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Broken Scale Invariance*

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We study the implications of approximate scale invariance for particle masses and the dynamics of the low-lying scalar and pseudoscalar mesons. Assuming that the form factor $\theta(t)$ (the trace of the energy-momentum tensor) obeys an unsubtracted dispersion relation, one may derive formulas of the Goldberger-Treiman type for hadron masses if a set of scalar mesons dominates the dispersion relation. The dependence of particle masses on the coupling strength and dimension of scale-breaking operators is studied. In addition to the scale-breaking operator u which breaks chiral symmetry, it seems necessary to introduce a scale-breaking $SU_3\times SU_3$ singlet δ, as advocated by Wilson. The absence of δ would contradict the analysis of meson-baryon scattering lengths by von Hippel and Kim, would make the large mass of η' and the baryons difficult to understand, and would require the operator u to have a dimension -2 or mixed. At present, the evidence for δ is not completely compelling, however. In the absence of all scale breaking, we distinguish two types of theories. In the first, all masses vanish in the limit of scale invariance and the vacuum is normal under scale transformations. This is the situation envisioned in most previous research. The second alternative is that of spontaneous breakdown of scale invariance, in which case the vacuum is degenerate and most particles have masses. The latter case is especially interesting in that it requires the existence of a massless "dilaton" and further permits other masses to be not so inaccessible to extrapolations from the real world. Lagrangian models of spinless mesons are investigated to understand the interplay of scale and $SU_3\times SU_3$ transformations. Working in the limit of $SU_3\times SU_3$ symmetry, the scale breaking is due to the trilinear interactions. We consider two linear models in which there are (a) 18 meson fields belonging to $(3,\bar{3})+(\bar{3},3)$ and (b) 20 meson fields, with two extra mesons S and P belonging to (1,1) in addition to the 18 of case (a). The scale-invariant part of the Lagrangian is found to be $U_3\times U_3$ invariant, so that the trilinear coupling δ breaks $F_0{}^5$ as well as scale. We speculate that the real world is $U_3\times U_3$ invariant in the scale-invariant limit (although there exist meson models in which $F_0{}^5$ does not even exist). Spontaneous breakdown is considered for cases in which the ninth scalar (σ_0) and the extra scalar (S) have nonvanishing vacuum expectation values. In the 18-meson model, spontaneous breakdown is permitted in the scale-invariant limit only when the quartic couplings have a relation prohibiting $\langle\sigma_0\rangle\neq 0$ in the presence of the scale-breaking term δ. The 20-meson model allows spontaneous scale breaking which survives when explicit scale breaking is introduced. In the latter model, the "dilaton" is a mixture of the S and σ_0 mesons. The most interesting qualitative feature of the "spontaneous-breakdown" solutions is that even the canonical dimension of an operator can be somewhat ambiguous. In particular, an operator having a given dimension appropriate to a normal ground state may have different (usually several) dimensions under scale transformations appropriate to the physical ground state.

I. INTRODUCTION

THE possible relation of scale transformations to the dynamical structure of field theories has been the subject of many investigations.[1–5] Although the real world does not seem to exhibit scale invariance, we suggest that the breaking of scale invariance and the concomitant interplay with chiral symmetry (and *its* breaking) are important for the understanding of the dynamics of hadrons. The present paper extends the considerations of Ref. 3 and also considers in some detail the breaking of scale invariance (and $SU_3\times SU_3$) by spontaneous-breakdown mechanisms. We are concerned with low-energy dynamics rather than possible manifestations of scale invariance in deep-inelastic electroproduction.

Wilson[2] has emphasized that the dimension of field operators can change drastically owing to the singular behavior of field theory at short distances. We wish to emphasize another mechanism by which interactions cause changes in dimension, i.e., by spontaneous breakdown. Although our analysis is limited to the usual self-consistent approximations characteristic of such theories, the qualitative difference from "normal" ground states leads to many interesting and suggestive results. In this case, a given power of a field is changed to a polynomial in the new fields, the latter having their ordinary dimension in the broken theory before the new interactions are taken into account. Thus many "free-field" properties are retained in the zeroth-order approximation.

Many of the aforementioned considerations are illustrated by the self-coupled scalar field studied in Sec. IV. Perhaps the most interesting point is that when a field has nonvanishing vacuum expectation value, it is useful to choose an energy-momentum tensor $\theta_{\mu\nu}$ different from that appropriate to the normal ground state. Further, an operator with given dimension in the

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[1] J. Wess, Nuovo Cimento **18**, 1086 (1960); H. Kastrup, Phys. Rev. **142**, 1060 (1966), and earlier papers cited therein; G. Mack, Nucl. Phys. **B5**, 499 (1968). Many references to early work in this field may be traced from the paper of G. Mack and A. Salam, Ann. Phys. (N. Y.) **53**, 174 (1969).
[2] K. Wilson, Phys. Rev. **179**, 1499 (1969).
[3] M. Gell-Mann, University of Hawaii summer school lectures, 1969 (California Institute of Technology Report No. CALT-68-244) (unpublished).
[4] C. G. Callan, S. Coleman, and R. Jackiw, Ann. Phys. (N. Y.) **59**, 42 (1970).
[5] D. J. Gross and J. Wess, Phys. Rev. D **2**, 753 (1970).

normal ground state may have several dimensions in the "displaced" theory. The dependence of $\theta_{\mu\nu}$ and of dimensions on dynamics is an interesting and crucial result which deserves further study. Since the "spontaneous-breakdown" solution represents only an approximate solution of the original theory, we see that its ground state is, properly speaking, an intermediate point from which one can hopefully do perturbation theory in the modified interactions.

Callan, Coleman, and Jackiw have recently shown[4] how to redefine the energy-momentum tensor $\theta_{\mu\nu}$ in such a manner that in renormalizable field theories, the matrix elements of $\theta_{\mu\nu}$ are finite, and further, that the operators inducing scale and special conformal transformations are space integrals of moments of $\theta_{\mu\nu}$. For example, the dilatation operator D is expressible as

$$D=\int d^3x\, x^\mu\theta_{\mu 0}\,, \tag{1.1}$$

and the nonconservation of D is connected with the trace of $\theta_{\mu\nu}$ by means of

$$\frac{dD}{dt}=\int d^3x\, \theta^\mu{}_\mu\,. \tag{1.2}$$

Further details can be found in Refs. 3 and 4. The dimension of a field ϕ (assumed to have a unique dimension l) is related to (1.1) by

$$i[D(t),\phi(x,t)]=(-l+x\cdot\partial)\phi(x,t)\,. \tag{1.3}$$

Equation (1.3) is the infinitesimal operator version of the classical scale transformation $\phi'(x')=\rho^l\phi(x)$, $x'=\rho x$. For example, $l=-1$ for spin-0 fields. For free spinor fields, one has $l=-\frac{3}{2}$.

As explained in Ref. 3, it is reasonable to suppose that the energy density θ_{00} has two parts, one ($\bar{\theta}_{00}$) of dimension -4 which corresponds to a scale-invariant theory and a *world scalar* (w) responsible for the violation of D. Breaking w into pieces w_n of dimension l_n, we have

$$\theta_{00}=\bar{\theta}_{00}+\sum_n w_n\,. \tag{1.4}$$

It is then possible[3] to express the trace $\theta^\mu{}_\mu$ in terms of the scale-breaking operators according to the "virial theorem"

$$\theta^\mu{}_\mu=\sum_n(l_n+4)w_n\,. \tag{1.5}$$

This result, which has many applications, is reminiscent of the traditional virial theorem in that it allows expectation values of kinetic-energy terms (which appear in $\bar{\theta}_{00}$) to be expressed in terms of "potential-energy" operators (scale-breaking polynomials in the fields).

A scale-invariant theory is characterized by vanishing trace $\theta^\mu{}_\mu$. The local form of (1.2) is

$$\partial^\mu\mathfrak{D}_\mu=\theta^\mu{}_\mu\,, \tag{1.6}$$

where $\mathfrak{D}_\mu=x^\nu\theta_{\mu\nu}$ is the scale current. In analogy with the dispersion-theoretic formulation of the partial conservation of axial-vector current (PCAC), we may define approximate scale invariance to mean that the matrix elements of $\theta^\mu{}_\mu$ obey an unsubtracted dispersion relation.[6] This assumption could be true even if scale invariance were badly broken (as seems to be the case). In practice, the relation is useful only if the matrix elements can be dominated by a set of scalar mesons. We refer to this hypothesis as PCDC, i.e., partial conservation of dilatation current.

In the case of spontaneous breakdown of scale invariance, we can have massive particles even though $\theta^\mu{}_\mu$ vanishes, provided there is a massless "dilaton" σ. The latter particle causes many ambiguities in the limiting theory. In order to interpret such ambiguities which occur as $k\to 0$ in various matrix elements, one should keep m_σ finite until the end of the calculation. The conformal generators defined in terms of $\theta_{\mu\nu}$ do not have the expected property of carrying one-particle states to one-particle states when there is a massless field with nonvanishing vacuum expectation value. On the other hand, the redefined tensor $\theta_{\mu\nu}'$ gives nonconserved conformal generators which do not add massless dilatons to single-particle states.

