Multiparticle Relativistic K-Matrix Equations and Unitarity*

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(Received 30 April 1970)

An extension of the work of Baker and Blankenbecler is presented in which $2 \to 2$, $2 \to n$, and $m \to n$ particle amplitudes satisfy unitarity, including both elastic and inelastic terms. Disconnected parts of the $m \to n$ amplitudes are, however, not included. Dynamical applications of the model to the unitarization of meromorphic Born terms are briefly discussed. In particular, pole degeneracies are shown to be broken, and factorization of output resonance poles is proved.

I. INTRODUCTION

THE construction of a class of scattering amplitudes which satisfy the unitarity equations, including the inelastic part, was accomplished in 1962 by Baker and Blankenbecler (BB).[1] The prescription for generating unitarized amplitudes involves the choice of "Born terms" which are arbitrary Hermitian analytic functions having no right-hand cuts. However, the BB construction is not the most general, because it assumes that certain of these Born terms are zero, and this introduces some undesired constraints among the scattering amplitudes. For example, a high-energy model which successfully incorporates an infinite set of cut corrections to the Pomeranchuk Regge pole and which is consistent with the multi-Regge bootstrap ideas, can be written in a unitarized K-matrix fashion but falls outside the class of BB amplitudes.[2]

The purpose of the present paper is to present a systematic generalization of the BB method. No Born terms are set to zero, the full dependence of the amplitude on its variables is taken into account, and no high-energy approximations are performed. As in Ref. 1, we include only a discussion of the amplitudes which do not contain disconnected terms in their cluster decompositions.

In Sec. II we describe the generalized partial-wave decomposition and the unitarity equation for an arbitrary $(i \to j)$-body scattering amplitude. Sections III and IV contain the generalized K-matrix equations and their solutions, and Sec. V is devoted to a proof that the solutions satisfy unitarity. Section VI contains remarks about symmetries, inversion of the K-matrix equations, a formulation which does not use partial waves, and the problem of disconnected pieces in the multiparticle amplitudes. Dynamical applications of the formalism, such as unitarization of partial-wave projections of meromorphic Born terms, are suggested in Sec. VII. Appendix A contains proofs of the splitting of degeneracies and factorization of output resonance pole terms, and Appendix B shows how different choices of Born terms can compensate for different choices of some kinematical functions that appear in the formalism.

* Work supported in part by the National Science Foundation and the U. S. Atomic Energy Commission.

[1] M. Baker and R. Blankenbecler, Phys. Rev. **128**, 415 (1962).

[2] Jan W. Dash, J. R. Fulco, and A. Pignotti, Phys. Rev. D **1**, 3164 (1970).

II. GENERALIZED PARTIAL-WAVE EXPANSION AND UNITARITY EQUATIONS

We describe a state of n_k particles by $3n_k-3$ variables in its c.m. frame. We choose them to be the following:

(i) The total energy squared s.

(ii) $3n_k-7$ variables v_k which specify the momenta in a standard orientation. For concreteness we choose this orientation such that the momentum of particle 1 points in the z direction, and that of particle 2 in the yz plane. In the case of two-particle states, the total energy is sufficient to specify the momenta in a standard orientation.

(iii) Three Euler angles, α_k, β_k, γ_k, collectively denoted by r_k, which label a rotation R_k that transforms the momenta in the standard orientation into the actual momenta.

(iv) In addition we introduce the intrinsic helicities $\{\mu_k\}$ of the particles which have arbitrary spins $\{\sigma_k\}$.

The scattering amplitude for an initial state i and a final state f with n_i and n_f particles, respectively, is written in the over-all c.m. system as $M_{fi}(v_f\mu_f,s,r_{fi},v_i\mu_i)$, the $3(n_i+n_f)-10$ variables being accounted for by $3n_i-7$ variables v_i, $3n_f-7$ variables v_f, and four variables s and r_{fi}. Here r_{fi} denotes the Euler angles of the rotation $R_{fi}=R_i^{-1}R_f=R_{if}^{-1}$, which specifies the relative orientation of the final and initial momenta. The relation of M_{fi} to the S-matrix element S_{fi} is

$$S_{fi}=N_fN_i\delta_{fi}+2i\delta^4(p_f-p_i)M_{fi}, \qquad (2.1)$$

where

$$N_i^2=\prod_{\alpha=1}^{ni}(2\pi)^3 2E_\alpha.$$

We use Hermitian analyticity ($s_\pm=s\pm i\epsilon$)

$$M_{if}^*(v_i\mu_i,s_+,r_{if},v_f\mu_f)=M_{fi}(v_f\mu_f,s_-,r_{fi},v_i\mu_i)$$

and write unitarity for M_{fi} in the form[3]

[3] Since we are assuming all M_{fi} to be connected, no cuts are present in the physical region of the variables v_f and v_i. See R. J. Eden, P. V. Landshoff, D. I. Olive, and J. C. Polkinghorne, *The Analytic S-Matrix* (Cambridge U. P., Cambridge, England, 1966), p. 232.

$$(2i)^{-1}[M_{fi}(v_f\mu_f,s_+,r_{fi},v_i\mu_i)-M_{fi}(v_f\mu_f,s_-,r_{fi},v_i\mu_i)]$$
$$=\sum_{k=2}^{N(s)}\sum_{\mu_k}\int dv_k dr_k\,\rho_k(s,v_k)$$
$$\times M_{fk}(v_f\mu_f,s_-,r_{fk},v_k\mu_k)M_{ki}(v_k\mu_k,s_+,r_{ki},v_i\mu_i)\,,\quad(2.2)$$

where

$$\rho_k(s,v_k)dv_k dr_k$$
$$=\prod_{\alpha=1}^{n_k}\frac{d^3p_\alpha}{(2\pi)^3 2E_\alpha}\delta^3(\sum_{\alpha=1}^{n_k}\mathbf{p}_\alpha)\delta(\sum_{\alpha=1}^{n_k}E_\alpha-\sqrt{s})\quad(2.3)$$

for $n_k>2$, and $N(s)$ labels the highest open channel.

In the case of two-particle states, no variables v_k are present, and an integration over the solid angle Ω_k is sufficient to span all the possible orientations. Therefore, for $n_k=2$ we replace $\rho_k(s,v_k)$ by $\rho_2(s)$ and dr_k by $2\pi d\Omega_k$ in Eqs. (2.2) and (2.3) and throughout the paper. In the following, we assume for simplicity that the number of particles in a channel uniquely specifies the channel, and we set $k=n_k$.

We now write the generalized partial-wave expansion[4]

$$M_{fi}(v_f\mu_f,s,r_{fi},v_i\mu_i)$$
$$=\sum_{J\lambda_f\lambda_i}\frac{2J+1}{8\pi^2}H_{fi}^{J\lambda_f\lambda_i}(v_f\mu_f,s,v_i\mu_i)D_{\lambda_i\lambda_f}^{J*}(r_{fi})\,.\quad(2.4)$$

Here J is the total angular momentum, and λ_i (λ_f) its projection in the direction of motion of particle 1 in the initial (final) state. If only two particles are present in the initial (final) state, λ_i (λ_f) is the difference of the helicities of these two particles, and there is no summation over λ_i (λ_f).

