limiting process.[6] From this fact we may suppose that a consistent theory describing higher-spin particles can be obtained only by a limiting process if one takes the Lagrangian approach.

The case of massless particles must be discussed separately, because terms such as $1/\mu^2$ do not exist in this case. Another important point in this case is the fact that the current is conserved, which is the necessary condition for Eq. (1) to hold.

After calculations similar to those in the case of massive particles, we find two solutions of Eq. (1) in the case of the spin-2 particle:

$$(1/s)\left[\tfrac{1}{2}(\delta_{\mu_1\nu_1}\delta_{\mu_2\nu_2}+\delta_{\mu_1\nu_2}\delta_{\mu_2\nu_1})-\tfrac{1}{2}\delta_{\mu_1\mu_2}\delta_{\nu_1\nu_2}\right], \tag{5}$$

$$(1/s)\left[\tfrac{1}{2}(\delta_{\mu_1\nu_1}\delta_{\mu_2\nu_2}+\delta_{\mu_1\nu_2}\delta_{\mu_2\nu_1})-\tfrac{1}{3}\delta_{\mu_1\mu_2}\delta_{\nu_1\nu_2}\right]. \tag{6}$$

Solution (5) corresponds to the propagator of the linearized Einstein theory and contains only helicity-$(\pm 2)$ states as intermediate states.

On the other hand, the propagator of the scalar-tensor theory[7] of the graviton also contains the helicity-0 state as an intermediate state and can be written, using one parameter $\alpha$, as

$$(1/s)\left\{\left[\tfrac{1}{2}(\delta_{\mu_1\nu_1}\delta_{\mu_2\nu_2}+\delta_{\mu_1\nu_2}\delta_{\mu_2\nu_1})-\tfrac{1}{2}\delta_{\mu_1\mu_2}\delta_{\nu_1\nu_2}\right]+\alpha\delta_{\mu_1\mu_2}\delta_{\nu_1\nu_2}\right\}, \tag{7}$$

where the limit of $\alpha$ determined by the deflection of light near the sun is $\alpha<0.1$. Hence solution (6) can be rejected as the propagator of the graviton. No scalar-tensor theory,[8] except Eq. (5), can satisfy Eq. (1). Thus, *the only solution which satisfies both Eq. (1) and the experimental facts is solution (5).*

We want to comment here that there is a gap between propagators of massless particles and those of massive particles in the case of higher-spin particles, and that the propagator of massless particles does not correspond to an irreducible representation of the $O(4)$ symmetry.

A full account of the properties of propagators of higher-spin particles, including self-energy parts, will be published elsewhere.[9]

[6] See footnote 9 of Ref. 1.

[7] C. Brans and R. H. Dicke, Phys. Rev. **124**, 925 (1961); W. Thirring, Acta Phys. Austriaca Suppl. **5**, 391 (1968); R. H. Sexl, Fortschr. Physik **15**, 269 (1967).

[8] Even if the interaction between the spin-0 part and the source and that between the spin-2 part and the source are introduced independently, the parts of spin 0 and spin 2 are coupled when the sources are the same.

[9] Y. Iwasaki, Progr. Theoret. Phys. (Kyoto) **44**, No. 5 (1970).

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# General Treatment of the Multiple Factorizations in the Dual Resonance Models; the $N$-Reggeon Amplitudes*

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The author presents a compact way of carrying out repeated factorizations on the dual amplitude. Prescriptions are given for writing down the multiply-factorized tree amplitudes. As an application of the prescriptions, the $N$-Reggeon amplitude is derived.

## I. INTRODUCTION

In this paper, we use the fact that the projective transformation of cross ratio is invariant under changing of the projective frames (duality) to generalize the multiple-factorization technique developed in a previous paper.[1] We find a neat and compact way of carrying out the mutliple factorizations on the dual amplitude. As a consequence of this, we obtain a set of prescriptions which enables us to write down directly the multiply-factorized tree amplitudes by simply examining the corresponding tree diagrams. Applying the prescriptions in a particular case, we obtain the formula for the $N$-Reggeon amplitudes.