In Sec. II we study gravitational form factors $\langle p'|\theta_{\mu\nu}|p\rangle$ for spin 0 and spin $\frac{1}{2}$ in order to find the implications of PCDC. Relations of the Goldberger-Treiman type are derived which express particle masses (baryons), or (mass)^2 for mesons, as sums $\sum F_\sigma g_{\sigma PP}$, where F_σ measures the strength of the matrix element $\langle 0|\theta_{\mu\nu}|\sigma\rangle$ and $g_{\sigma PP}$ is the coupling of the 0^+ meson σ to the particle P.

Section III considers the dependence of the particle masses on the parameters measuring the strength of the various scale-breaking operators. Powerful constraints emerge when one combines the trace theorem with the vanishing of the self-stress[7] and uses Feynman's theorem to express the parameter variation of the masses in terms of the various scale-breaking operators. In particular, it is noticed that the $SU_3\times SU_3$-breaking operator $\lambda u=\lambda(u_0+cu_8)$ of Gell-Mann,[7] and Gell-Mann, Oakes, and Renner[8] gives half the total pseudoscalar octet mass as a consequence of the quadratic mass formula $[m^2(\lambda)\propto\lambda]$. In Ref. 3 it was proved that a *single* scale-breaking operator of unique dimension must have $l=-2$ in order to give a quadratic mass formula when one perturbs from the zero-mass limit. Applying this theorem to the pseudoscalar octet, one cannot have for u_0+cu_8 a unique dimension l_u different from

[6] M. Gell-Mann, Phys. Rev. **125**, 1062 (1962). In this paper the derivation of Goldberger-Treiman formulas for scalar mesons was considered, although the connection with scale transformations was not noted.

[7] H. Umezawa, *Quantum Field Theory* (North-Holland, Amsterdam, 1956); J. M. Jauch and F. Rohrlich, *Theory of Photons and Electrons* (Addison-Wesley, Reading, Mass., 1955).

[8] M. Gell-Mann, R. J. Oakes, and B. Renner, Phys. Rev. **175**, 2195 (1968). Also see S. L. Glashow and S. Weinberg, Phys. Rev. Letters **20**, 224 (1968).

-2 (such as -3 in the naive quark model or -1 in lowest order in the σ model) unless another scale-breaking operator (called δ) is present in the $SU_3\times SU_3$ limit. Arguments in favor of the existence of δ are summarized below.

Although a detailed theory of strong interactions does not yet exist, it is reasonable to suppose that $SU_3\times SU_3$ is nearly a good symmetry. We write the energy density θ_{00} in the form

$$\theta_{00}=\hat{\theta}_{00}+u\,, \tag{1.7}$$

where $\hat{\theta}_{00}$ is invariant under $SU_3\times SU_3$ and u contains SU_3 singlet and octet components. From the work of Gell-Mann, Oakes, and Renner,[8] it seems plausible that u has the simple form u_0+cu_8 with c near $-\sqrt{2}$ (so that the pion mass is nearly zero), the u_i being scalar components of a $(3,\bar{3})\oplus(\bar{3},3)$ representation. Under standard assumptions,[3,7] $\partial^\mu\mathfrak{F}_{i\mu}{}^5=i[F_i{}^5,u]$ $(i=1,\ldots,8)$ and the axial-vector current divergences are dominated by a pseudoscalar octet in $(3,\bar{3})+(\bar{3},3)$.

When u goes to zero, the axial-vector currents are conserved, and the pseudoscalar octet becomes massless. However, other particles apparently remain massive in this limit (many with masses of the order of the nucleon mass). The numerical analysis of von Hippel and Kim[9] indicates that the *nucleon* mass remains essentially unchanged as $u\to 0$. It also seems that the η' remains massive in this limit. Thus it is reasonable to assume that $\hat{\theta}_{00}$ contains a scale-breaking (scalar) operator δ in addition to a part of dimension 4, $\bar{\theta}_{00}$,

$$\begin{aligned}\hat{\theta}_{00}&=\bar{\theta}_{00}+\delta\,,\\ \theta_{00}&=\bar{\theta}_{00}+\delta+u\,.\end{aligned} \tag{1.8}$$

δ is to be $SU_3\times SU_3$ invariant, as is $\bar{\theta}_{00}$. A similar argument has been advanced by Wilson.[2] Mass shifts of the order of 1 GeV are to be attributed to the operator δ, while u causes shifts of a few hundred MeV at most.

At present the most compelling evidence for the existence of δ is the result of Ref. 9 that the nucleon matrix element $\langle N|u_0+cu_8|N\rangle$ is very small. However, this result is based on a complicated sequence of approximations. If $\langle N|u_0|N\rangle$ is not 215 MeV but is too small by a factor of 3, say, then one would not be compelled to introduce δ. Moreover, the arguments given below based on the special status of η' are highly model dependent. These problems deserve further research.

Let λ denote the coefficient of the operator u which breaks $SU_3\times SU_3$. When $\lambda\to 0$ the baryon octet is massive, according to Ref. 9. In addition, the pseudoscalar octet obeys $m^2\propto\lambda$. From these facts we conclude that at least two SU_3 singlet scalar mesons are needed to satisfy the Goldberger-Treiman formulas for masses (based on scalar meson dominance of an unsubtracted dispersion relation for $\theta^\mu{}_\mu$). This situation is in accord with the consideration following Eq. (1.10) below. However, this deduction depends on the vanishing of the form factor $\theta(t)$ as $t\to\infty$.

[9] F. von Hippel and J. K. Kim, Phys. Rev. Letters **22**, 740 (1969); Phys. Rev. D **1**, 151 (1970).

Thus it is reasonable to temporarily omit u and to consider models which exhibit scale breaking in the $SU_3\times SU_3$ limit. The representation $(3,\bar{3})+(\bar{3},3)$, to which the pseudoscalar octet fields belong, has 18 components. Since there is some evidence for (heavy) spin-zero mesons, we suppose the full multiplet exists rather than initially search for a nonlinear realization. The absence of degeneracy implies that the vacuum is not invariant under $SU_3\times SU_3$. This situation is vividly exhibited by "spontaneously broken" solutions to effective Lagrangian models, in which the eight pseudoscalars are the Goldstone bosons associated with the symmetry breaking. These models are very instructive and will be analyzed in detail. In these models we find that contributions to a Lagrangian which are *both* scale invariant and $SU_3\times SU_3$ invariant are automatically $U_3\times U_3$ invariant. This suggests that $F_0{}^5$ is a useful operator in the limit of scale invariance. It is therefore interesting to assume that $\bar{\theta}_{00}$ is $U_3\times U_3$ invariant. Then the current $\mathfrak{F}_{0\mu}{}^5$ has divergence

$$\partial^\mu\mathfrak{F}_{0\mu}{}^5=i[F_0{}^5,\delta]\,. \tag{1.9}$$

Since there is no light pseudoscalar with the quantum numbers of the η', we see that δ must break $F_0{}^5$ explicitly.[10] It should be stressed that there exist purely mesonic models in which $F_0{}^5$ does not occur.[11] Hence the assumption of $U_3\times U_3$ invariance in the scale-invariant limit is a hypothesis, not a theorem. In particular, the nonlinear models of chiral symmetry deserve further study.

We note that δ belongs to (1,1), so that if (1.9) were dominated by a meson state P, P would not belong to $(3,\bar{3})+(\bar{3},3)$ but would be dynamically independent. Similarly, we note that PCDC gives in the $SU_3\times SU_3$ limit

$$\partial^\mu\mathfrak{D}_\mu=\theta^\mu{}_\mu=(l_\delta+4)\delta \tag{1.10}$$

(the latter equality holding when δ has only one dimension), where $\theta^\mu{}_\mu$ is (1,1) and might be dominated by a meson S. One is thus led to consider the possibility that there be two extra meson-field operators S and P which are invariant under $SU_3\times SU_3$, in addition to the 18 belonging to $(3,\bar{3})+(\bar{3},3)$. In the 20-meson picture, there are three $I=0$, nonstrange 0^+ mesons associated with the scale-breaking operators δ, u_0, u_8, and three analogous 0^- mesons. Kastrup[12] and Chang and Freund[13] have independently suggested the existence of a "tenth scalar" meson.

[10] Essentially the same argument was given by S. L. Glashow; see *Hadrons and Their Interactions*, edited by A. Zichichi (Academic, New York, 1968), p. 102.

[11] S. Coleman and L. Maiani (private communication); C. J. Isham, A. Salam, and J. Strathdee, Phys. Rev. D **2**, 685 (1970); W. A. Bardeen (unpublished).

[12] H. A. Kastrup, Nucl. Phys. **B15**, 179 (1970).

