lorentz natural parity. To the extent that leading power models are physically meaningful in the multiperipheral scheme, this represents a constraint which is relevant to the understanding of the dynamics in the complex angular momentum plane. Note that, when $M=0$, only channels having the same value of $\sigma_\gamma(-)^\nu$ can be coupled at $t=0$. This is consistent, due to (3.9), with the explicit expressions of the approximate diagonalized kernel[1,4]:

$$\sinh\tilde{q}\sum_m R_m{}^{\gamma,\gamma'}(u,u')\hat{d}_{\alpha_{\gamma'}\alpha_\gamma m}{}^{\lambda M}(\tilde{q}^{-1}), \quad (3.18)$$

because for $M=0$ the $\hat{d}$ function (Ref. 4, Appendix B) is symmetric under $m\to-m$. Note finally that only channels with $(-)^\nu=\sigma_\gamma$ can contribute to the spin-averaged cross section ($j_a=0$).

IV. GENERALIZATION TO NONLEADING ORDERS AND $t=0$ LIMIT

The problem of nonleading orders at $t\neq0$ arises in the asymptotic expansion of the unitarity integrand in terms of three-dimensional BCP variables. In fact, an asymptotic, factorized expansion[2,4] of each Regge-pole contribution [Eqs. (3.5) and (3.6)] in terms of $O(1,1)$-pole contributions is required to get a simple partial-wave expansion. Note also that the leading-order model

[28] This relation differs from Toller's (Ref. 23) due to the difference in the phase conventions, that we assume to be those of Refs. 14 and 15.

[29] To introduce improper transformations in $M\neq0$ representations we have to double the space by putting M and $-M$ together (see Ref. 15). We do not treat this case because the covariance condition does not give any additional constraints.

is not unambiguously defined, the diagonalized equation (2.11) being dependent on the use of (2.3) at intermediate and low subenergies. Therefore, the previously quoted result, which allows in principle the treatment of all the nonleading powers, is essential for the consistency of the diagonalization procedure.

For a given $\alpha_\gamma=\alpha_{\gamma u}+\alpha_{\gamma l}-\nu$ in the Clebsch-Gordan series, we define[4] a set of functions $B_{m_a,n\kappa}{}^\gamma(a,\mathcal{T})$ satisfying the asymptotic convariance condition

$$B_{m_a,n\kappa}{}^\gamma(a\zeta) \underset{|\zeta|\to\infty}{\simeq} B_{m_a,n\kappa}{}^\gamma(a)e^{\kappa(\alpha_\gamma-n)\zeta}\theta(\kappa\zeta), \quad (4.1)$$

where $n=0, 1, \ldots$ is the index of the $O(1,1)$-pole contribution, and $m_a=m_{al}-m_{au}$. The function $R_{\kappa\kappa'}{}^{\gamma\gamma'}$ of Eq. (2.1) is now replaced by

$$U_{n\kappa,n'\kappa'}{}^{\gamma\gamma'}(\mathcal{T},\mathcal{T}')\equiv\sum_{k,j'} W_{n\kappa,k}{}^{\alpha\gamma}U_{\kappa j'}{}^{\gamma\gamma'}V_{j',n'\kappa'}{}^{\alpha\gamma'}, \quad (4.2)$$

where

$$V_{j,n\kappa}{}^l\equiv \lim_{\mu\to-\kappa(\alpha-n)}(-2\pi\kappa)[\mu+\kappa(\alpha-n)]K_{j,\mu+}{}^l, \quad (4.3)$$

$K_{j,\mu r}{}^i$ being the transformation function between $O(2)$ and $O(1,1)$ states of Eq. (A2), and the W function is defined in Eq. (A50) of Ref. 4. The function $U_{kj'}{}^{\gamma\gamma'}$, defined in Eq. (2.22) of Ref. 4, is schematically shown in Fig. 3. We want to show now that the reflection symmetry (2.8) is replaced by

$$U_{n,-\kappa;n',-\kappa'}{}^{\gamma\gamma'}(\mathcal{T},\mathcal{T}') =\sigma_\gamma\sigma_{\gamma'}(-)^{\nu+\nu'}U_{n\kappa,n'\kappa'}{}^{\gamma\gamma'}(\mathcal{T},\mathcal{T}'). \quad (4.4)$$