In Sec. II we explicitly carry out the third and the fourth factorizations and prove the factorization of the quadruply-factorized tree into two triply-factorized trees. We thus discuss the application of the quadruply-

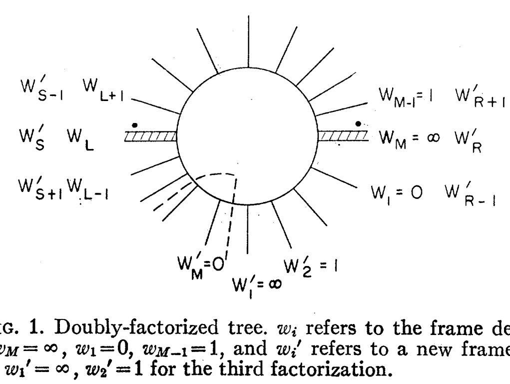


FIG. 1. Doubly-factorized tree. $w_i$ refers to the frame defined by $w_M=\infty$, $w_1=0$, $w_{M-1}=1$, and $w_i'$ refers to a new frame $w_M'=0$, $w_1'=\infty$, $w_2'=1$ for the third factorization.

* Work supported in part by the U. S. Atomic Energy Commission.

[1] L. P. Yu, Phys. Rev. D **2**, 1010 (1970).

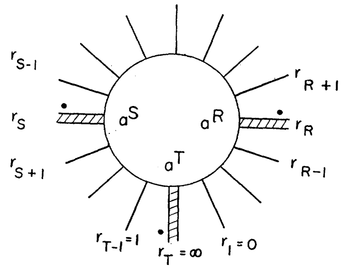


FIG. 2. Triply-factorized tree.

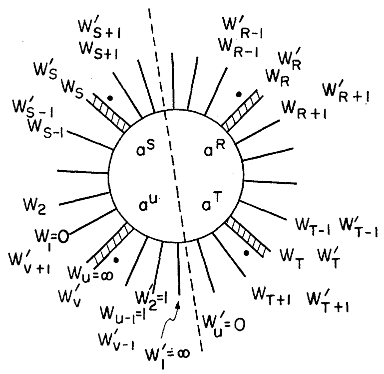


FIG. 3. Quadruply-factorized tree.

factorized tree to the general double-loop calculation. In Sec. III we give the prescriptions for writing down an arbitrarily multiply-factorized tree amplitude. And in Sec. IV we consider a particular case of multiply-factorized tree to obtain the $N$-Reggeon amplitude. In the Conclusion we discuss the difficulties due to linear dependences.

## II. THIRD AND FOURTH FACTORIZATIONS

It is convenient to follow the notations of Kikkawa and Sato.[2] They define

$$
\begin{aligned}
|p_i) &\equiv \left(\frac{p_i}{1^{1/2}}, \frac{p_i}{2^{1/2}}, \frac{p_i}{3^{1/2}}, \ldots\right), \\
|a^R) &\equiv (a_1{}^R, a_2{}^R, a_3{}^R, \ldots), \\
(M_+)_{nm} &\equiv (\tilde{M}_+)_{nm} = \left(\frac{n}{m}\right)^{1/2} (-)^n \binom{-m}{n}, \\
(\tilde{M}_-)_{nm} &\equiv (M_-)_{mn} = \left(\frac{n}{m}\right)^{1/2} (-)^n \binom{m}{n}, \qquad (1)\\
(a|x &\equiv \sum_{n=1}^{\infty} a_n x^n, \\
(a|x|b) &\equiv \sum_{n=1}^{\infty} a_n x^n b_n, \\
(a|xM_\pm yM_\pm z|b) &\equiv \sum_{n,m,k=1}^{\infty} a_n x^n (M_\pm)_{nm} y^m (M_\pm)_{mk} z^k b_k .
\end{aligned}
$$