Substituting (2.4) into (2.2) and using the group, unitarity, and orthogonality properties of the D matrices in the form

$$\int dr_k\,D_{\lambda_f\lambda_k}{}^J(r_{kf})D_{\lambda_i\lambda_{k'}}{}^{J_1*}(r_{ki})$$
$$=\sum_{\lambda_\alpha\lambda_\beta}D_{\lambda_f\lambda_\alpha}{}^J(r_f^{-1})D_{\lambda_i\lambda_\beta}{}^{J_1*}(r_i^{-1})$$
$$\times\int D_{\lambda_\alpha\lambda_k}{}^J(r_k)D_{\lambda_\beta\lambda_{k'}}{}^{J_1*}(r_k)dr_k$$
$$=\frac{8\pi^2}{2J_1+1}D_{\lambda_i\lambda_f}{}^{J*}(r_{fi})\delta_{JJ_1}\delta_{\lambda_k\lambda_{k'}}\,,$$

we obtain the unitarity relation for partial waves:

$$(2i)^{-1}[H_{fi}^{J\lambda_f\lambda_i}(v_f\mu_f,s_+,v_i\mu_i)-H_{fi}^{J\lambda_f\lambda_i}(v_f\mu_f,s_-,v_i\mu_i)]$$
$$=\sum_{k=2}^{N(s)}\int dv_k\,\rho_k(s,v_k)\sum_{\lambda_k\mu_k}H_{fk}^{J\lambda_f\lambda_k}(v_f\mu_f,s_-,v_k\mu_k)$$
$$\times H_{ki}^{J\lambda_k\lambda_i}(v_k\mu_k,s_+,v_i\mu_i)\,.\quad(2.5)$$

[4] J. Werle, *Relativistic Theory of Reactions* (Wiley, New York, 1966), see especially Chap. 5.

III. GENERALIZED K-MATRIX EQUATIONS—INELASTIC UNITARIZATION

We now write the K-matrix equations. This is done in two steps, not because of any inherent difficulty in proceeding in one step, but because the application of the formalism to phenomenology seems to be facilitated by such an approach.[2]

The equations are written in an N/D form. We simplify the notation by including the particle helicities $\{\mu_i\}$ and the "total helicity" λ_i in the v_i variables (therefore, for two particle states we introduce $v_2=\{\mu_i\}$). The symbol $\sum_k\int dv_k$ is then to be interpreted to include sums over the intrinsic helicities and total helicities, and $\delta(v_k-v_i)$ similarly includes Kronecker deltas in the helicity indices. We use the indices f, i, j, and k to run from 2 to ∞, whereas n, m, and r take values from 3 to ∞. For the first "inelastic" stage we define what we call "modified Born terms" $\tilde{B}_{ji}$ by the equations

$$\sum_k\int dv_k\,\tilde{B}_{fk}{}^J(v_f,s,v_k)D_{ki}{}^J(v_k,s,v_i)=B_{fi}{}^J(v_f,s,v_i)\,,\quad(3.1)$$

where

$$D_{ni}{}^J(v_n,s,v_i)=\delta_{ni}\delta(v_n-v_i)-I_n(s,v_n)B_{ni}{}^J(v_n,s,v_i)\quad(n\geqslant3)\,,$$
$$D_{2i}{}^J(v_2,s,v_i)=\delta_{2i}\delta(v_2-v_i)\,.\quad(3.2)$$

Here $B_{fi}{}^J(v_f,s,v_i)$ is the input Born term. It is an arbitrary Hermitian analytic function of s with no right-hand cuts but with possible left-hand cuts and poles on the positive real axis, and such that the integral equations (3.1) are of Fredholm type. $I_n(s,v_n)$ is a real analytic function of s whose imaginary part is $\rho_n(s,v_n)$ above the threshold for channel n. It is of course independent of the helicities. The functions $D_{ni}{}^J(v_n,s,v_i)$ for this first stage have both right-hand s cuts due to I_n and left-hand cuts due to the Born terms $B_{ni}{}^J$ and possible left-hand cuts in I_n.[3] The two-body unitarity cut is not present at this stage. (Formally we imagine that $I_2=0$ for this stage.) Note that $D_{ni}{}^J$ is defined using the product of I_n and $B_{ni}{}^J$ rather than a dispersive integral over $\rho_n B_{ni}{}^J$.

Equations (3.1) and (3.2) are equivalent to the following integral equation for $\tilde{B}_{nm}$ (n, $m>2$):

$$\tilde{B}_{nm}{}^J(v_n,s,v_m)=B_{nm}{}^J(v_n,s,v_m)$$
$$+\sum_{r=3}^{\infty}\int\tilde{B}_{nr}{}^J(v_n,s,v_r)I_r(s,v_r)B_{rm}{}^J(v_r,s,v_m)dv_r\,.\quad(3.3)$$

We assume that this equation is of Fredholm type and approximate $B_{nm}{}^J$ by a finite sum of factorized terms, thus approximating the kernel with kernels of finite rank:

$$B_{nm}{}^J(v_n,s,v_m)=\lim_{L\to\infty}\sum_{l=1}^{L}\bar{g}_n{}^{Jl}(s,v_n)g_m{}^{Jl}(s,v_m)\,.\quad(3.4)$$

The functions $\bar{g}_n^{Jl}$ and g_m^{Jl} may have left-hand s cuts and poles on the right-hand s axis, but no right-hand s cuts. Hermitian analyticity is enforced for B_{nm}^J by setting $\bar{g}_n^{Jl}=g_n^{Jl*}$ for physical s [thus $B_{nm}^J(s)=B_{mn}^{J*}(s)$].

We define

$$C^{Jll'}(s)=\sum_{r=3}^{\infty}\int I_r(s,v_r)g_r^{Jl}(s,v_r)\bar{g}_r^{Jl'}(s,v_r)dv_r. \quad (3.5)$$

Since $I_r(s,v_r)$ is real analytic in s, we have

$$C^{Jll'}(s_+)=C^{Jl'l*}(s_-). \quad (3.6)$$

Note that for physical values of s the sum for $\mathrm{Im}C^{Jll'}(s)$ terminates at $r=N(s)$, whereas the sum for $\mathrm{Re}C^{Jll'}$ is *a priori* infinite at all values of s. Further $C^{Jll'}$ has no helicity indices. For convenience we adopt a matrix notation in the rank indices l, l'. In this notation, g_n^J is a vector with L components, and C^J is an $L\times L$ matrix, Hermitian analytic in rank space. Omitting variables for the sake of brevity, we have

$$B_{nm}^J=(\bar{g}_n^J)^T g_m^J, \quad (3.4')$$

where T means transpose in rank space. The solution to Eq. (3.3) is then

$$\tilde{B}_{nm}^J=(\bar{g}_n^J)^T(1-C^J)^{-1}g_m^J. \quad (3.7)$$

Hermitian analyticity of $\tilde{B}_{nm}^J$ follows from the Hermitian analyticity of C^J in the rank indices.