[13] L. N. Chang and P. G. O. Freund, Ann. Phys. (N. Y.) (to be published).

In Secs. V and VI we construct models for 18- or 20-meson systems, and study the dynamics of the spontaneously broken solutions. The 20-meson solution is especially interesting in that it permits spontaneous breaking of scale invariance in a way that survives when the scale-breaking interaction δ is turned on.

In this paper we emphasize models in which $0^{\pm}$ mesons occur, because of the central role such mesons play in determining the structure of the vacuum. Moreover, certain of the 0^+ mesons determine the masses of the particles, and the massless pseudoscalars are manifestations of the underlying chiral symmetry.

It is interesting, though probably not meaningful, to compare the models with the experimental meson spectrum, although there is little firm information about scalar mesons. It is quite easy to fit the experimental situation with the 18-meson model when $SU_3\times SU_3$-breaking terms are added to the previous analysis. Quark-model considerations[14] suggest that one of the quartic couplings is absent (or small), in which case the SU_3 multiplet masses of the spontaneously broken $SU_3\times SU_3$ model are in the ratio $m_S^2(8):m_P^2(1):m_S^2(1):m_P^2(8)=4:3:1:0$. Breaking of symmetry leads to a reasonable spectrum, with a substantial admixture of the "forbidden" coupling. The 20-meson model is interesting but has too many parameters to give predictions.

II. APPROXIMATE SCALE INVARIANCE; MASS FORMULAS

The breaking of scale invariance is characterized by the nonvanishing of the trace $\theta^\mu{}_\mu$. In this section we consider how to formulate the idea of approximate scale invariance, in order that the implications of the presumed underlying symmetry may be derived. We suggest that the matrix elements of $\theta^\mu{}_\mu$ are sufficiently well behaved for large momentum transfer that one may write unsubtracted dispersion relations for the form factors $\theta(t)$. From general principles we know that $\theta(0)$ gives the mass [$\theta(0)=2m^2$ for spin 0, M for spin $\frac{1}{2}$ with the normalization adopted below]. One can guarantee this by writing

$$\theta(t)=\theta(0)+\frac{t}{\pi}\int\frac{\mathrm{Im}\theta(t')dt'}{t'(t'-t)}. \qquad (2.1)$$

If we further assume that θ approaches a limit as $t\to\infty$, we have the unsubtracted form

$$\theta(t)=\theta(\infty)+\frac{1}{\pi}\int\frac{\mathrm{Im}\theta(t')dt'}{t'-t}. \qquad (2.2)$$

It is tempting to further speculate that $\theta(\infty)$ vanishes, in which case one has a direct analogy with ordinary PCAC. Qualitatively one expects $\theta(\infty)$ to vanish because of the enormous number of competing channels open for large t. Of course, electromagnetic form factors vanish strongly at large momentum transfer. The vanishing or nonvanishing of $\theta(\infty)$ is an important, model-dependent question. In either case, it is of interest to dominate (2.2) with a set of scalar mesons. The case of spontaneous breakdown of scale invariance is of particular interest since a zero-mass scale meson, or "dilaton," is required by the vanishing of θ. The scale-breaking operators raise the dilaton mass from zero, and if $\theta(\infty)$ vanishes, the particle mesons are given by Goldberger-Treiman formulas dominated by the scale meson and perhaps some other scalar mesons.

We consider the matrix elements $\langle p'|\theta_{\mu\nu}|p\rangle$ for spin 0 and spin $\frac{1}{2}$. For spin-0 mesons,[15,16]

$$(4\omega\omega')^{1/2}\langle p'|\theta_{\mu\nu}|p\rangle =2P_\mu P_\nu F(t)+(k_\mu k_\nu-g_{\mu\nu}k^2)H(t), \qquad (2.3)$$
$$P=\tfrac{1}{2}(p+p'),\quad k=p'-p,\quad t=k^2.$$

If there are no zero-mass scalar particles coupling to $\theta_{\mu\nu}$, we have $H(0)=$finite and $F(0)=1$. Taking the trace gives

$$(4\omega\omega')^{1/2}\langle p'|\theta^\mu{}_\mu|p\rangle =(2m^2-\tfrac{1}{2}t)F(t)-3tH(t)\equiv\theta(t). \qquad (2.4)$$

For $H(t)$ finite, we see that $\langle p|\theta^\mu{}_\mu|p\rangle=m^2/\omega$ as expected; also (2.1) gives $\langle p|\theta_{00}|p\rangle=\omega$. But if $\theta^\mu{}_\mu=0$, $H(t)$ has a pole according to (2.4), *provided* that the external masses are nonzero. This pole corresponds to a massless 0^+ meson. Even when $\theta^\mu{}_\mu\neq0$, such mesons may exist (though massive) and dominate the dispersion relation for $\theta^\nu{}_\nu$.

To calculate the form factors we find the absorptive part $[M_{\mu\nu}]$ of the annihilation amplitude $M_{\mu\nu}$:

$$M_{\mu\nu}=(4\omega\bar{\omega}')^{1/2}\langle 0|\theta_{\mu\nu}|p\bar{p}'\text{ in}\rangle$$
$$[M_{\mu\nu}]=\tfrac{1}{2}(2\pi)^4\sum_n\delta(p_n-p-\bar{p}') \times\langle 0|\theta_{\mu\nu}|n\rangle\langle n|j_m{}^\dagger|m(p)\rangle(2\omega)^{1/2}. \qquad (2.5)$$

The states $|n\rangle$ have $J^P=0^+$, 2^+, corresponding to the trace and traceless parts of $\theta_{\mu\nu}$. We may have several (or many) mesons of both types. Denoting a 0^+ meson by σ, we define the strength F_σ for creating σ from the vacuum by

$$(2\omega_\sigma)^{1/2}\langle 0|\theta_{\mu\nu}|\sigma\rangle=-\tfrac{1}{3}F_\sigma(k_\mu k_\nu-g_{\mu\nu}k^2),$$
$$(2\omega_\sigma)^{1/2}\langle 0|\theta^\mu{}_\mu|\sigma\rangle=m_\sigma{}^2F_\sigma,$$

where $k^2=m_\sigma{}^2$, and the $mm\sigma$ coupling $G_{\sigma mm}$ by

$$(4\omega_\sigma\omega_m)^{1/2}\langle\sigma|j_m{}^\dagger|m(p)\rangle=G_{\sigma mm}, \qquad (2.6)$$

where j_m is the meson current. ($G_{\sigma mm}$ has dimension -1.) We could treat the 2^+ contribution similarly (and

[14] J. Rosner (private communication). The coupling $(\mathrm{Tr}\mathfrak{M}^\dagger\mathfrak{M})^2$ involves graphs in which quark spin indices are not irreducible linked.

[15] We use a metric tensor $g_{\mu\nu}$ with elements $g_{00}=1$, $g_{ii}=-1$, $i=1, 2, 3$.

[16] H. Pagels, Phys. Rev. **144**, 1250 (1966).

obtain universal coupling of a single f meson when unsubtracted dispersion relations are assumed for F and its analog for other particles). Here we are only concerned with the trace, whose form factor $\theta(t)$ is assumed to obey an unsubtracted dispersion relation, giving

$$\theta(t)=\sum_\sigma \frac{m_\sigma^2 F_\sigma G_{\sigma mm}}{m_\sigma^2-t}+\text{background}, \tag{2.7}$$

where the sum is over the σ mesons and the background could be small. Using $\theta(0)=2m^2$ and ignoring the background, we get the Goldberger-Treiman relations

$$2m^2=\sum_\sigma F_\sigma G_{\sigma mm}. \tag{2.8}$$

In general, we will be able to express particle masses in terms of the scalar meson couplings and the (universal) constants F_σ. However, the problem of determining these quantities, and even determining the number of the scalars, is not trivial.