In the Appendix we give all the identities used in this paper. We start with the doubly-factorized tree of Eq. (17) of Ref. 1:

$$
\left\langle 0 \left| \tilde{G}_{(W)}{}^{(2)}(a,b) \right| \begin{matrix}\lambda_a \\ \lambda_b\end{matrix} \right\rangle = \int \prod dw_i \{W_M\} \left\langle 0 \right| \exp\left[ \sum_{mn} \frac{a_n b_m}{(nm)^{1/2}} \left(\frac{w_{L+1}-w_L}{w_{L+1}-w_{L-1}}\right)^n B_{nm}\left(\frac{w_{L-1}}{w_L}\right) w_L{}^m \right]
$$
$$
\times \exp\left[ \sum_n \frac{a_n}{n^{1/2}} \sum_{i=1\ (i\neq L)}^{M} p_i \left(\frac{w_{L+1}-w_L}{w_{L+1}-w_{L-1}}\right)^n \left(\frac{w_{L-1}-w_i}{w_L-w_i}\right)^n + \sum_m \frac{b_m}{m^{1/2}} \left( \sum_{i=1}^{M-1} p_i w_i{}^m \right) \right] \left| \begin{matrix}\lambda_a \\ \lambda_b\end{matrix} \right\rangle, \qquad (2)
$$

where

$$
B_{nm}(x) \equiv \sum_{i=0}^{m} \binom{m}{i} \binom{-n}{m-i} (-)^m m x^i, \qquad (3)
$$

and $w_M=\infty$, $w_{M-1}=1$, and $w_1=0$ (Fig. 1). In our new notations, we write the formula as follows:

$$
\begin{aligned}
\tilde{G}_{(W)}{}^{(2)}(a^L,a^M) = \int \prod w_i \{W_M\} \exp\Big[ &\sum_{i=1\ (i\neq L)}^{M} (a^L| P(w_L, w_{L-1}, w_{L+1}, w_i) |p_i) \\
&+ \sum_{i=1\ (i\neq M)}^{M} (a^M| P(w_M, w_1, w_{M-1}, w_i) |p_i) + (a^L| P(w_L, w_{L-1}, w_{L+1}, w_M) \\
&\times M_- P(w_{L-1}, w_1, w_L, w_M) \tilde{M}_- P(w_M, w_1, w_{M-1}, w_L) |a^M) \Big], \qquad (4)
\end{aligned}
$$

where $a^L \equiv a$, $a^M \equiv b$, and

$$
P(w_i, w_j, w_k, w_m) \equiv \frac{(w_i-w_k)(w_j-w_m)}{(w_i-w_m)(w_j-w_k)}. \qquad (5)
$$

[2] K. Kikkawa and H. Sato, University of Toyko Report No. UT-36, 1970 (unpublished).

In Eq. (4), we have omitted the coherent states $|\lambda_a\rangle$ and $|\lambda_b\rangle$, and we will continue to do so throughout the paper if no confusion arises. We observe that the cross ratios $P(w_i,w_j,w_k,w_l)$ appear in a rather regular fashion, depending on the dot positions of the excited legs. This turns out to be a very general feature as we go on to higher factorizations.

Let us now do the third factorization. In Eq. (4) we change from the frame $w_M=\infty$, $w_{M-1}=1$, $w_1=0$ to the frame $w_1'=\infty$, $w_2'=1$, $w_M'=0$ (Fig. 1), i.e.,

$$w_i=P(w_R',\, w_{R-1}',\, w_{R+1}',\, w_i');$$

then Eq. (4) becomes

$$\begin{aligned}\tilde{G}_{(W')}{}^{(2)}(a^R,a^S)=\int\prod w_i'\{W_M'\}\exp\Big[\sum_{i=1\ (i\neq R)}^{M}(a^R|P(w_R',w_{R-1}',w_{R+1}',w_i')|p_i)\\+\sum_{i=1\ (i\neq S)}^{M}(a^S|P(w_S',w_{S+1}',w_{S-1}',w_i')|p_i)+(a^S|P(w_S',w_{S+1}',w_{S-1}',w_R')\\\times M\_P(w_{S+1}',w_{R-1}',w_S',w_R')\tilde{M}\_P(w_R',w_{R-1}',w_{R+1}',w_S')|a^R)\Big].\qquad(6)\end{aligned}$$