To do this, we need the symmetry properties under helicity inversion of all the kinematical factors appearing in Fig. 3. First we have

$$V_{-j;n,-\kappa}{}^\alpha=V_{j;n\kappa}{}^\alpha, \quad (4.5)$$

which follows from (4.3) and (A3), and the analogous relation for W. Moreover, the x boosts h and f' contribute symmetrical representation functions; while for the y rotation χ, we have[30]

$$D_{-m_u,-m_l}{}^s(\chi)=(-)^{m_u-m_l}D_{m_u,m_l}{}^s(\chi). \quad (4.6)$$

The phase factor in (4.6) just cancels the internal helicity dependence of (3.7), while the asymptotic Clebsch-Gordan coefficients contribute the factor $(-)^{\nu+\nu'}$ as in Eq. (3.9). Equation (4.4) then follows.

It is now clear that the covariance condition (2.15) has to be modified by a factor $(-)^\nu$, and therefore the parity P_κ under $\kappa\to-\kappa$ of the partial-wave residue amplitudes of definite natural parity σ is given by

$$P_\kappa=\sigma\sigma_\gamma(-)^\nu, \quad (4.7)$$

which generalizes (2.12) to nonleading orders. By applying the covariance condition on $\mathbf{s}'$ to the left-hand on-

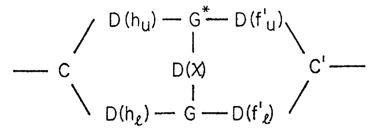

FIG. 3. Index summation scheme for the function $U_{kj'}{}^{\gamma\gamma'}$. The x boosts h and f' and the y rotation χ are defined in Ref. 4.

shell particles we get the same result as Toller's.[14] That is,

$$P_m=\Pi_{au}\Pi_{al}(-)^{s_{au}+s_{al}-m_a}\sigma, \quad (4.8)$$

where P_m is the parity under helicity inversion of the partial-wave amplitudes of definite natural parity σ. Equation (4.8), together with (4.7), also determines the parity under $\kappa\to-\kappa$ of the incomplete absorptive part $B_{m_a,n\kappa}{}^\gamma$, for a given helicity state of the on-shell particles.

For Eq. (4.7) to be consistent with (3.17) at $t=0$, one has to have $P_\kappa=+$ for $M=0$ states. The $t=0$ limit for pairwise equal masses has been worked out in Ref. 4. Referring to definite channel spin (j_a,m_a)[14] for the on-shell particles and to definite "total" Regge trajectory $(\alpha_\gamma,n\kappa)$ for the input Reggeons, we have

$$b_{m_a,n\kappa}{}^{l,\gamma}(u,\theta)=(\sin\theta)\sum_{M,s,s'}\int_{-i\infty}^{+i\infty}d[\lambda]K_{m_a}{}^{\lambda M}(j_a,ls') \times b_{\alpha_\gamma s}{}^{\lambda M}(u)\tilde{D}_{ls',\alpha_{n\kappa}+;\alpha_\gamma s,n\kappa}{}^{\lambda M}(\theta). \quad (4.9)$$

Here b^l and $b^{\lambda M}$ are the residue $O(2,1)$ and $O(3,1)$ partial-wave amplitudes, K is (for unitary representation) the complex conjugate of Scarrino and Toller's[27] transformation function, and the $\tilde{D}$ function is related to the $O(3,1)$ representation functions in noncompact basis[31] as follows:

$$\tilde{D}_{ls',\alpha_{n\kappa}r,\alpha s,n\kappa}{}^{\lambda M}=2\pi\lim_{\mu\to\alpha_{n\kappa}}(-\kappa)(\mu-\alpha_{n\kappa}) \times D_{ls',\mu r;\alpha s,\mu+}{}^{\lambda M}(\theta), \quad (4.10)$$

where $\alpha_{n\kappa}\equiv-\kappa(\alpha-n)$.