For the baryons (spin $\frac{1}{2}$), one has[15] [P and k are defined as in Eq. (2.3)]

$$\begin{aligned}&(EE'/M^2)^{1/2}\langle p'|\theta_{\mu\nu}|p\rangle=\bar{u}(p',\lambda')G_{\mu\nu}u(p,\lambda)/M,\\&G_{\mu\nu}=\tfrac{1}{2}G_1(t)(P_\mu\gamma_\nu+P_\nu\gamma_\mu)+G_2(t)P_\mu P_\nu/M\\&\qquad+G_3(t)(k_\mu k_\nu-g_{\mu\nu}k^2)/M,\\&G^\mu{}_\mu=MG_1(t)+(P^2/M)G_2(t)-(3t/M)G_3(t),\end{aligned} \tag{2.9}$$

with[5] $G_1(0)=M$ when there is no zero mass σ. G_3 would have a pole in the limit $\theta^\mu{}_\mu=0$, if the external masses remained nonzero. Only G_3 contains the σ coupling, as shown by computing the discontinuity $[J_{\mu\nu}{}^\sigma]$ for $N\bar{N}\rightarrow$ graviton due to a σ intermediate state

$$\begin{aligned}&J_{\mu\nu}=(E\bar{E}'/M^2)^{1/2}\langle 0|\theta_{\mu\nu}|N(p,\lambda)\bar{N}(p',\lambda')\rangle,\\&[J_{\mu\nu}{}^\sigma]=-\tfrac{1}{3}\pi\delta(t-m_\sigma^2)F_\sigma g_{\sigma NN}(k_\mu k_\nu-g_{\mu\nu}k^2)\\&\qquad\times\bar{v}(\bar{p}',\lambda')u(p,\lambda).\end{aligned} \tag{2.10}$$

Here λ and λ' are helicities, and u and v the usual spinors. $g_{\sigma NN}$ is defined by

$$\begin{aligned}&g_{\sigma NN}\bar{v}(\bar{p}',\lambda')u(p,\lambda)\\&\qquad=(2E\omega_\sigma/M)^{1/2}\bar{v}(\bar{p}',\lambda')\langle\sigma|j_N{}^\dagger|p\lambda\rangle.\end{aligned} \tag{2.11}$$

From (2.9)–(2.11) we find [$\theta_N(t)=G^\mu{}_\mu/M$]

$$\theta_N(t)=\sum_\sigma\frac{m_\sigma^2F_\sigma g_{\sigma NN}}{m_\sigma^2-t}+\text{background}. \tag{2.12}$$

Making the strong assumption that $\theta(\infty)=0$ yields

$$\theta_N(0)=M=\sum_\sigma F_\sigma g_{\sigma NN}. \tag{2.13}$$

The results (2.8) and (2.13) show the intimate connection between the mass of a particle and its couplings to the scalar mesons. Note that the mesons couple by m^2, while the baryons couple linearly.

We now consider the implications of the mass formulas (2.8) and (2.13), remembering, however, that even when $\theta(\infty)$ vanishes, scalar mesons might not dominate if scale breaking is not "gentle." Our first problem is to decide how many scalar mesons occur. Although direct evidence for 0^+ mesons is scanty and controversial, such a meson is needed for consistency of πN and NN scattering, and there is also some evidence for a broad 0^+ meson around 700 MeV. If we define the $\sigma\pi\pi$ coupling so that $\mathcal{L}_{\sigma\pi\pi}=m_\pi g_{\sigma\pi\pi}\sigma\pi^2$, so that $G_{\sigma\pi\pi}=2m_\pi g_{\sigma\pi\pi}$, Eqs. (2.8) and (2.13) give $g_{\sigma\pi\pi}/g_{\sigma NN}=m_\pi/m_N$, while the evidence suggests $g_{\sigma\pi\pi}{}^2/4\pi\approx g_{\sigma NN}{}^2/4\pi\approx 10$. This difficulty does not occur if at least one of the form factors requires a subtraction.

Even if further scalar mesons are included in Eqs. (2.8) and (2.13), very stringent constraints follow from imposing qualitative requirements such as the behavior of masses in the limit of $SU_3\times SU_3$ symmetry. For example, in this limit the pseudoscalar octet becomes massless whereas the baryon octet presumably remains massive. Further, the pseudoscalar octet mass depends on the chiral-symmetry-breaking energy density λu as $m^2(\lambda)\propto\lambda$, as elaborated in Sec. III.

The foregoing results may be used to show that at least two SU_3 singlet 0^+ mesons are needed to satisfy the Goldberger-Treiman formulas (2.8) and (2.13) consistent with the dependence $m^2(\lambda)\propto\lambda$ for the pseudoscalar octet and $M\propto\lambda^0$ for the baryon octet. We note that the scalar couplings to pseudoscalars or nucleons do not have to vanish as $\lambda\rightarrow 0$ and so may be taken to be $O(\lambda^0)$. (Since $G_{\sigma MM}$ is dimensional, its nonvanishing assumes the existence of a scale-breaking operator such as δ which does not go away when $\lambda\rightarrow 0$.) This means that neither of these couplings exist solely due to breaking of $SU_3\times SU_3$.

Consider the case of a single scalar meson σ. Applying (2.8) to the pseudoscalar octet then implies that $F_\sigma=O(\lambda)$. Thus $\langle 0|\delta|\sigma\rangle$ vanishes to lowest order in λ, and hence the nucleon mass vanishes in the $SU_3\times SU_3$ limit. This difficulty is not circumvented by having two mesons σ_0 and σ_8 belonging to a scalar nonet. In the presence of SU_3 breaking these states mix, giving mesons σ_1 and σ_2, say. Defining $F_{\sigma_i}{}^\delta=(2\omega)^{1/2}\langle 0|\delta|\sigma_i\rangle$ and $F_{\sigma_i}{}^u=(2\omega)^{1/2}\langle 0|u|\sigma_i\rangle$ and $m_i^2=xm_i^2(\delta)+ym_i^2(u)$, and generalizing to any number of scalar mesons σ_i, we find the relations

$$\begin{aligned}2m_i^2(\delta)&=\sum_j F_j{}^\delta G_{jii},\\2m_i^2(u)&=\sum_j F_j{}^u G_{jii},\\M_i(\delta)&=\sum_j F_j{}^\delta g_{jii},\\M_i(u)&=\sum_j F_j{}^u g_{jii},\end{aligned} \tag{2.14}$$

for the pseudoscalar octet and the baryon octet. Here we have assumed that δ and u *separately* obey unsubtracted dispersion relations. (This is tantamount to assuming that we can vary the coupling parameters ν and λ independently.)

Transforming (2.14) to the unmixed basis, we apply it to the case of two mesons σ_0 and σ_8. To lowest order in λ, we have $F^\delta(\sigma_0)=O(\lambda^0)$ while $F^\delta(\sigma_8)$, $F^u(\sigma_0)$, and $F^u(\sigma_8)$ are $O(\lambda)$. To lowest order in λ, $G_{\sigma_0 ii}$ and $G_{\sigma_8 ii}$ are nonvanishing $O(\lambda^0)$ and SU_3 symmetric. Thus the requirement that $m_i^2(\delta)$ be $O(\lambda)$ forces $F^\delta(\sigma_0)$ to vanish to leading order (λ^0). Again the nucleon mass is $O(\lambda)$ and vanishes as $\lambda\to 0$.

In order to allow $m_P^2\propto\lambda$ in the presence of finite nucleon mass $[O(\lambda^0)]$, we introduce another SU_3-singlet scalar meson, called S, which in general mixes with σ_0 and σ_8. Since $F^\delta(S)=O(\lambda^0)$, we can arrange $m_P^2\propto\lambda$ by arranging cancellation to lowest order:

$$F^\delta(S)G_{SPP}+F^\delta(\sigma_0)G_{\sigma_0 PP}=0\,. \tag{2.15}$$

The nucleon mass now becomes (in the $SU_3\times SU_3$ limit)

$$M=F^\delta(S)(G_{\sigma_0 PP}g_{SNN}-g_{\sigma_0 NN}G_{SPP})/G_{\sigma_0 PP}\neq 0\,. \tag{2.16}$$

The essential aspect of the foregoing is that there exist at least two SU_3 singlet 0^+ mesons coupled to the vacuum (by δ) and to pseudoscalars $[G=O(\lambda^0)]$ in such a manner that a cancellation such as (2.15) occurs.

If we assume that there is a scalar nonet, a tenth 0^+ meson is thus required by the above argument.[17] In Sec. VI we study a model of this sort.

There exist reasonable models in which the baryon mass remains finite in the $SU_3\times SU_3$ limit, even though only one σ_0 exists. In such cases either a subtraction is called for, or the dynamical approximations are completely misleading.

III. PARAMETER DEPENDENCE OF MASSES

The "virial theorem," Eq. (1.5), allows one to study the dependence of masses on the dimension and strength of the scale-breaking operators w_i of dimension l_i. We introduce a set of parameters λ_i $(0\leq\lambda_i\leq 1)$ such that

$$\begin{aligned}\theta_{00}&=\bar\theta_{00}+\sum_i \lambda_i w_i\,,\\ \theta^\mu{}_\mu&=\sum_i (l_i+4)\lambda_i w_i\,.\end{aligned} \tag{3.1}$$

The physical situation is represented by setting every λ_i equal to unity. However, it is useful to consider the entire family of systems which result when the λ_i are varied. The solutions corresponding to (3.1) will, in general, break up into sets not smoothly connected to each other, according to the values of the λ_i. Moreover, it is not *a priori* clear whether the various λ_i can all be varied independently. (It is even possible that an adiabatic change in the energy is prohibited, although we shall not consider this to be the case.)