Next we solve the inelastic first stage K-matrix equation for $\tilde{B}_{2n}^J$:

$$\tilde{B}_{2n}^J=B_{2n}^J+\sum_{r=3}^{\infty}\int\tilde{B}_{2r}^J I_r B_{rn}^J dv_r. \quad (3.8)$$

Using Eq. (3.4′) for B_{rn}^J, we obtain

$$\tilde{B}_{2n}^J=B_{2n}^J+(\bar{b}_2^J)^T(1-C^J)^{-1}g_n^J, \quad (3.9)$$

where $\bar{b}_2^J$ is the vector with components

$$\bar{b}_2^{Jl}=\sum_r\int B_{2r}^J I_r\bar{g}_r^{Jl}dv_r. \quad (3.10)$$

The equation for $\tilde{B}_{n2}^J$ reads

$$\tilde{B}_{n2}^J=B_{n2}^J+\sum_{r=3}^{\infty}\int\tilde{B}_{nr}^J I_r B_{r2}^J dv_r. \quad (3.11)$$

Substituting the solution for $\tilde{B}_{nr}^J$ already obtained in Eq. (3.7), we obtain

$$\tilde{B}_{n2}^J=B_{n2}^J+(\bar{g}_n^J)^T(1-C^J)^{-1}b_2^J, \quad (3.12)$$

where

$$b_2^{Jl}=\sum_{r=3}^{\infty}\int B_{r2}^J I_r g_r^{Jl}dv_r. \quad (3.13)$$

Hermitian analyticity [$\tilde{B}_{n2}^J(s_+)=\tilde{B}_{2n}^{J*}(s_-)$] is again guaranteed since I_r is real analytic.

Note that if B_{n2}^J is taken to factorize in the form

$$B_{n2}^J=(\bar{g}_n^J)^T g_2^J,$$

we obtain

$$\tilde{B}_{n2}^J=(\bar{g}_n^J)^T(1-C^J)^{-1}g_2^J. \quad (3.14)$$

The $2\to 2$ modified Born term $\tilde{B}_{22}^J$ is defined by the inelastic equation

$$B_{22}^J=B_{22}^J+\sum_{r>2}\int\tilde{B}_{2r}^J I_r B_{r2}^J dv_r. \quad (3.15)$$

Substituting $\tilde{B}_{2r}^J$ from Eq. (3.9), we obtain

$$\tilde{B}_{22}^J=B_{22}^J+\sum_{r>2}\int B_{2r}^J I_r B_{r2}^J dv_r +(\bar{b}_2^J)^T(1-C^J)^{-1}b_2^J. \quad (3.16)$$

If B_{2r}^J factorizes and B_{22}^J is also taken to factorize as $(\bar{g}_2^J)^T g_2^J$, we obtain

$$\tilde{B}_{22}^J=(\bar{g}_2^J)^T(1-C^J)^{-1}g_2^J. \quad (3.17)$$

Hence, factorization of all B_{fi}^J as

$$B_{fi}^J=(\bar{g}_f^J)^T g_i^J$$

yields the solution for all $\tilde{B}_{fi}^J$ as

$$\tilde{B}_{fi}^J(v_f,s,v_i)=[\bar{g}_f^J(s,v_f)]^T[1-C^J(s)]^{-1}g_i^J(s,v_i). \quad (3.18)$$

IV. GENERALIZED K-MATRIX EQUATIONS—ELASTIC UNITARIZATION

Next, we proceed with the second "elastic" stage. The K-matrix equations now use the modified Born terms as input, and produce the full partial-wave amplitudes. The equations are

$$\sum_{k=2}^{\infty}\int dv_k\, H_{fk}^J(v_f,s,v_k)\tilde{D}_{ki}^J(v_k,s,v_i)=\tilde{B}_{fi}^J(v_f,s,v_i), \quad (4.1)$$

where the $\tilde{D}_{ki}^J$ functions are defined as

$$\begin{aligned}\tilde{D}_{ni}^J(v_n,s,v_i)&=\delta_{ni}\delta(v_n-v_i),\quad n>2\\ \tilde{D}_{2i}^J(v_2,s,v_i)&=\delta_{2i}\delta_{v_2v_i}-I_2\tilde{B}_{2i}^J(v_2,s,v_i).\end{aligned} \quad (4.2)$$

The D_{ki}^J functions for the first stage were obtained by formally setting $I_2=0$ and using the Born terms B^J as input. Conversely, only I_2 contributes to the second stage. The function I_2 is real analytic and has the two-body phase-space function $\rho_2(s)$ as its imaginary part above the two-body threshold, but other possible (left-hand) cuts of $I_2(s)$ are unspecified. Note that $\tilde{D}_{n2}^J=0$, whereas $\tilde{D}_{2n}^J=-I_2\tilde{B}_{2n}^J$ for $n>2$. Consider the two-body amplitude H_{22}^J, defined by

$$H_{22}^J(v_2',s,v_2)=\tilde{B}_{22}^J(v_2',s,v_2) +\sum_{v_2''}H_{22}^J(v_2',s,v_2'')I_2(s)\tilde{B}_{22}^J(v_2'',s,v_2), \quad (4.3)$$

where, as before, v_2 only includes intrinsic helicities.

Using Eq. (4.2) with $i=2$, we find the solution of Eq. (4.3) to be

$$H_{22}{}^J(v_2{}',s,v_2)=\sum_{v_2{}''}\tilde{B}_{22}{}^J(v_2{}',s,v_2{}'')\times[\tilde{D}_{22}{}^J]^{-1}(v_2{}'',s,v_2)\,,\quad (4.4)$$

so

$$H_{22}{}^J(v_2{}',s_+,v_2)=H_{22}{}^{J*}(v_2,s_-,v_2{}')\,.$$

Here matrix inversion in the helicity indices is understood. The last equality follows from the Hermitian analyticity of $\tilde{B}_{22}{}^J$ and the fact that $\tilde{B}_{22}{}^J$ and $(\tilde{D}_{22}{}^J)^{-1}$ commute.

The amplitude $H_{2n}{}^J$ is formed by solving the equation

$$H_{2n}{}^J(v_2,s,v_n)=\tilde{B}_{2n}{}^J(v_2,s,v_n)+\sum_{v_2{}'}H_{22}{}^J(v_2,s,v_2{}')I_2(s)B_{2n}{}^J(v_2{}',s,v_n)\,.\quad (4.5)$$

Using Eqs. (4.4) and (4.2), we find

$$H_{2n}{}^J(v_2,s,v_n)=\sum_{v_2{}'}[\tilde{D}_{22}{}^J]^{-1}(v_2,s,v_2{}')\tilde{B}_{2n}(v_2{}',s,v_n)\,.\quad (4.6)$$

Similarly, from the equation

$$H_{n2}{}^J(v_n,s,v_2)=\tilde{B}_{n2}{}^J(v_n,s,v_2)+\sum_{v_2{}'}H_{n2}{}^J(v_n,s,v_2{}')I_2(s)\tilde{B}_{22}(v_2{}',s,v_2)\,,$$

we have

$$H_{n2}{}^J(v_n,s,v_2)=\sum_{v_2{}'}\tilde{B}_{n2}{}^J(v_n,s,v_2{}')\tilde{D}_{22}{}^{J-1}(v_2{}',s,v_2)$$

and (4.7)

$$H_{n2}{}^J(v_n,s_+,v_2)=H_{2n}{}^{J*}(v_2,s_-,v_n)\,.$$

Finally, H_{nm} is defined by

$$H_{nm}(v_n,s,v_m)=\tilde{B}_{nm}{}^J(v_n,s,v_m)+\sum_{v_2}H_{n2}{}^J(v_n,s,v_2)I_2(s)\tilde{B}_{2m}{}^J(v_2,s,v_m)\,.\quad (4.8)$$