Since the $\tilde{D}$ function is left unchanged[32] by the transformation $M\to-M$, $\kappa\to-\kappa$, two conclusions follow: (a) The $M=0$ contribution has $P_\kappa=+$. This is required for the consistency of (4.7) and (3.17). (b) The $M\neq0$ contributions do not have definite P_κ or σ. This is the usual statement of parity doubling at $t=0$, and no further constraints are obtained.

A particularly simple illustration of these results is the evaluation of the leading Regge-pole wave function resulting from a given Lorentz pole in $b_{\alpha s}{}^{\lambda M}$ at $\lambda=\pm\lambda_0$,

[30] The y rotation χ is the transformation between the frames in which upper and lower helicities are measured (Ref. 4, footnote 18).

[31] In Eq. (4.10) the subscripts ls, μr denote an $O(2,1)$ state in a noncompact basis where $(-i\mu)$ is the eigenvalue of the y-boost generator K_y. Since both $O(2,1)$ and $O(1.1)$ are doubled, we have also the indices $s=\pm$ and $r=\pm$. Note that a function $\delta(\mu-\mu')$ has been factored out from the right-hand side of Eq. (4.10), due to the fact that θ and K_y commute.

[32] See Ref. 4, Eq. (C6).

$M = M_0$. After some algebra (see Appendix C), we get

$$f_{m_a,n\kappa}^{(0)}(u,\theta) = C_{j_a m_a}{}^{\lambda_0 M_0}[\Gamma(2\lambda_0)]^{-1}\Gamma(\lambda_0+\alpha_\gamma-n) \times\Gamma(\lambda_0-\alpha_\gamma+n)f(u)(2\sin\theta)^{\lambda_0}\tilde{D}_{-M_0,n\kappa}{}^{\alpha_\gamma}[f'(\theta)], \quad (4.11)$$

where f and $f^{(0)}$ are the wave functions of the Lorentz pole and of the Regge pole, $C_{j_a m_a}{}^{\lambda_0 M_0}$ is some coefficient depending on the spin state of the on-shell particles, and

$$\tilde{D}_{m,n\kappa}{}^{\alpha}(f') = 2\pi \lim_{\mu\to\alpha_{n\kappa}} (-\kappa)(\mu-\alpha_{n\kappa})D_{m,\mu+}{}^{\alpha}(f'), \quad (4.12)$$

f' being an x boost such that $\exp[f'(\theta)] = \cot\frac{1}{2}\theta$. In particular, for $n=0$, we get (see Appendix C)

$$\tilde{D}_{M,0\kappa}{}^{\alpha}[f'(\theta)] = (\sin\theta)^{-\alpha}\, e^{iM\kappa\theta}. \quad (4.13)$$

The last result implies that, in the weak-coupling approximation ($\lambda_0 \simeq \alpha_\gamma$), for which $n=0$ is the important contribution, the θ dependence of the $t=0$ equation for the Regge pole can be diagonalized by a Fourier transform.[33]

V. DISCUSSION

The relation (3.17) obtained at $t=0$ can be regarded as a selection rule on the important Regge exchanges contributing to an output $M=0$ Regge pole. In fact, the kernel of our equations[2,3] has singularities at $+1-\lambda = \alpha_{\gamma u}(t_u)+\alpha_{\gamma l}(t_l)-\nu$ that, when integrated over the internal momenta, give rise to cuts of the AFS type.[5] Output poles are generated by the coherent superposition of these branch cuts arising from the iteration of the kernel in the integral equation. The resulting eigenvalue equations show[34] that the output poles are basically determined by the position of the cut and by the internal coupling strengths. Lower-lying cuts are less important provided the couplings are of the same order of magnitude. Now Eq. (3.17) just says that channels with $\sigma = -\sigma_\gamma$ give rise to singularities which do not contribute to leading order. Therefore, it can rule out some possibilities. Note that the neglect of lower-lying singularities is better justified when the output Regge pole is expected to be close enough (say, within one unit) to the input branch point. This is also the case in which the multi-Regge pole approximation is expected to be a sensible one. On the other hand, the selection rule seems to be unimportant for cut phenomenology, because these cuts do not have, in general, a definite M quantum number. In some cases (photoproduction, nucleon-nucleon scattering) the parity-doubled $M=1$ contribution seems to be the most important one.