Within each domain we may investigate the parameter dependence of the masses. The effective Lagrangian models of Secs. IV–VI provide instructive examples. According to a well-known result of Feynman, we know that the change in energy $\partial E/\partial\lambda_i$ depends only on the change in the Hamiltonian and not on the change in state vectors. This result is a simple consequence of the variational principle. Thus we have for a single-particle state $|p(\lambda)\rangle$,

$$\partial E/\partial\lambda_i=\langle p(\lambda)|w_i|p(\lambda)\rangle\,. \tag{3.2}$$

Multiplying by $2E(\lambda)$ and writing $E^2(\lambda)=p^2+m^2(\lambda)$ gives

$$\partial m^2/\partial\lambda_i=(p(\lambda)|w_i|p(\lambda))\,, \tag{3.3}$$

where $|p(\lambda))=(2E)^{1/2}|p(\lambda)\rangle$ correspond to covariantly normalized states. The right-hand side is indeed independent of p since w_i is a scalar operator, so that we may take $p\to 0$ even for values of λ_i such that $m^2\to 0$, a step not permitted in (3.2).

Apart from the exceptional λ_i giving $m=0$ (which we shall regard as the limit point), the trace theorem may be employed to give

$$2m^2(\lambda)=\sum_i (l_i+4)\lambda_i(\partial m^2/\partial\lambda_i)\,, \tag{3.4}$$

or, writing $x_i=l_i+4$,

$$m(\lambda)=\sum_i x_i\lambda_i(\partial m/\partial\lambda_i)\,. \tag{3.5}$$

One should remember that the dimensions l_i in general depend on the parameters λ_i.

As a first example suppose there is only one scale-breaking interaction δ with parameter ν. Also consider the case in which the mass of the particle under consideration vanishes as $\nu\to 0$. The perturbation expansion has the form

$$\begin{aligned}E&=p+\nu E_1+\nu^2E_2+\cdots\\ &=p+\frac{\nu}{2p}\frac{\partial m^2}{\partial\nu}\bigg|_0\\ &\qquad+\frac{\nu^2}{4p}\left[\frac{\partial^2m^2}{\partial\nu^2}\bigg|_0-\frac{1}{2p^2}\left(\frac{\partial m^2}{\partial\nu}\right)^2\bigg|_0\right]+\cdots\,. \end{aligned}\tag{3.6}$$

Since this formula must agree with $E=p+m^2/2p+\cdots$, we see that a first-order shift $E_1\neq 0$ corresponds to a quadratic mass formula $m^2\propto\nu$ (provided higher-order corrections are small), while a second-order energy shift $(E_2\neq 0, E_1=0)$ corresponds to a linear mass formula $(m\propto\nu)$.

If the mass shift occurs in first order (vanishing as $\nu\to 0$), then consistency with the virial theorem, Eq. (1.5), requires that $l=-2$. Since $\nu\langle\mathbf{p}|\delta|\mathbf{p}\rangle=m^2/Ex$

[17] After this work was completed, R. Crewther and M. Gell-Mann noted that the PCAC relation $F_\pi\, G_{\sigma\pi\pi}=(m_\pi^2-m_\sigma^2)F_1(0)$ [F_1 is the form factor multiplying $(p+p')_\mu$ in the matrix element $\langle\pi|\mathfrak{F}_\mu{}^5|\sigma\rangle$] indicates that $G_{\sigma\pi\pi}$ remains large in the $SU_3\times SU_3$ limit. A dispersion-theory study of F_1 indicates that F_1 remains finite in this limit. Thus it is entirely possible that for the pseudoscalar octet, scale breaking is not adequately expressed by formula (2.8). Thus it is possible to avoid an extra scalar meson.

($x=l+4$) for the exact states, we find in lowest order

$$E=\langle\mathbf{p}|\bar{\theta}_{00}+\nu\delta|\mathbf{p}\rangle$$
$$=p+m^2/px, \qquad (3.7)$$

showing that $l=-2$ under the indicated assumptions.[3] The boson mass term $\frac{1}{2}m^2\phi^2$ illustrates this situation. For a fermion mass operator the mass shift is second order, so that $l=-3$ does not conflict with the foregoing result.

As an example, suppose that $\delta=0$ and that the baryon octet masses vanished as $u\to 0$ (i.e., $\delta=0$) so that they obeyed a quadratic mass formula. In this case $l=-2$, and one can easily estimate $\langle B|u_0|B\rangle$ to be roughly $\frac{1}{2}m_B$, or about 600 MeV, since the average of u_8 over the octet vanishes. Although the approximation is drastic, the result is not sufficiently different from that of Ref. 9 to totally exclude the vanishing of δ.

Another interesting case is the cubic boson interaction $g\phi^3$, which has $l=-3$, and gives no first-order shift. However, δm^2 in second order is negative, which indicates an instability. The system can be stabilized by going to a new ground state if the scale-invariant theory contains a term like $f\phi^4$ (with $f<0$). The new ground state has $\langle\phi\rangle_0\neq 0$ and $m^2>0$; as explained in Sec. IV, the dimensions of operators appropriate to the new ground state are changed from those in the original theory. The new ground state is only approximate, since all interactions have not been taken into account. But if the "new" interactions are regarded as small, the expectation value $\langle p|\delta'|p\rangle$ between physical states merely picks out the mass term $\frac{1}{2}m^2\phi'^2$. We note that the self-coupled massless scalar field gives rise to spontaneous breakdown, and that an ordinary mass term occurs in the physical fields $\phi'=\phi-\langle\phi\rangle_0$.

The result (3.7) is not applicable to the pseudoscalar octet since other scale-breaking operators remain in the massless limit ($\lambda u\to 0$). As an example, consider the model studied by Gell-Mann, Oakes, and Renner[8] in which $\theta_{00}=\hat{\theta}_{00}+\lambda u$, where $\hat{\theta}_{00}$ is $SU_3\times SU_3$ invariant and $u=u_0+cu_8$, where the u_i are in the representation $(3,\bar{3})+(\bar{3},3)$. As $\lambda\to 0$, we have conservation of the octet axial-vector currents and an octet of massless pseudoscalars, which obey $m^2(\lambda)\propto\lambda$, whatever their dependence on other parameters. Given the quadratic mass formula, we find

$$\langle p|u|p\rangle_{\text{rest}}=\left.\frac{\partial m}{\partial\lambda}\right|_{\lambda=1}=\tfrac{1}{2}m, \qquad (3.8)$$

giving better foundation to the result $(p|u|p)=m,^2$ previously based[8] on extrapolating a single pion to zero four-momentum. [If the pseudoscalar mesons were to obey a *linear* formula, the extrapolated matrix element $(p|u|p')|_{p'=0}$ would lose half the actual value of $(p|u|p)$.] If $u=u_0+cu_8$ were a quark mass term, one would have $l_u=-3$, so that according to the virial theorem, $\langle u\rangle$ and $\langle\delta\rangle$ each account for half the pseudoscalar mass, and $l_\delta=-3$. On the other hand, if l_u were -2 (and the mass formula quadratic), u would give the entire mass, and $\langle\delta\rangle=0$. This example illustrates how one may use Eq. (3.5) in a semiempirical way to correlate the dimension of operators and the structure of the underlying theory.

To amplify the foregoing, we consider the scheme proposed in the Introduction in which the $SU_3\times SU_3$ energy density $\hat{\theta}_{00}$ is broken up into $\bar{\theta}_{00}+\delta$, where $\bar{\theta}_{00}$ has $l=-4$, and δ breaks scale. δ is assumed to be $SU_3\times SU_3$ invariant and $\bar{\theta}_{00}$ may be $U_3\times U_3$ invariant. We suppose δ to have a single dimension l_δ (though this would not be true if we have both mass terms and meson interaction terms), and write

$$\theta_{00}=\bar{\theta}_{00}+\nu\delta+\lambda u,$$
$$\theta^\mu{}_\mu=x\nu\delta+y\lambda u, \qquad (3.9)$$

where $x=l_\delta+4$, $y=l_u+4$. The solution of Eq. (3.5) is

$$m(\nu,\lambda)=c_3\nu^{1/2x}\lambda^{1/2y}g(\lambda^{1/2y}/\nu^{1/2x}) \qquad (3.10)$$

or (in a useful equivalent form)

$$m(\nu,\lambda)=c_1\nu^{1/x}+c_2\lambda^{1/y}+c_3\nu^{1/2x}\lambda^{1/2y}f(\lambda^{1/2y}/\nu^{1/2x}), \qquad (3.11)$$

provided the dimensions are independent of ν and λ. Here we assumed that all masses go to zero in the scale-invariant limit and that the theory is analytic in $0\leq(\lambda,\nu)\leq 1$. $g(z)$ is analytic except for a possible c_1/z term.

We may use Eq. (3.11) to understand how the presence of two interactions δ and u avoids the $l=-2$ restriction [cf. Eq. (3.7)] which would occur if δ were absent. Setting $c_1=0$ for the pseudoscalar octet, we see that there are essentially two cases: (A) ($c_3=0$, $c_2\neq 0$) and (B) ($c_2=0$, $c_3\neq 0$). In case (A) the mass is independent of ν, so that as expected, $m^2\propto\lambda$ implies $l_u=-2$, and $m\propto\lambda$ requires $l_u=-3$, exactly as for the case of a single scale-breaking interaction. But for case (B), in which the (ν,λ) dependence is closely connected, we find $m^2\propto\lambda$ is possible for $l_u=-3$ (f=constant; $m^2\propto\lambda\nu^{1/x}$) as well as for $l_u=-2$ [$f(z)=z$; $m^2\propto\lambda$].