Using Eqs. (4.7), we find

$$H_{nm}{}^J(v_n,s,v_m)=\tilde{B}_{nm}{}^J(v_n,s,v_m)+\sum_{v_2{}'v_2{}''}B_{n2}{}^J(v_n,s,v_2{}')(\tilde{D}_{22}{}^J)^{-1}(v_2{}',s,v_2{}'')\times I_2(s)\tilde{B}_{2m}{}^J(v_2{}'',s,v_m)$$

and

$$H_{nm}{}^J(v_n,s_+,v_m)=H_{mn}{}^{J*}(v_m,\,s_-,v_n)\,.\quad (4.9)$$

If we use finite-rank approximations for all Born terms $B_{fi}{}^J$, we obtain the general expression for all the partial-wave amplitudes $H_{fi}{}^J(v_f,s,v_i)$:

$$H_{fi}{}^J(v_f,s,v_i)=[\bar{g}_f{}^J(s,v_f)]^T[1-\tilde{C}^J(s)]^{-1}g_i{}^J(s,v_i)\,,\quad (4.10)$$

where

$$\tilde{C}^{Jll'}(s)=\sum_{k=2}^{\infty}\int I_k(s,v_k)g_k{}^{Jl}(s,v_k)\bar{g}_k{}^{Jl'}(s,v_k)dv_k\quad (4.11)$$

now includes the two-body sum $\sum_{v_2}$ as well as the multiparticle sums $\sum_r\int dv_r$.

V. UNITARITY AND GENERALIZED *K*-MATRIX EQUATIONS

In this section we prove that the solutions to the K-matrix equations satisfy unitarity. Specifically, we show that the functions $\tilde{B}_{fi}$, solutions to the first stage equations, satisfy inelastic unitarity (i.e., unitarity without two-body intermediate states). The partial-wave amplitudes $H_{fi}{}^J$ are shown to satisfy the full unitarity equation, Eq. (2.5), not only for the $2\to 2$ amplitude but also for the $2\to n$, $n\to 2$, and $n\to m$ amplitudes (n, $m>2$). Thus, a dynamical scheme utilizing this formalism has unitarity for all amplitudes directly built in. Note, however, that disconnected terms in M_{nm} are not included. Some of the problems that arise in connection with these terms are discussed in Sec. VI.

A. Inelastic Unitarity and $\tilde{B}_{fi}{}^J$

We first consider the function $\tilde{B}_{nm}{}^J$, for n, $m>2$. We denote $\tilde{B}_{fi}{}^J(v_f,s_\pm,v_i)$ by $\tilde{B}_{fi}{}^J(\pm)$, $C^J(s_\pm)$ by $C_\pm{}^J$, and $I_r(s_\pm,v_r)$ by $I_r(\pm)$. Because $g_n{}^J$ has no right-hand s cuts by assumption, the right-hand s discontinuity of $\tilde{B}_{nm}{}^J$ is calculated as $[\Delta\tilde{B}_{fi}{}^J\equiv\tilde{B}_{fi}{}^J(+)-\tilde{B}_{fi}{}^J(-)]$

$$\begin{aligned}\Delta\tilde{B}_{nm}{}^J&=\Delta[(\bar{g}_n{}^J)^T(1-C^J)^{-1}g_m{}^J]\\&=(\bar{g}_n{}^J)^T[(1-C_+{}^J)^{-1}-(1-C_-{}^J)^{-1}]g_m{}^J\\&=(\bar{g}_n{}^J)^T(1-C_-{}^J)^{-1}(C_+{}^J-C_-{}^J)(1-C_+{}^J)^{-1}g_m{}^J\,.\quad (5.1)\end{aligned}$$

We next construct the inelastic unitarity sum for $B_{nm}{}^J$ and, using Eq. (3.7), obtain

$$\sum_{r=3}^{N(s)}\int B_{nr}{}^J(-)\rho_rB_{rm}{}^J(+)dv_r=(\bar{g}_n{}^J)^T(1-C_-{}^J)^{-1}\times\sum_{r=3}^{N(s)}\int g_r{}^J\rho_r(\bar{g}_r{}^J)^Tdv_r(1-C_+{}^J)^{-1}g_m{}^J\,.\quad (5.2)$$

Note that the sum is finite because ρ_r vanishes for $r>N(s)$. We identify $C_+{}^J-C_-{}^J$ from Eq. (3.5) and obtain

$$\frac{1}{2i}\Delta\tilde{B}_{nm}{}^J=\sum_{r=3}^{N(s)}\int\tilde{B}_{nr}{}^J(-)\rho_r\tilde{B}_{rm}(+)dv_r\,.\quad (5.3)$$

The same relation with the (+) and (−) signs interchanged follows by modifying the last step in Eq. (5.1).

Next we consider $\tilde{B}_{n2}{}^J$. From the integral equation (3.11) we obtain

$$\Delta\tilde{B}_{n2}{}^J=\sum_{r=3}^{\infty}\int[\tilde{B}_{nr}{}^J(+)I_r(+)-\tilde{B}_{nr}{}^J(-)I_r(-)]B_{r2}{}^Jdv_r\,.\quad (5.4)$$

Adding and subtracting $\tilde{B}_{nr}{}^J(-)I_r(+)B_{r2}{}^J$ in the integrand and using Eq. (5.3), we obtain after a change of variables in the first term

$$\frac{1}{2i}\Delta\tilde{B}_{n2}{}^J=\sum_{s=3}^{N(s)}\int\tilde{B}_{ns}{}^J(-)\rho_s \times\left[B_{s2}{}^J+\sum_{r=3}^{\infty}\int\tilde{B}_{sr}{}^J(+)I_r(+)B_{r2}{}^J dv_r\right]dv_s.$$

Using Eq. (3.11), we get inelastic unitarity for $\tilde{B}_{n2}{}^J$:

$$\frac{1}{2i}\Delta\tilde{B}_{n2}{}^J=\sum_{r=3}^{N(s)}\int\tilde{B}_{nr}{}^J(-)\rho_r\tilde{B}_{r2}{}^J(+)dv_r. \quad (5.5)$$

The relation with the $(+)$ and $(-)$ sign interchanged follows by adding and subtracting $\tilde{B}_{nr}{}^J(+)I_r(-)B_{r2}{}^J$ to the integrand in Eq. (5.4). Using Eq. (5.5) and the Hermitian analyticity of $\tilde{B}_{fi}{}^J$, we get

$$\frac{1}{2i}\Delta\tilde{B}_{2n}{}^J=\sum_{r=3}^{N(s)}\int\tilde{B}_{2r}{}^J(-)\rho_r\tilde{B}_{rn}{}^J(+)dv_r. \quad (5.6)$$

Similarly, inelastic unitarity for $\tilde{B}_{22}$,

$$\frac{1}{2i}\Delta\tilde{B}_{22}{}^J=\sum_{r=}^{N(s)}\int\tilde{B}_{2r}{}^J(-)\rho_r\tilde{B}_{r2}{}^J(+)dv_r, \quad (5.7)$$

follows from the same procedure as for $\tilde{B}_{n2}{}^J$ with n replaced by 2.