By using Eq. (3.17) to select channels in a bootstrap scheme, we now suggest an explanation of why meson trajectories of natural parity lie higher than the unnatural parity ones in a Chew-Frantschi plot. The argument is as follows: It is clear that $M=0$, $\sigma=+1$, can receive contributions for which upper and lower Reggeons are identical (apart from other quantum-number considerations) whereas $\sigma=-1$ cannot. We can imagine initiating a bootstrap program where we start with only two trajectories, one natural parity and the other unnatural parity (we put the Pomeranchukon aside in these considerations).[35] In general, one of them will lie higher than the other. In either case, in view of our selection rule, the kernel for the natural-parity channel will have a higher-lying branch point than the corresponding kernel for the unnatural-parity channel. In addition, we expect a maximum coherence due to the matching of phases of the signature factors for the natural-parity channel. Therefore, the only consistent bootstrap solution would be for the natural-parity singularity to lie higher than the unnatural-parity singularity.

To make these points more concrete, we take the Regge exchanges to be ρ and π, and we look at the isospin-1 crossed channel. The kernel for the natural- and unnatural-parity channels are shown in Fig. 4, where we have explicitly taken into into account that pions are the most copiously produced particles. The resulting eigenvalue equations of the separable models are[34]

$$\begin{aligned} &1-g^4K_{\pi\pi}(j)K_{\rho\rho}(j)=0 \quad \text{(natural parity)},\\ &1-g^4|K_{\pi\rho}(j)|^2=0 \quad \text{(unnatural parity)}, \end{aligned} \quad (5.1)$$

where $K_{\pi\pi,\rho\rho,\pi\rho}$ are the Regge propagator functions for $\pi\pi$, $\rho\rho$, and $\pi\rho$ exchanges, respectively.[13] If we are working in a region which is not too close to the branch points, we may make the approximation[13,34]

$$K_{mn}(j) \sim (j-\beta_{mn})^{-1}, \quad (5.2)$$

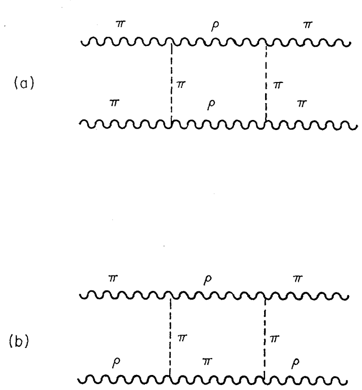

FIG. 4. The kernels for (a) natural-parity and (b) unnatural-parity channels for the $M=0$ model of the text.

[33] This result had been obtained by Pinsky and Weisberger (Ref. 19), not taking into account the index κ.

[34] G. F. Chew and W. R. Frazer, Phys. Rev. **181**, 1914 (1969); W. R. Frazer and C. H. Mehta, Phys. Rev. Letters **23**, 258 (1969).

[35] The reasons for the leading trajectory having vacuum quantum numbers have been investigated by several authors. See, e.g., L. L. Foldy and R. F. Peierls, Phys. Rev. **130**, 1585 (1963); D. Amati, L. L. Foldy, A. Stanghellini, and L. Van Hove, Nuovo Cimento **32**, 1685 (1964); R. F. Peierls and T. L. Trueman, Phys. Rev. **134**, B1365 (1964).

where

$$\beta_{mn}=\alpha_m(0)+\alpha_n(0)-1 \tag{5.3}$$

is the position of the AFS branch point. Our equations then become

$$\begin{aligned}(j-\beta_{\pi\pi})(j-\beta_{\rho\rho})&=g^4,\\ (j-\beta_{\rho\pi})^2&=\eta g^4, \quad \eta\leq 1\end{aligned} \tag{5.4}$$

where the factor η takes into account the lack of coherence in the unnatural-parity channel. The difference between the leading output singularities in the respective channels is given by

$$\begin{aligned}\Delta j=j_+-j_-&=[g^4+(\beta_{\rho\rho}-\beta_{\pi\pi})^2/4]^{1/2}-(\sqrt{\eta})g^2\\ &=\{g^4+[\alpha_\rho(0)-\alpha_\pi(0)]^2\}^{1/2}-(\sqrt{\eta})g^2\geq 0,\end{aligned} \tag{5.5}$$