IV. AMBIGUOUS DIMENSIONS IN LAGRANGIAN MODELS

When the dynamics is such that one or more fields in a theory has nonvanishing expectation value, the canonical dimension of an operator depends somewhat on one's point of view. To illustrate this interesting phenomenon, consider the scalar meson model defined by the Lagrangian density

$$\mathcal{L}=\tfrac{1}{2}(\partial\phi)^2+f\phi^4+f_0\phi. \qquad (4.1)$$

The "improved" $\theta_{\mu\nu}$ of Ref. 4 is

$$\theta_{\mu\nu}=\partial_\mu\phi\partial_\nu\phi-g_{\mu\nu}\mathcal{L}-\tfrac{1}{6}(\partial_\mu\partial_\nu-g_{\mu\nu}\partial^2)\phi^2, \qquad (4.2)$$

and ϕ has canonical dimension -1 when the definitions (1.1) and (1.4) are employed. Now the field ϕ has no preferred status; the presence of the linear term makes it likely that the physical field is $\phi'=\phi-\phi_0$, where $\phi_0=\langle\phi\rangle_0\neq 0$. Writing (4.1) in terms of ϕ', we find

$$\mathcal{L}'=\tfrac{1}{2}(\partial\phi')^2-\tfrac{1}{2}m^2\phi'^2+g\phi'^3+f\phi'^4, \quad (4.3)$$

where the extremal condition is $f_0+4f\phi_0{}^3=0$ and $m^2=-12f\phi_0{}^2$. For $f<0$, one has a minimum (and stability). To the extent that the new interactions are "small," the field ϕ' "has" mass m.

The Lagrangian (4.3) is just as good (perhaps better) than (4.1), and we are entitled to define $\theta_{\mu\nu}'$ and a corresponding dilatation D' under which ϕ' has dimension -1. One may argue that the latter choice is better if the self-consistent determination of the mass really gives a good approximate solution. Under D we find $i[D,\phi']=(\phi'+\phi_0)$, $i[D',\phi]=(\phi-\phi_0)$, where $x=0$. Polynomial operators of degree n in ϕ break up into a sum of terms of dimensions $-l=0,+1,\ldots,n$.

The model (4.1) gives an operator statement of PCDC:

$$\partial^\mu\mathfrak{D}_\mu=\theta^\mu{}_\mu=-3f_0\phi=Fm^2\phi, \quad (4.4)$$

the latter equality following from the relations after (4.3). F is equal to $-\phi_0$. Hence Eq. (2.8) follows exactly in this model, when one replaces ϕ by ϕ' in (4.4), as is appropriate for the physical ground state in which $\langle\phi'\rangle_0=0$.

A less unrealistic model is defined by the Lagrangian

$$\mathcal{L}=\tfrac{1}{2}(\partial\phi)^2-\tfrac{1}{2}m_0{}^2\phi^2+g\phi^3+f\phi^4. \quad (4.5)$$

$\theta_{\mu\nu}$ is again given by Eq. (4.2); using the equation of motion, one finds

$$\theta^\mu{}_\mu=m_0{}^2\phi^2-g\phi^3, \quad (4.6)$$

giving a simple illustration of the virial theorem, Eq. (1.5) ($\sum w=\tfrac{1}{2}m_0{}^2\phi^2-g\phi^3$).

We now consider the modifications which occur in the case of spontaneous breakdown. For certain values of the parameters there may exist other ground states in which $\langle\phi\rangle_0\neq 0$. These correspond to minima ϕ_0 in the (classical) function $\tfrac{1}{2}m_0{}^2\phi^2+g\phi^3+f\phi^4$ for $\phi_0\neq 0$. To illustrate, we set $m_0{}^2=0$, which assures the existence of such a solution, provided f is negative. To find the minima, we seek ϕ_0 such that variations $\phi'=\phi-\phi_0$ leave $\mathcal{L}$ unchanged to first order. Eliminating the linear term in ϕ' requires that

$$3g+4f\phi_0=0 \quad (4.7)$$

and gives the Lagrangian

$$\mathcal{L}'=\tfrac{1}{2}(\partial\phi')^2-\tfrac{1}{2}m^2\phi'^2+g'\phi'^3+f\phi'^4, \quad m^2=3g\phi_0, \quad g'=g+4f\phi_0. \quad (4.8)$$

To the extent that the interactions among the new fields are small, we have self-consistently induced a mass m^2. The modified energy-momentum tensor $\theta'_{\mu\nu}$ has trace

$$\theta'^\mu{}_\mu=m^2\phi'^2-g'\phi'^3, \quad (4.9)$$

while the scale-breaking term δ' is

$$\delta'=\tfrac{1}{2}m^2\phi'^2-g'\phi'^3. \quad (4.10)$$

A number of explanatory remarks are in order. First we note that the establishment of equilibrium at $\phi_0\neq 0$ involves a balance between the scale-breaking and scale-preserving ($l=-4$) parts of $\mathcal{L}$. Each of these becomes a polynomial in the new fields ϕ'; e.g., $f\phi'^4$ has terms from ϕ' up to ϕ'^4. The ϕ' term cancels against the corresponding part of $g\phi^3$; the mass term ϕ'^2 receives contributions from both cubic and quartic interactions, while the cubic interaction strength is modified. The new fields ϕ' have dimension -1 with respect to the dilation constructed from the *new* energy-momentum tensor $\theta_{\mu\nu}'$. The operator ϕ^4, which had dimension -4 for the "normal" theory, breaks up into a sum of terms of dimension -4 to 0 in the displaced theory. This behavior is characteristic of spontaneous breakdown and will be investigated later for less unrealistic models.

The tensor $\theta'_{\mu\nu}$ differs from $\theta_{\mu\nu}$ only in the second-derivative term.

To investigate PCDC, we expand $\theta^\mu{}_\mu$ [Eq. (4.6) with $m_0=0$] in terms of ϕ' and normal order with regard to this field. We get $\partial^\mu\mathfrak{D}_\mu=-3g\phi_0{}^2\phi'+\cdots$, so that $m^2F=(0|\theta|\phi')=-3g\phi_0{}^2$. Thus $F=-\phi_0$ and $(p|\theta|p)=-3g\phi_0{}^2g'/m^2=Fg'$, as expected.

In the preceding discussion the self-consistent determination of a stable ground state played an essential role. It might seem that the starting point is a matter of luck. However, one usually has some idea about the symmetry of the energy density or Lagrangian (even if the solutions lack that symmetry, as in the case of spontaneous breakdown). Therefore, one constructs invariant densities $\mathcal{L}$ (or $\theta_{\mu\nu}$) from fields transforming as irreducible representations of the symmetry group in the usual way. The construction of a solution to a given dynamical problem will, in general, proceed through three stages. At each stage the appropriate dimension of a given operator will be different. In the first stage, one formulates the symmetrical theory. The basic dynamical quantity, say $\mathcal{L}$ or $\theta_{\mu\nu}$, is built from operators transforming irreducibly under the suspected symmetry group. The bulk of the theory is invariant under the group. In addition, a small set of symmetry-breaking operators will transform according to various "rows" of simple representations of the group. The next stages necessarily involve dynamical approximations. When spontaneous breakdown occurs, it is appropriate to reformulate the symmetrical theory about an approximate self-consistent ground state as in effective Lagrangian models. The approximate normal modes of this intermediate "reference" theory are usually the most useful coordinates in terms of which to approximate the true solution.

V. 18-MESON MODEL

The model of Eq. (4.5) is now generalized to accommodate $SU_3\times SU_3$ transformations. The 18 mesons in $(3,\bar{3})+(\bar{3},3)$ are described by the 3×3 matrix $\mathfrak{M}=\sigma+i\phi$ where σ and ϕ are, respectively, the usual scalar and pseudoscalar nonet matrices. The effective Lagrangian is

$$\mathcal{L}=\tfrac{1}{2}\,\mathrm{Tr}\partial_\mu\mathfrak{M}^\dagger\partial^\mu\mathfrak{M}+f_1(\mathrm{Tr}\mathfrak{M}^\dagger\mathfrak{M})^2+f_2\,\mathrm{Tr}\mathfrak{M}^\dagger\mathfrak{M}\mathfrak{M}^\dagger\mathfrak{M}+g_1(\det\mathfrak{M}+\mathrm{H.c.})\,. \quad (5.1)$$

The chiral properties of this model have been studied by Lévy[18] and by Bardeen and Lee.[19] We are interested in the interplay of scale and chiral properties in the dynamics of spontaneous breakdown.[20] It is easy to add mass terms, if desired.