B. Full Unitarity and $H_{fi}{}^J$

We first consider $H_{22}{}^J$. From Eq. (4.4) the discontinuity of $H_{22}{}^J$ is

$$\Delta H_{22}{}^J=\tilde{B}_{22}{}^J(+)[\tilde{D}_{22}{}^J(+)]^{-1} -\tilde{B}_{22}{}^J(-)[\tilde{D}_{22}{}^J(-)]^{-1}, \quad (5.8)$$

where the sums over intermediate helicities are understood, $H_{22}{}^J$, $\tilde{B}_{22}{}^J$, and $\tilde{D}_{22}{}^J$ being treated as matrices in helicity space.

Using (4.6) and (4.7), we rewrite the full unitarity sum, obtaining

$$\begin{aligned}2i\sum_{k=2}^{N(s)}&\int H_{2k}{}^J(-)\rho_k H_{k2}{}^J(+)dv_k\\ &=[\tilde{D}_{22}{}^J(-)]^{-1}\tilde{B}_{22}{}^J(-)[I_2(+)-I_2(-)]\\ &\times\tilde{B}_{22}{}^J(+)[\tilde{D}_{22}{}^J(+)]^{-1}+2i[\tilde{D}_{22}{}^J(-)]^{-1}\\ &\times\sum_{r=3}^{N(s)}\int\tilde{B}_{2r}{}^J(-)\rho_r\tilde{B}_{r2}{}^J(+)dv_r[\tilde{D}_{22}{}^J(+)]^{-1}.\end{aligned}$$

The inelastic sum is given by Eq. (5.7) and therefore

$$\begin{aligned}2i\sum_{k=2}^{N(s)}&\int H_{2k}{}^J(-)\rho_k H_{k2}{}^J(+)dv_k\\ &=[\tilde{D}_{22}{}^J(-)]^{-1}\{\tilde{B}_{22}{}^J(-)[-1+I_2(+)\tilde{B}_{22}{}^J(+)]\\ &\quad+[1-I_2(-)\tilde{B}_{22}{}^J(-)]\tilde{B}_{22}{}^J(+)\}[\tilde{D}_{22}{}^J(+)]^{-1}\\ &=\tilde{B}_{22}{}^J(+)[\tilde{D}_{22}{}^J(+)]^{-1}-[\tilde{D}_{22}{}^J(-)]^{-1}\tilde{B}_{22}{}^J(-)\\ &=\Delta H_{22}{}^J.\end{aligned} \quad (5.9)$$

This verifies unitarity for $H_{22}{}^J$.

Next we consider $H_{2n}{}^J$, as given by Eq. (4.6). Its discontinuity is

$$\Delta H_{2n}{}^J=[\tilde{D}_{22}{}^J(+)]^{-1}\tilde{B}_{2n}{}^J(+) -[\tilde{D}_{22}{}^J(-)]^{-1}\tilde{B}_{2n}{}^J(-). \quad (5.10)$$

We rewrite the relevant full unitarity sum with Eqs. (4.4), (4.6), and (4.9), obtaining

$$2i\sum_{k=2}^{N(s)}\int H_{2k}{}^J(-)\rho_k H_{kn}{}^J(+)dv_k=E_{2n}{}^{(1)}+E_{2n}{}^{(2)}, \quad (5.11)$$

where $E_{2n}{}^{(1)}$ is the $k=2$ term

$$E_{2n}{}^{(1)}=\tilde{B}_{22}{}^J(-)[\tilde{D}_{22}{}^J(-)]^{-1}[I_2(+)-I_2(-)] \times[\tilde{D}_{22}{}^J(+)]^{-1}\tilde{B}_{2n}{}^J(+)$$

and $E_{2n}{}^{(2)}$ is the rest of the sum

$$E_{2n}{}^{(2)}=2i[\tilde{D}_{22}{}^J(-)]^{-1}\sum_{r=3}^{\infty}\int\tilde{B}_{2r}{}^J(-)\rho_r\{\tilde{B}_{rn}{}^J(+) +\tilde{B}_{r2}{}^J(+)[\tilde{D}_{22}{}^J(+)]^{-1}I_2(+)\tilde{B}_{2n}{}^J(+)\}dv_r. \quad (5.12)$$

Using Eqs. (5.6) and (5.7), we have

$$\begin{aligned}E_{2n}{}^{(2)}=&[\tilde{D}_{22}{}^J(-)]^{-1}[\tilde{B}_{2n}{}^J(+)-\tilde{B}_{2n}{}^J(-)]\\ &+[\tilde{D}_{22}{}^J(-)]^{-1}[\tilde{B}_{22}{}^J(+)-\tilde{B}_{22}{}^J(-)]\\ &\times[\tilde{D}_{22}{}^J(+)]^{-1}I_2(+)\tilde{B}_{2n}{}^J(+).\end{aligned} \quad (5.13)$$

We group the terms in (5.12) and (5.13), and using the relation

$$\tilde{B}_{22}{}^J(+)I_2(+)-\tilde{B}_{22}{}^J(-)I_2(-)=\tilde{D}_{22}{}^J(-)-\tilde{D}_{22}{}^J(+),$$

we obtain from (5.10) and (5.11)

$$\frac{1}{2i}\Delta H_{2n}{}^J=\sum_{k=2}^{N(s)}\int H_{2k}{}^J(-)\rho_k H_{kn}{}^J(+)dv_k. \quad (5.14)$$

In a similar way, using Eqs. (4.6)–(4.8) and the fact that $\tilde{B}_{fi}{}^J$ satisfies inelastic unitarity, we obtain the unitarity equation for $H_{nm}{}^J$ after some algebra, i.e.,

$$\frac{1}{2i}\Delta H_{nm}{}^J=\sum_{k=2}^{N(s)}\int H_{nk}{}^J(-)\rho_k H_{km}{}^J(+)dv_k. \quad (5.15)$$

The unitarity relations can all be verified with the $(+)$ and $(-)$ signs interchanged.

Finally, if the Born terms $B_{fi}{}^J$ have poles on the right-hand s axis at $s=\{s_k\}$, we first verify the unitarity relations for all $\{s\neq s_k\}$, that is, almost everywhere. Since the output poles are shifted away from the real s axis (see Appendix A), the equations hold throughout the physical region.

VI. EXTENSIONS OF FORMALISM

A. Inclusion of Symmetries

The generalization of the formalism to include symmetries is straightforward. We first note that the extension to more than one kind of particle, i.e., involving several channels for each number of particles n_k, can be incorporated by including appropriate extra internal variables in the variables v_k; the k sums $\sum_k \int dv_k$ then include sums over these internal variables. If there are N_{n_k} channels with n_k particles, then the dimensionality of the amplitude is correspondingly increased. Matrix products and inverses in channel space are to be used, and $I_k(s,v_k)$ becomes a diagonal matrix in the channel indices. In particular, if we have an internal symmetry group with total diagonal quantum numbers $\{a_\alpha{}^{(k)}\}$ for the kth channel, we label the Born terms with these numbers. Including sums over possible values of the quantum numbers for the individual particles in the k sums, the formalism remains unchanged except that $(\tilde{D}_{22}{}^J)^{-1}(v_2',s,v_2)$ now includes an inverse in the internal symmetry labels as well as the internal helicities, and the output amplitudes are also labeled with the total diagonal quantum numbers.