where $+$ and $-$ refer to natural and unnatural parity, respectively, and we have used the relation

$$\beta_{\rho\rho}+\beta_{\pi\pi}=2\beta_{\rho\pi}. \tag{5.6}$$

Note that Δj does *not* depend on the sign of $(\alpha_\rho-\alpha_\pi)$, as was pointed out earlier. Degenerate trajectories are allowed in this crude model, but the natural-parity trajectory has to be, otherwise, the higher ranking. Releasing some of the approximations like (5.2), (5.3), or the separability itself seems only to lower the factor η rather than to modify the conclusion.

A point open to discussion is whether the selection rule is approximately valid at $t\neq 0$. It is clear that this is possible to the extent that parity-doubled $M\neq 0$ states are low-lying. To leading order, the mixing is likely to be due to matrix elements of the kernel $R^{\gamma\gamma'}$ for which one of the two vertices has parity change ($\sigma_\gamma=-\sigma_{\gamma'}$). However, no general conclusions seem to be possible without making reference to a definite model for the Toller-angle dependence of the vertex functions.

ACKNOWLEDGMENTS

We thank Dale Snider for reading the manuscript. One of us (M.C.) wishes to thank Carleton DeTar and Michael Misheloff for helpful discussions. The other of us (H.J.Y.) is grateful to William R. Frazer for his advice and encouragement.

APPENDIX A: y REFLECTION IN NONCOMPACT BASIS

The matrix elements of $\mathbf{s}'$ in the unitary representation $\{l,\sigma,\epsilon\}$ of the $O(2,1)$ group are given by[14,15]

$$\begin{aligned}D_{mm'}{}^{l\sigma\epsilon}(g\mathbf{s}')&=\sigma D_{m,-m'}{}^{l\sigma\epsilon}(g),\\ \sigma^2=\mathbf{s}'^2&=\mathbf{s}^2e^{2i\pi J_y}=1, \quad \epsilon=0,\tfrac{1}{2}.\end{aligned} \tag{A1}$$

The $O(1,1)$ basis $|\mu,r\rangle$ is labeled by the K_y eigenvalue $(-i\mu)$ and by a discrete index $r=\pm$ needed for the completeness relation. The transformation function is[2–4]

$$K_{m,\mu r}{}^l=\frac{\Gamma(-l+\mu)\Gamma(-l-\mu)}{2\pi 2^l\Gamma(-2l)}e^{i\pi(l+m-\mu)/2}\times F(-l-rm,\,-l+\mu,\,-2l;\,2+i0). \tag{A2}$$

From standard properties of the hypergeometric function,[36] it follows that

$$K_{-m,-\mu+}{}^l=K_{m,\mu+}{}^l, \quad K_{-m,-\mu-}{}^l=(-)^{2\epsilon}K_{m,\mu-}{}^l. \tag{A3}$$

We have, therefore,

$$\begin{aligned}D_{m,\mu r}{}^{l\sigma\epsilon}(g\mathbf{s}')&=\sigma\sum_{m'}D_{m,-m'}{}^{l\sigma\epsilon}(g)K_{m',\mu r}{}^l\\ &=\sigma_r D_{m,-\mu r}{}^l(g),\end{aligned} \tag{A4}$$

where, as in Eq. (2.16), $(-)^{2\epsilon}\sigma_-=\sigma_+=\sigma$. Note that the factor $(-)^{2\epsilon}$ for $r=-$ is required from consistency with the relation

$$e^{i\pi J_z}|\mu+\rangle=|-\mu,-\rangle. \tag{A5}$$

The equivalence relations of Eq. (B1) of Refs. 3 and (A3) of Ref. 4 have to be slightly modified for fermions. We have

$$D_{m,\mu r}{}^{l'\sigma'\epsilon}(g)=\sum_{r'=\pm}U_m{}^lD_{m,\mu r'}{}^{l\sigma\epsilon}(g)\gamma_{r'\mu r}{}^{l\epsilon}, \tag{A6}$$