The scale-breaking term δ is

$$\delta=-g_1(\det\mathfrak{M}+\mathrm{H.c.})\,. \quad (5.2)$$

The meson fields σ_i, ϕ_i, which transform like the quark operators u_i, v_i, under $SU_3\times SU_3$, also transform as the latter under the operator $F_0{}^5$. Then we can compute $i[F_0{}^5,\delta]$, which generates a new operator δ_5 which closes as follows:

$$\begin{aligned} i[F_0{}^5,\delta]&=3\delta_5\,,\\ i[F_0{}^5,\delta_5]&=-3\delta\,. \end{aligned} \quad (5.3)$$

Thus δ gives nonconservation of $\mathfrak{F}_{0\mu}{}^5$, as required by previous considerations, and induces changes $\Delta F_0{}^5=\pm3$. However the remainder of $\mathcal{L}$, which corresponds to $\bar{\theta}_{00}$, conserves $F_0{}^5$ and is $U_3\times U_3$ symmetric. The symmetries are

$$\begin{aligned} \bar{\theta}_{00}&:\quad U_3\times U_3,\quad \text{scale},\\ \delta&:\quad SU_3\times SU_3\,. \end{aligned} \quad (5.4)$$

We study "spontaneous-breakdown" solutions for which $\langle\sigma_0\rangle\neq0$ but insist that the vacuum be SU_3 symmetric, i.e., $\langle\sigma_8\rangle=0$. It is possible to relax the latter, but for clarity we choose this alternative. (The near equality $F_\pi\cong F_K$ indicates[8] that $\langle\sigma_8\rangle$ is small compared with $\langle\sigma_0\rangle$.) (To reach the real world, one finally has to introduce "explicit" breaking to raise the pseudoscalar masses from zero.) Following the notation of Gasiorowicz and Geffen,[19] we write $\sigma_0=\sigma_0'+\sqrt{3}a$, where σ_0' is the physical field. The equilibrium condition requires that the new Lagrangian have no linear term and that all the (mass)^2 terms be positive. (Without the quartic terms, one m^2 is always negative because of the $SU_3\times SU_3$ structure of the cubic term.) Then the equilibrium condition is

$$a[2a(3f_1+f_2)+g_1]=0 \quad (5.5)$$

and the mass terms are

$$\begin{aligned} m_P{}^2(8)&=-12f_1a^2-4f_2a^2-2ag_1=0\,,\\ m_P{}^2(1)&=-12f_1a^2-4f_2a^2+4ag_1=6ag_1\,,\\ m_S{}^2(8)&=-12f_1a^2-12f_2a^2+2ag_1=4ag_1-8f_2a^2\,,\\ m_S{}^2(1)&=-36f_1a^2-12f_2a^2-4ag_1=2ag_1\,, \end{aligned} \quad (5.6)$$

where the right-hand side follows on using Eq. (5.5). The condition $g_1a>0$ implies that $3f_1+f_2<0$ for $a\neq0$. If the residual interactions among the displaced fields are small, the above quantities should be good zeroth-order approximations to the masses. Even if the interactions are large and the masses distorted from (5.6), it may be essential to formulate the dynamical problem about the asymmetric ground state.

Next consider the following question: Is spontaneous breakdown of scale invariance possible if $g_1=0$, i.e., $\delta=0$? Equation (5.5) shows this to be possible, *provided* the special condition $3f_1+f_2=0$ is satisfied. In this case the masses are given by [set $g_1=0$ after the first equality in Eqs. (5.6)]

$$\begin{aligned} m_P{}^2(1)&=m_P{}^2(8)=m_S{}^2(1)=0\,,\\ m_S{}^2(8)&=-8f_2a^2\,, \end{aligned} \quad (5.7)$$

indicating that $f_2<0$. Although $a\neq0$ and $m_S(8)\neq0$, the *value* of a is not fixed, and there is nothing to fix it in a scale-invariant theory. The theories with different a simply differ by a scale transformation.

However, if one now turns on scale breaking ($g_1\neq0$) keeping $3f_1+f_2=0$ as demanded by the above, Eq. (5.5) shows that the only solution is the *normal* one with $a=0$. Thus in this model it does not seem appropriate to arrange the couplings so that spontaneous breakdown of scale invariance (and concomitant breakdown of $F_0{}^5$ and $SU_3\times SU_3$) occurs. Instead, the most reasonable solution (which occurs *unless* $3f_1+f_2$ vanishes) is normal in the scale-invariant limit; when δ is turned on, there is spontaneous breakdown of $SU_3\times SU_3$, while scale and $F_0{}^5$ are broken explicitly.

It is also of interest to consider the perturbation of the foregoing solution by an *explicit* $SU_3\times SU_3$ breaking term $f_0'\sigma_0$ in the Lagrangian. In this case, (5.5) is modified to

$$a'[f_0+4a'^2(3f_1+f_2)+2a'g_1]=0\,, \quad (5.8)$$

where $f_0=f_0'/\sqrt{3}a'$. The new $\langle\sigma_0\rangle$ ($\sqrt{3}a'$) is related to the spontaneously determined one by the equation

$$f_0'+2\sqrt{3}g_1a(a'-a'^2/a)=0\,, \quad (5.9)$$

whose perturbation solution is

$$a'=a(1+f_0/2g_1a+\cdots)\,, \quad (5.10)$$

giving $\delta a=f_0/2g_1$ in first order. The pseudoscalar masses (5.6) increase by f_0, while the scalar masses increase by $3f_0$. This calculation is included to illustrate the nature of the change when an explicit symmetry breaking is included in the spontaneously broken theory. SU_3 breaking is induced by adding $f_8\sigma_8$ in lowest-order perturbation theory.

It is simple and interesting to exhibit the consequences of the 18-meson model. Although this model is surprisingly good in its qualitative structure, other possibilities (such as the 20-meson model) may be forced by closer study.

[18] M. Lévy, Nuovo Cimento 52A, 23 (1967).
[19] W. A. Bardeen and B. W. Lee, Phys. Rev. 177, 2389 (1969); S. Gasiorowicz and D. Geffen, Rev. Mod. Phys. 41, 531 (1969).
[20] S. P. de Alwis and P. J. O'Donnell, Phys. Rev. D 2, 1023 (1970), have considered a different model in which scale breaking is connected to spontaneous breakdown.

The mass formulas (5.6) were based on the energy density having $SU_3 \times SU_3$ symmetry and so are not in proper form for comparison with experiment. In particular, we have eight massless pseudoscalar Goldstone bosons. As a first step, we break $SU_3 \times SU_3$ explicitly by introducing a term $f_0' \sigma_0$ into the Lagrangian as in the discussion following Eqs. (5.8)–(5.10). Secondly, we break SU_3 by introducing a breaking term transforming as the hypercharge, treated as a first-order perturbation.

Without SU_3 breaking, the masses of the SU_3 multiplets contained in the representation $(3,\bar{3})+(\bar{3},3)$ have the values

$$\begin{aligned} m_P^2(8) &= y, & m_P^2(1) &= 3x+y, \\ m_S^2(1) &= x+3y, & m_S^2(8) &= 4x+3y+z, \end{aligned} \tag{5.11}$$

where $x=2ag_1$, $y=f_0=f_0'/\sqrt{3}a$, and $z=24f_1a^2$. This decomposition is motivated by quark-model considerations, which suggest that $f_1=0$ since the corresponding coupling involves disconnected quark graphs.[14] When $y=z=0$, the spontaneously broken $(\text{mass})^2$ ratios are $m_S^2(8):m_P^2(1):m_S^2(1):m_P^2(8)=4:3:1:0$. From Eqs. (5.11) we find the mass formulas

$$\begin{aligned} 3m_S^2(1) &= m_P^2(1)+8m_P^2(8), \\ m_S^2(8)+m_P^2(8) &= m_S^2(1)+m_P^2(1)+z. \end{aligned} \tag{5.12}$$

Assuming unmixed pseudoscalars $m_P^2(1)=0.917(\eta')$ and $m_P^2(8)=0.162$ GeV^2, we find $m_S^2(1)=0.739$ GeV^2, i.e., $m_S(1)=860$ MeV.

The experimental situation with regard to the scalar nonet is very confused.[21] There is probably a scalar σ near 700 MeV and another (σ') at 1060 MeV. We choose $\delta(960)$ to be the $I=1$ member of the nonet, although $\pi_N(1016)$ could qualify. The κ is especially uncertain, although bumps are seen in $K\pi$ spectra in the range 1100–1200 MeV. Dispersion-theory analyses of K_{l3} form factors suggest $m_\kappa \cong 1$ GeV.[22] Using $m_\sigma^2+m_{\sigma'}^2 = m_{\sigma_0}^2+m_{\sigma_8}^2$, we see that $m_\sigma=700$ MeV would give $m_{\sigma_8}=940$ MeV, while $m_\sigma=730$ MeV would give $m_{\sigma_8}=960$ MeV. In the former case, $m_\kappa=920$ MeV, while in the latter $m_\kappa=960$ MeV. We choose the latter case (degenerate unmixed octet) for simplicity. This corresponds to a mixing angle of 35°, which is close to the canonical value. Thus σ' should be ϕ-like, decaying strongly into $K\bar{K}$, as observed. If the κ mass should be as high as 1100 MeV, either the nonet scheme is wrong, or an extra scalar is required. From the foregoing numbers we can estimate $z=-0.57$, and hence f_1/f_2 is 0.72 in the model, rather than zero.