Next we consider time-reversal (T) and parity (P) invariance. T and P invariance of the Born terms $B_{fi}{}^J$ are written as

$$\begin{aligned} B_{fi}{}^J(v_f\mu_f,s,v_i\mu_i) &= \bar{\eta}_f{}^T\eta_i{}^T B_{if}{}^J(v_i{}^R\mu_i,s,v_f{}^R\mu_f) \quad (T) \\ &= \bar{\eta}_f{}^P\eta_i{}^P B_{fi}{}^J(v_f{}^R\mu_f{}^R,s,v_i{}^R\mu_i{}^R) \quad (P), \end{aligned} \tag{6.1}$$

where the reversed helicities are $\{\mu_i{}^R\}=\{-\mu_i\}$, and $\{v_i{}^R\}$ are the $\{v_i\}$ with reversed 3-momenta. (Since we have defined our partial-wave amplitudes with the use of two azimuthal angles in the r_{fi}, not all the $\{v_i,v_f\}$ are dot products of momenta.) The η^T and η^P are phase factors.

For the finite-rank approximation of all $B_{fi}{}^J$, we have

$$\begin{aligned} H_{fi}{}^J(v_f\mu_f,s,v_i\mu_i) \\ = [\bar{g}_f{}^J(s,v_f\mu_f)]^T[1-\tilde{C}^J(s)]^{-1}g_i{}^J(s,v_i\mu_i). \end{aligned} \tag{6.2}$$

We satisfy Eq. (6.1) by choosing the $g_i{}^{Jl}$ such that

$$\begin{aligned} g_i{}^{Jl}(s,v_i\mu_i) &= \eta_i{}^T\bar{g}_i{}^{Jl}(s,v_i{}^R\mu_i) \quad (T) \\ &= \eta_i{}^P g_i{}^{Jl}(s,v_i{}^R\mu_i{}^R) \quad (P). \end{aligned}$$

Hence we obtain

$$H_{fi}{}^J(v_f\mu_f,s,v_i\mu_i) = \bar{\eta}_f{}^T\eta_i{}^T H_{if}{}^J(v_i{}^R\mu_i,s,v_f{}^R\mu_f) \quad (T),$$

provided that $\tilde{C}^J$ is symmetric in rank space,

$$\tilde{C}^{Jll'}(s) = \tilde{C}^{Jl'l}(s).$$

This requires that $I_r(s,v_r)=I_r(s,v_r{}^R)$.

Similarly, P invariance of $B_{fi}{}^J$ implies P invariance of $H_{fi}{}^J$.

We further note that Bose or Fermi statistics can easily be incorporated since symmetry properties of, e.g., $g_i{}^{Jl}(s,v_i\mu_i)$ propagate through to the solution [Eq. (6.2)].

B. Inversion of Equations

We now discuss the question of how general the above representation for unitary connected amplitudes is. This amounts to solving the inverse problem, i.e., given arbitrary unitary connected $H_{fi}{}^J$, find the corresponding Born terms $B_{fi}{}^J$ and prove that they have no right-hand cuts. The equations for $H_{fi}{}^J$ (written in one stage instead of two) are

$$H_{fi}{}^J = B_{fi}{}^J + \sum_k \int H_{fk}{}^J I_k B_{ki}{}^J dv_k,$$

which can be interpreted as equations determining the $B_{fi}{}^J$ for given $H_{fi}{}^J$. They can be solved as above if they are of Fredholm type, except that the finite-rank approximations are now applied to the $H_{fi}{}^J$. The absence of right-hand cuts in $B_{fi}{}^J$ follows from the unitarity of $H_{fi}{}^J$.

Notice that if we set $B_{nm}{}^J=0$ (n, $m>2$) as Baker and Blankenbecler did, we see that the $H_{nm}{}^J$ amplitude is completely determined in terms of $H_{n2}{}^J$ and $H_{22}{}^J$ as

$$H_{nm}{}^J = H_{n2}{}^J I_2 H_{2m}{}^J/(1+I_2H_{22}{}^J).$$

Hence the BB formalism does not allow a solution of the inverse problem for an arbitrary set of unitary $H_{fi}{}^J$.

C. Full-Amplitude K-Matrix Formalism

The K-matrix formalism presented above can also be carried through without using the J partial-wave decomposition. The equations written in one stage for the given total Born term $\mathcal{B}_{fi}(v_f,s,r_{fi},v_i)$ are

$$\begin{aligned} M_{fi}(v_f,s,r_{fi},v_i) &= \mathcal{B}_{fi}(v_f,s,r_{fi},v_i) \\ &+\sum_k \int M_{fk}(v_f,s,r_{fk},v_k) I_k \mathcal{B}_{ki}(v_k,s,r_{ki},v_i) dv_k dr_k. \end{aligned}$$

We then write the finite-rank decomposition,

$$\mathcal{B}_{ki}(v_k,s,r_{ki},v_i) = \lim_{L\to\infty}\sum_{l=1}^{L} \bar{G}_k{}^l(s,v_k,r_k) G_i{}^l(s,v_i,r_i{}^{-1}),$$

and solve the equations as above. The solution in matrix notation is

$$H_{fi}(v_f,s,r_{fi},v_i) = [\bar{G}_f(s,v_f,r_f)]^T[1-\mathcal{C}(s)]^{-1}G_i(s,v_i,r_i{}^{-1}),$$

where

$$\mathcal{C}^{ll'}(s)=\sum_{k=2}^{\infty}\int dv_k dr_k\, G_k{}^{l}(s,v_k,r_k{}^{-1})I_k(s,v_k)\bar{G}_k{}^{l'}(s,v_k,r_k)\,.$$

D. Inclusion of Disconnected Pieces

The inclusion of disconnected parts would require a much more sophisticated treatment than that used here. The essential difficulty is that the disconnected structure of M_{nm} and $\mathcal{B}_{nm}$ produce extra intractable terms in the integral equations. Some terms in the equations are integrals of the form

$$\sum_k \int M_{fk}(\cdots,\xi_i)I_k(\prod_l \mathcal{B}_{k_l i_l})dv_k\,.$$

In the case of *connected* amplitudes and Born terms, we saw that M_{fk} was independent of all initial variables. Now, however, this is no longer true. The variables $\{\xi_i\}$ depend on the initial momenta and result from the extra δ functions from the disconnected structure of the Born terms, and they prevent a simple solution of the equations. Similar terms appear due to the disconnected pieces of the M_{fk}, which, in fact, couple the various equations together.

VII. DYNAMICAL APPLICATIONS

Although we have a formalism guaranteeing full unitarity for all connected amplitudes, the dynamics has yet to be specified by the choice of Born terms $B_{fi}{}^J(v_f,s,v_i)$ and possible left-hand cuts of the $I_k(s,v_k)$ functions. One could imagine using partial-wave projections of s-meromorphic Born terms with parallel trajectories of the form (Re$s>0$)

$$B_{fi}{}^J(v_f,s,v_i)=\sum_{l=1}^{\infty}\frac{1}{s_l-s}\sum_{\alpha=1}^{D_l}h_f{}^{Jl\alpha}(s,v_f)h_i{}^{Jl\alpha}(s,v_i)\,,\quad(7.1)$$

where D_l is the degeneracy of the lth pole with spin J. Cutting the sum off at $l=l_0$ provides a natural choice for the finite-rank approximation (the rank being $L=\sum_{l=1}^{l_0}D_l$). Figure 1 shows the relation of the finite-rank approximation to the input particle spectrum. In Appendix A we show that the resonance poles move off the real s axis when the $B_{fi}{}^J$ are unitarized, although the ghosts probably move up onto the first sheet. A description of the pole movement is complicated, and depends on the rank used. However, we show that the pole degeneracies are in general split and the output residues factorize.