where $l'\equiv -l-1$, $\sigma'\equiv(-)^{2\epsilon}\sigma$. Substituting (A4) and (A5) into (A6) yields the symmetry properties

$$\gamma_{r',-\mu,r}{}^{l\epsilon}=\begin{cases}(-)^{2\epsilon}\gamma_{r'\mu r}{}^{l\epsilon} & (r'=r)\\ \gamma_{r'\mu r}{}^{l\epsilon} & (r'=-r),\end{cases} \tag{A7}$$

$$\gamma_{-r',\mu,-r}{}^{l\epsilon}=(-)^{2\epsilon}\gamma_{r'\mu r}{}^{l\epsilon}.$$

Therefore, $\gamma^{l\epsilon}$ is a unitary matrix in the r basis of the form

$$\gamma_\mu{}^{l\epsilon}=\begin{pmatrix}\gamma_{1\mu}{}^{l\epsilon} & (-)^{2\epsilon}\gamma_{2\mu}{}^{l\epsilon}\\ \gamma_{2\mu}{}^{l\epsilon} & (-)^{2\epsilon}\gamma_{1\mu}{}^{l\epsilon}\end{pmatrix}, \tag{A8}$$

with

$$\gamma_{1\mu}{}^{l\epsilon}=(-)^{2\epsilon}\gamma_{1,-\mu}{}^{l\epsilon}, \quad \gamma_{2\mu}{}^{l\epsilon}=\gamma_{2,-\mu}{}^{l\epsilon}. \tag{A9}$$

Using (A6), one gets the asymptotic $O(2,1)$ expansion of Eq. (2.17) much in the same way as in Ref. 4, Appendix A.

APPENDIX B: REFLECTION SYMMETRY OF ASYMPTOTIC CLEBSCH-GORDAN COEFFICIENTS

We want to prove Eq. (3.8). After dividing some factors $i^{m_i-n_i}$ out, the definition (2.18) of Ref. 4 reduces to

$$\begin{aligned}&\tilde{a}_{m_1n_1}{}^{-l_1-1}[b_x(\zeta)]\tilde{a}_{m_2n_2}{}^{-l_2-1}[b_x(\zeta)]\\ &\sim\sum_{\nu=0}^{\infty}C'(l_1,l_2\nu;\,m_1,m_2,m)\\ &\qquad\times\tilde{a}_{mn}{}^{-(l_1+l_2-\nu)-1}[b_x(\zeta)]C(l_1,l_2,\nu;\,n_1,n_2,n).\end{aligned} \tag{B1}$$

Here C and C' are nonzero only when $m_2-m_1=m$ and

[36] *Higher Transcendental Functions*, Bateman Manuscript Project, edited by A. Erdélyi (McGraw-Hill, New York, 1953), Vol. I, formula 2.9(3).

$n_2-n_1=n$. Putting $z=\cosh\zeta$ and using the explicit representation[27] of Toller's a function, we find the expression

$$\tilde{a}_{mn}^{-l-1}(z)=\frac{\Gamma(2l+1)}{\Gamma(l+1+n)\Gamma(l+1-n)}\left(\frac{z-1}{2}\right)^{2l}\left(\frac{z+1}{z-1}\right)^{(m+n)/2}\times F(-l+m,\,-l+n,\,-2l;\,2/(1-z)),\quad \text{(B2)}$$

and we can prove that[36]

$$\tilde{a}_{mn}^{-l-1}(-z+i0)=e^{i\pi l}\tilde{a}_{-m,n}^{-l-1}(z).\quad \text{(B3)}$$

Continuing (B1) analytically in the z plane to the upper negative z axis, and using (B3) we get a factor $\exp[i\pi(l_1+l_2)]$ on the left-hand side, a factor $\exp[i\pi(l_1+l_2-\nu)]$ on the right-hand side, and all m's reversed. This proves (3.8) for C'. The analogous relation for C follows from the property

$$\tilde{a}_{-m,-n}^{-l-1}[b_x(\zeta)]=\tilde{a}_{m,n}^{-l-1}[b_x(\zeta)].\quad \text{(B4)}$$