VI. 20-MESON MODEL

We next consider the model having 20 mesons, as suggested in the Introduction. By assumption, the extra mesons S and P commute with the generators of $SU_3 \times SU_3$, but their behavior under F_0^5 has yet to be discussed. The simplest possibility is that F_0^5 turn S into P, and vice-versa, according to

$$i[F_0^5,S]=P, \quad i[F_0^5,P]=-S, \tag{6.1}$$

exactly as for (σ_0,ϕ_0) or for $\delta^{1/3}$, $\delta_5^{1/3}$. However, much of our discussion is independent of the coefficient of the right-hand side of (6.1), which can be multiplied by a numerical factor q without changing basic $U_3 \times U_3$ invariants such as S^2+P^2.

We now allow cubic and quartic interactions among the 20 mesons of $(3,\bar{3})\oplus(\bar{3},3)$ and S and P which either conserve F_0^5 or have the simple property of changing it by $\Delta F_0^5 = \pm q$, where q is some integer. (The trilinear coupling $\det\mathfrak{M}$ was of this type with $q=\pm 3$.) In this way the Lagrangian of the 18-meson model is augmented by

$$\begin{aligned} \Delta\mathfrak{L} = {} & f_3(S^2+P^2)\,\mathrm{Tr}\mathfrak{M}^\dagger\mathfrak{M} + f_4(S^2+P^2)^2 \\ & + g_2 S\,\mathrm{Tr}\mathfrak{M}^\dagger\mathfrak{M} + g_3 S(S^2+P^2). \end{aligned} \tag{6.2}$$

The quartic terms are $U_3 \times U_3$ invariant while the cubic interactions break F_0^5, as noted earlier. When nonvanishing $\langle S\rangle$ and $\langle\sigma_0\rangle$, are allowed, the S and σ_0 fields mix by virtue of the interactions involving f_3 and g_2. (When SU_3 is broken, there will be mixing of the three scalars S, σ_0, and σ_8.) In this model, P and ϕ_0 do not mix.

If we allow both S and σ_0 to have nonvanishing expectation values $\langle S\rangle = s$, $\langle\sigma_0\rangle=\sqrt{3}a$, the physical fields S' and σ_0' are

$$S'=S-s, \quad \sigma_0'=\sigma_0-\sqrt{3}a. \tag{6.3}$$

The condition that the full Lagrangian be extremal with respect to S' and σ_0' gives a new Lagrangian having mass terms and modified cubic interactions. The cross term $S'\sigma_0'$ gives rise to mixing so that $m^2(s')$ and $m_S^2(1)$ are not $(\text{mass})^2$ eigenvalues when one neglects the cubic and quartic interactions in the physical fields. We have neglected possible bare-mass terms for the 20 mesons, although it is easy to include their effect. The resulting interaction has seven arbitrary constants which one might hope to eliminate by means of further symmetry relations.

The extremal conditions are

$$\begin{aligned} 2s(3a^2f_3+2s^2f_4)+3(a^2g_2+s^2g_3) &= 0, \\ a[2a^2(3f_1+f_2)+s^2f_3+ag_1+sg_2] &= 0. \end{aligned} \tag{6.4}$$

We consider solutions with $s\neq 0$, $a\neq 0$. (When $s=0$ the S and P mesons decouple from the others and one is essentially left with the 18-meson problem.) The second of Eqs. (6.4) assures us that the pseudoscalar octet mass is zero. The (unmixed) masses are expressible as

[21] Particle Data Group, Rev. Mod. Phys. **42**, 87 (1970).
[22] J. C. Pati and K. C. Sebastian, Phys. Rev. **174**, 2033 (1968). This paper contains references to earlier work.

$$\begin{aligned} m_P^2(8) &= 0\,,\\ m_P^2(1) &= 6ag_1\,,\\ m_S^2(8) &= 4ag_1 - 8f_2a^2\,,\\ m_S^2(1) &= 2ag_1 + 4sg_2 + 4s^2f_3\,,\\ m^2(P) &= 3a^2g_2/s + sg_3\,,\\ m^2(S') &= 3(3a^2g_2/s + sg_3) - 12f_3a^2\,,\\ m^2(S'\sigma_0') &= -2\sqrt{3}a(g_2 + 2sf_3)\,. \end{aligned} \tag{6.5}$$

Of the 18 mesons in $(3,\bar{3})\oplus(\bar{3},3)$, only σ_0' has a mass differing from that in the 18-meson model (although the numerical value of a will be different). The coupling constants are constrained to ranges in which the unmixed (mass)^2 terms are positive, and the mixed masses are also positive.

The above model has little predictive power but exhibits some interesting theoretical features. We study the breakdown of the scale-invariant theory by setting $g_1 = g_2 = g_3 = 0$.

Solution zero: $a = s = 0$. This is the normal solution which involves 20 degenerate mesons. This solution is uninteresting for the present purposes.

Solution 1: $s = 0$, $a \neq 0$. This solution only exists when $3f_1 + f_2 = 0$, as in the 18-meson model. The masses are then

$$\begin{aligned} m_P^2(8) &= m_P^2(1) = m_S^2(1) = 0\,,\\ m_S^2(8) &= -8f_2a^2\,,\\ m^2(P) &= m^2(S) = -6f_3a^2\,, \end{aligned} \tag{6.6}$$

showing that $f_2, f_3 < 0$ for stability (otherwise we have the normal solution). The S and P mesons play no role in the dynamics but acquire a mass by virtue of interacting with the σ_0 tadpole. We have a nonet of pseudoscalars with the conserved currents $\mathfrak{F}_{i\mu}{}^5$ $(i = 0, \ldots, 8)$ and a massless 0^+ Goldstone boson (σ_0') associated with the conserved dilatation current $\mathfrak{D}_\mu$. The present solution is essentially the same as the 18-meson model. Because of the spontaneous breakdown, there are nonzero masses but their magnitude is not determined, since $\langle\sigma_0\rangle$ is not fixed by anything in the theory.

Solution 2: $s \neq 0$, $a = 0$. This solution cannot exist unless $f_4 = 0$, as shown by the first of Eqs. (6.4). The masses are given by

$$\begin{aligned} m^2(S) &= m^2(P) = 0\,,\\ m_P^2(8) &= m_P^2(1) = m_S^2(8) = m_S^2(1)\,, \end{aligned} \tag{6.7}$$

so that $f_3 < 0$. Here S is the scale meson giving conservation of $\mathfrak{D}_\mu$ while P is the Goldstone boson resulting from the condition $\langle S\rangle \neq 0$. $SU_3 \times SU_3$ is satisfied by degeneracy of the $(3,\bar{3})\oplus(\bar{3},3)$ multiplet. The vacuum is invariant under $SU_3 \times SU_3$ but not D nor $F_0{}^5$.

Solution 3: $s \neq 0$, $a \neq 0$.

The external conditions give two equations determining the ratio s/a. Consistency then requires that the couplings be related as follows:

$$3f_3^2 = 4f_4(3f_1 + f_2)\,. \tag{6.8}$$

The masses are given by

$$\begin{aligned} m_P^2(8) &= m_P^2(1) = m^2(P) = 0\,,\\ m_S^2(8) &= -8f_2a^2\,,\\ m^2(S') &= 12a^2f_3\,,\\ m_S^2(1) &= m^2(\sigma_0') = 4s^2f_3\,,\\ m^2(S'\sigma_0') &= -4\sqrt{3}saf_3\,. \end{aligned} \tag{6.9}$$

The eigenvalues of the $S' - \sigma_0'$ mass matrix are

$$m_1^2 = 0\,, \quad m_2^2 = 4(s^2 + 3a^2)f_3\,, \tag{6.10}$$

corresponding to the mixed states σ_1, σ_2 whose mixing angle is given by

$$\begin{aligned} \tan 2\chi &= 2\xi/(1 - \xi^2)\,,\\ \xi &= \frac{\langle\sigma_0\rangle}{\langle S\rangle} = \frac{\sqrt{3}a}{s} = \left(\frac{2|f_4|}{|f_3|}\right)^{1/2}. \end{aligned} \tag{6.11}$$

Note that equal mixing occurs when $\langle\sigma_0\rangle = \langle S\rangle$. Equations (6.4) and (6.9) indicate that $f_3 > 0$, $f_4 < 0$ for stable solutions. Equation (6.8) in turn requires $3f_1 + f_2 < 0$.

In the present model σ_1 is the scale meson. The P meson is massless because it is the Goldstone boson associated with $\langle S\rangle \neq 0$. Both become massive when the cubic scale-breaking interactions are taken into account.

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