Questions of asymptotic behavior and crossing symmetry are clearly related through the (as yet) unspecified left-hand cuts of the I_k functions. While these cuts can be interpreted as modifying the Born terms (see Appendix B), it is not clear to what extent they can be used to ensure a crossing-symmetric output for arbitrary

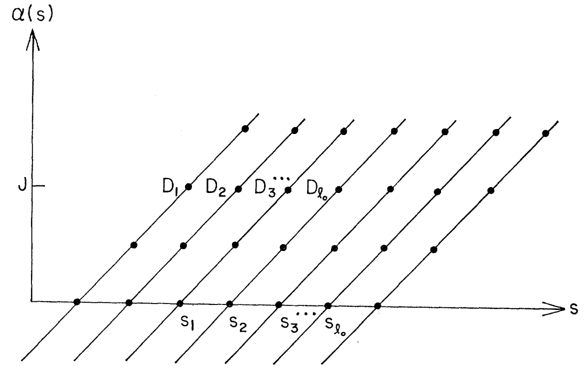

FIG. 1. Possible pole spectrum of the nonunitary Born terms $B_{fi}{}^J$ and the relation to the finite-rank approximation $L=\sum_{l=1}^{l_0}D_l$, where D_l is the degeneracy of the lth pole with spin J.

Born terms. For given Born terms, the asymptotic behavior may be crucially related to the I_k. For example, the phenomenological applications in Ref. 2 required that asymptotically

$$I_2(s)\to i\rho_2(s)=[(4m^2-s)/s]^{1/2}.$$

The case of the elastic K-matrix unitarization of the 4-point Veneziano amplitude was suggested by Lovelace.[5] It is tempting to try to use the Veneziano n-point functions for the multiparticle Born terms. They factorize in the form of Eq. (7.1).[6] However, in addition to poles in s, these functions have poles in *subenergies*, which make the K-matrix equations of non-Fredholm type. There is a connection here with the inclusion of disconnected terms, since these terms appear in the discontinuity equations across subenergies[3] and make the K-matrix equations of non-Fredholm type. It is reasonable to argue that this disconnected unitarization is responsible for moving poles in the subenergies off the real subenergy axes, in analogy to the movement of the poles in the total energy (see Appendix A).

APPENDIX A

In this appendix we investigate the movement of poles of a Born term undergoing unitarization. For simplicity we consider a single pole in $B_{ij}{}^J$ above the two-body threshold at $s=s_0$ of degeneracy D. Choosing the rank $L=D$ and suppressing the s dependence of $h_i{}^{Jl}$, we write, for Re$s>0$

$$B_{ij}{}^J=\sum_{l=1}^{D}\frac{\bar{h}_i{}^{Jl}(v_i)h_j{}^{Jl}(v_j)}{s_0-s}=\frac{1}{s_0-s}[\bar{h}_i{}^J(v_i)]^T h_j{}^J(v_j)\,.\quad(\mathrm{A1})$$

The unitarized amplitude $H_{ij}{}^J$ is then

$$H_{ij}{}^J=[\bar{h}_i{}^J(v_i)]^T[s_0-s-C^J(s)]^{-1}h_j{}^J(v_j)\,,\quad(\mathrm{A2})$$

[5] C. Lovelace, CERN Report No. TH1041 (unpublished); Nucl. Phys. **B12**, 253 (1969).
[6] See, e.g., S. Fubini, D. Gordon, and G. Veneziano, Phys. Letters **29B**, 679 (1969).

where

$$C^J(s)=\sum_k \int dv_k\, I_k(s,v_k)h_k{}^J(v_k)[h_k{}^J(v_k)]^T \quad \text{(A3)}$$

or

$$H_{ij}{}^J=\frac{1}{\Delta_D{}^J(s)}\sum_{ll'} \bar{h}_i{}^{Jl}(v_i)\Delta_{D-1}{}^{Jll'}(s)h_j{}^{Jl'}(v_j)\,, \quad \text{(A4)}$$

where

$$\Delta_D{}^J(s)=\det[s_0-s-C^J(s)] \quad \text{(A5)}$$

and $\Delta_{D-1}{}^{Jll'}$ is the (l,l')th minor of $\Delta_D{}^J$.

Regarding $C^J(s)$ as relatively constant in s over the region where the poles move, we have

$$\Delta_D{}^J(s)\approx\prod_{i=1}^{D}(s_i-s)\,. \quad \text{(A6)}$$

For rank $D=1$,

$$s_1\approx(s_0-\mathrm{Re}C^J)-i(\mathrm{Im}C^J)\,,$$

where

$$\mathrm{Im}C^J=\sum_k \int dv_k\, \rho_k h_k{}^J\bar{h}_k{}^J\,.$$

Hence in the rank-1 approximation, resonance poles with $\bar{h}_k{}^J=h_k{}^{J*}$ move onto the second sheet, as they should. Ghosts with the non-Hermitian-analytic condition $\bar{h}_k{}^J=-h_k{}^{J*}$ move up onto the first sheet, however. For rank D, we have

$$\sum_{i=1}^{D}\mathrm{Im}s_i=-\sum_{l=1}^{D}\mathrm{Im}C^{Jll}\,.$$

If all poles have $\bar{h}_k{}^{Jl}=h_k{}^{Jl*}$, the unitarized poles will again all be on the second sheet, although $\mathrm{Im}s_i$ now depends both on the real and imaginary parts of the $C^{Jll'}$.

In general, the degeneracy will be broken, the pole positions s_i being distinct. For example, if $D=2$, we obtain poles at $s_\pm$ where

$$s_\pm=s_0-\tfrac{1}{2}(C^{J11}+C^{J22})\pm\tfrac{1}{2}[(C^{J11}-C^{J22})^2+4(C^{J12})^2]^{1/2}\,. \quad \text{(A7)}$$

The pole positions found in the rank-D approximation will change as the rank is increased to include other poles. The s dependence of $C^J(s)$ will also change their positions. The movement of poles is thus seen to be quite complicated.