Note that, from (B2), we have

$$\tilde{a}_{mn}^{-l-1}(z)=-\pi^{-1}\tan\pi(l-m)e_{-l-1}^{m,n}(z+i0)\times e^{i(\pi/2)(m-n)}\left[\frac{\Gamma(-l+n)\Gamma(-l-n)}{\Gamma(-l+m)\Gamma(-l-m)}\right]^{1/2}\quad (z>1),\quad \text{(B5)}$$

where the e function is given by Andrews and Gunson.[37] Using (B5) and formula (13.8) of Ref. 37, the coefficients C and C' can be given in terms of complex generalizations of the usual Clebsch-Gordan coefficients of the rotation group.

APPENDIX C: CALCULATION OF DAUGHTERS WAVE FUNCTION

The u- and θ-dependent residue function of the Regge daughters (that we call "wave function") is obtained by distorting the λ contour of Eq. (4.9), as indicated at the end of Sec. III of Ref. 4. Concentrating on the θ-dependent part (which gives the "off-shell" effect), we obtain a contribution

$$C_{N,\alpha n\kappa}^{\lambda_0 M_0}\equiv(\sin\theta)\times\lim_{l\to l_N}[(l-l_N)\tilde{D}_{l+,\alpha n\kappa+;\,\alpha-,n\kappa}^{\lambda_0,M_0}(\theta)]\quad \text{(C1)}$$

to the Nth Regge daughter at $l_N\equiv\lambda_0-1-N$. Here the relevant function is[32]

$$D_{l+,\mu+;\alpha-,\mu+}^{\lambda M}(\theta)=\int_{\cot\theta}^{\infty}d\sinh a\; d_{\mu+,M}^{l}(a^{-1})\times(\sin\theta\sinh a-\cos\theta)^{\lambda-1}d_{-M,\mu+}^{\alpha}(a'),\quad \text{(C2)}$$

where a and a' are x boosts related by

$$e^{a'}=\frac{\cos\frac{1}{2}\theta\, e^{a}-\sin\frac{1}{2}\theta}{\sin\frac{1}{2}\theta\, e^{a}+\cos\frac{1}{2}\theta}\underset{a\to\infty}{\sim}\cot\tfrac{1}{2}\theta,\quad \text{(C3)}$$

and[2,3]

$$d_{\mu r,M}^{l}(a^{-1})=\frac{i^M}{2\pi}\int_{-\infty}^{+\infty}d\zeta\, e^{-\mu\zeta}(\cosh\zeta\cosh a+r\sinh a)^{-l-1}e^{im\phi_r(\zeta)},\quad \text{(C4)}$$

$$\tan\tfrac{1}{2}\phi_r(\zeta)=(e^{\zeta}+r\tanh\tfrac{1}{2}a)/(e^{\zeta}\tanh\tfrac{1}{2}a+r).$$

For $N=0$, the residue (C1) is given by the coefficients of the leading asymptotic behavior of the integrand of (C2) when $a\to+\infty$. It follows that

$$C_{0,\alpha n\kappa}^{\lambda M}=(2\sin\theta)^{\lambda}\frac{\Gamma(\lambda+\alpha-n)\Gamma(\lambda-\alpha+n)}{\Gamma(2\lambda)}\times\tilde{D}_{-M,n\kappa}^{\alpha}[f'(\theta)],\quad \text{(C5)}$$

$$e^{f'(\theta)}\equiv\cot\tfrac{1}{2}\theta.$$

When also $n=0$, the $\tilde{D}$ function defined by (4.10) is given by the leading asymptotic behavior of the integrand of (C4) when $\zeta\to\infty$. Noting that

$$\cosh f'(\theta)=(\sin\theta)^{-1},$$
$$\tan\tfrac{1}{2}\phi_+(\infty)=[\tanh\tfrac{1}{2}f'(\theta)]^{-1}=\tan\tfrac{1}{2}(\tfrac{1}{2}\pi-\theta),\quad \text{(C6)}$$

we get Eq. (4.16).

[37] M. Andrews and J. Gunson, J. Math. Phys. 5, 1391 (1964).