To prove factorization of the residues, we note the theorem on determinants[7] proved by pivoting the $D\times D$ determinant $\Delta_D{}^J$ $D-2$ times:

$$\Delta_D{}^J\Delta_{D-2}{}^{Jll'll'}=\Delta_{D-1}{}^{Jll}\Delta_{D-1}{}^{Jl'l'}-\Delta_{D-1}{}^{Jll'}\Delta_{D-1}{}^{Jl'l}\,, \quad \text{(A8)}$$

where $\Delta_{D-2}{}^{Jll'll'}$ is the determinant $\Delta_D{}^J$ with rows ll' and columns ll' deleted. $\Delta_{D-1}{}^{Jll'}$, as before, is the determinant $\Delta_D{}^J$ with row l and column l' deleted. Now with T invariance, $\Delta_{D-1}{}^J$ is a symmetric determinant. Hence if $\Delta_D{}^J$ vanishes, we obtain

$$\Delta_{D-1}{}^{Jll}\Delta_{D-1}{}^{Jl'l'}=(\Delta_{D-1}{}^{Jll'})^2\,. \quad \text{(A9)}$$

But then at the pole $s=s_k$

$$\lim_{s\to s_k}(s_k-s)H_{ij}{}^J=[\prod_{j\neq k}(s_j-s_k)]^{-1}[\sum_{l=1}^{D}\Delta_{D-1}{}^{Jll}(s_k)^{1/2}\bar{h}_i{}^{Jl}(v_i)]\times[\sum_{l'=1}^{D}\Delta_{D-1}{}^{Jl'l'}(s_k)^{1/2}h_j{}^{Jl'}(v_j)]\,. \quad \text{(A10)}$$

Thus we see that the residues of unitarized poles factorize.

APPENDIX B

In this appendix we investigate the effective change in the Born terms when left-hand s cuts in the I_k functions are added. Specifically, we write (subtractions must be added in general)

$$I_k(s,v_k)=I_k{}^0(s,v_k)+E_k(s,v_k)\,,$$

where

$$I_k{}^0(s,v_k)=\frac{1}{\pi}\int_{s_k}^{\infty}\frac{\rho_k(s',v_k)ds'}{s'-s} \quad \text{(B1)}$$

and

$$E_k(s,v_k)=\frac{1}{\pi}\int_{-\infty}^{0}\frac{\tilde{\rho}_k(s',v_k)ds'}{s'-s}\,, \quad \text{(B2)}$$

where $\tilde{\rho}_k$ is arbitrary.

For simplicity we assume finite-rank factorization of all the Born terms, including $B_{22}{}^J$. The $j\to i$ amplitude is (suppressing variables)

$$H_{ij}{}^J(s)=[\bar{g}_i{}^J(s)]^T[1-\tilde{C}^J(s)]^{-1}g_j{}^J(s)\,, \quad \text{(B3)}$$

where

$$\tilde{C}^J(s)=\sum_k\int dv_k(I_k{}^0+E_k)g_k{}^J(\bar{g}_k{}^J)^T \equiv C_{I^0}{}^J+C_E{}^J\,, \quad \text{(B4)}$$

and we want to find $\gamma_k{}^J$ such that

$$H_{ij}{}^J(s)=[\bar{\gamma}_i{}^J(s)]^T[1-\hat{C}^J(s)]^{-1}\gamma_j{}^J(s)\,, \quad \text{(B5)}$$

where

$$\hat{C}^J(s)=\sum_k\int dv_k\, I_k{}^0\gamma_k{}^J(\bar{\gamma}_k{}^J)^T\,. \quad \text{(B6)}$$

The solution is

$$\gamma_i{}^J=(1-C_E{}^J)^{-1/2}g_i{}^J$$

and

$$\bar{\gamma}_i{}^J=[(1-C_E{}^J)^{-1/2}]^T\bar{g}_i{}^J\,. \quad \text{(B7)}$$

[7] A. C. Aitken, *Determinants and Matrices* (Oliver and Boyd, London, 1956), p. 48.

For rank one, we obtain

$$(\gamma_i{}^J)^2=(g_i{}^J)^2\bigg/1-\sum_k\int dv_k\,|g_k{}^J|^2E_k. \qquad \text{(B8)}$$

The proof is simple. Plugging (B7) into (B5) and noting that $C_E{}^J$ and $C_{I^0}{}^J$ commute, we find

$$\begin{aligned}H_{ij}{}^J&=(\bar{g}_i{}^J)^T(1-C_E{}^J)^{-1/2}[1-(1-C_E{}^J)^{-1}C_{I^0}{}^J]^{-1}\\&\qquad\times(1-C_E{}^J)^{-1/2}g_j{}^J\\&=(\bar{g}_i{}^J)^T(1-C_{I^0}{}^J-C_E{}^J)g_i{}^J,\end{aligned}$$

which is (B3) as desired. Since $C_E{}^J=(C_E{}^J)^{T*}$ has no right-hand s cuts, $\bar{\gamma}_i{}^J\to\gamma_i{}^{J*}$ as s becomes physical.

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Regge Slopes and Residues for a Spin-1+Spin-0 System in the Content of $O(4)$ Symmetry and Bethe-Salpeter Model*

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(Received 22 December 1969)

The Bethe-Salpeter equation for a spin-1 + spin-0 system at vanishing total 4-momentum K is separated by means of the $O(4)$ vector spherical harmonics, which are defined and discussed briefly. Perturbation expressions for the $M=1$ residues near $K=0$ are derived and then examined near $J=0$, where it is found that only the nonsense part of the $O(3)$ content of the $M=1$ residues survives. Perturbation expressions are given also for the slopes of Regge trajectories at $s=-K^2=0$, the signs of the slopes are investigated for certain classes of potentials, and a sum rule satisfied by slopes of contiguous daughters is derived.

I. INTRODUCTION

CONSIDERABLE progress has been achieved in understanding the $O(4)$ properties of scattering amplitudes since it was first discovered that these amplitudes acquire $O(4)$ symmetry at vanishing total 4-momentum K of the interacting particles ($K^2=-s=0$ in the s channel corresponds to forward scattering in the t channel for processes with pairwise-equal masses).[1–3] The Bethe-Salpeter (BS) model,[4] which exhibits this type of symmetry, has been popular because of its mathematical tractability. Thus, many attempts have been made to study the $O(4)$ properties of scattering amplitudes in the context of the BS model and several methods have been proposed for the expansion of these amplitudes.[5,6]

This work is devoted to investigating the residues near $J=0$ at small values of K and the signs of the slopes of Regge trajectories at $s=-K^2=0$ for a spin-1 +spin-0 system in the context of the BS model. We start off in Sec. II by separating the BS equation using the $O(4)$ vector spherical harmonics (VSH), which are defined and discussed briefly in the Appendix. The definition of these functions is motivated by the four-dimensional vector character of the BS wave function and the analogy with $O(3)$ VSH.[7,8]

Assuming nondegenerate perturbation theory to hold near $K=0$, we derive, in Sec. III, expressions for the $M=1$ residues at small values of K and investigate the behavior of these residues near $J=0$. The assumption that nondegenerate perturbation theory holds near $K=0$ is consistent with an $M=0$ classification of the pion since the $J=0$, $M=1$ residue chooses nonsense at $K=0$ and continues to choose nonsense near $K=0$.[9] The possibility of an $M=1$ classification of the pion which attracted interest for some time[10] could be realized if nondegenerate perturbation theory failed near $K=0$ and the pion residue were allowed to recieve a considerable contribution from the $M=1$ states.[11] The realization of this situation in the context of the BS model did not produce satisfactory results.[6]

In Sec. IV we study in some detail the signs of the slopes of Regge trajectories at $s=0$. Our tool is again nondegenerate perturbation theory, which we apply to the BS equation after putting it in a simpler form by

* Work supported in part by the National Science Foundation and based in part on a thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Physics, University of California.
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