Divide $w_i'$ into $r_i$, $s_j$, and $t$ such that

$$\begin{aligned}&R\text{ frame:}\quad r_T=\infty\,,\quad r_{T-1}=1\,,\quad r_1=0;\quad r_{T+i}=r_i\,,\\&S\text{ frame:}\quad s_{T-1}=\infty\,,\quad s_T=1\,,\quad s_M=0;\quad s_{M+i}=s_{T-2+i}\,,\\&W'\text{ frame:}\quad w_1'=\infty\,,\quad w_2'=1\,,\quad w_M'=0;\quad w_{M+i}'=w_i'\,.\end{aligned}\qquad(7)$$

Thus

$$w_i'=r_2/r_i\,,\quad i=1,\,2,\,\ldots,\,T-1\,,\qquad w_j'=r_2ts_j\,,\quad j=T,\,T+1,\,\ldots,\,M\,.\qquad(8)$$

We introduce the creation and destruction harmonic oscillator operators $a^{T\dagger}$ and $a^T$ to remove $t$ and $s_j$, $j=T$, $T+1$, $\ldots$, $M$. As discussed in Ref. 1, we have from Eqs. (6)–(8)

$$\begin{aligned}\exp\Big[\sum_{i=1\ (i\neq R)}^{M}(a^R|P(w_R',w_{R-1}',w_{R+1}',w_i')|p_i)\Big]\Rightarrow\exp\Big\{\sum_{i=1\ (i\neq R)}^{T}\big[(a^R|P(r_R,r_{R-1},r_{R+1},r_i)|p_i)\big]\\+(a^R|P(r_R,r_{R-1},r_{R+1},r_T)M\_P(r_{R-1},r_1,r_R,r_T)\tilde{M}\_P(r_T,r_1,r_{T-1},r_R)|a^T)\Big\}\,,\qquad(9a)\end{aligned}$$

$$\begin{aligned}\exp\Big[\sum_{i=1\ (i\neq S)}^{M}(a^S|P(w_S',w_{S+1}',w_{S-1}',w_i')|p_i)\Big]\Rightarrow\exp\Big\{\sum_{i=1\ (i\neq S)}^{T}\big[(a^S|P(r_S,r_{S+1},r_{S-1},r_i)|p_i)\big]\\+(a^S|P(r_S,r_{S+1},r_{S-1},r_T)M\_P(r_{S+1},r_1,r_S,r_T)\tilde{M}\_P(r_T,r_1,r_{T-1},r_S)|a^T)\Big\}\,,\qquad(9b)\end{aligned}$$

and

$$\begin{aligned}\exp(a^S|P(w_S',\,w_{S+1}',\,w_{S-1}',\,w_R')M\_P(w_{S+1}',\,w_{R-1}',\,w_S',\,w_R')\tilde{M}\_P(w_R',\,w_{R-1}',\,w_{R+1}',\,w_S')|a^R)\\=\exp(a^S|P(r_S,\,r_{S+1},\,r_{S-1},\,r_R)M\_P(r_{S+1},\,r_{R-1},\,r_S,\,r_R)\tilde{M}\_P(r_R,\,r_{R-1},\,r_{R+1},\,r_S)|a^R)\,.\qquad(9c)\end{aligned}$$

Substituting Eqs. (8) and (9) in Eq. (6), and defining

$$\tilde{G}_{(W')}{}^{(2)}(a^R,a^S)=\langle 0_T|\tilde{G}_{(R)}{}^{(3)}(a^R,a^S,a^T)D(R_{a^T},s_3)G_{(S)}{}^{(1)}(a^{T\dagger})|0_T\rangle\,,$$

we thus obtain the triply-factorized tree (Fig. 2):

$$\begin{aligned}\tilde{G}_{(R)}{}^{(3)}(a^R,a^S,a^T)=\int\prod r_i\{R_T\}\exp\Big[\sum_{i=1\ (i\neq R)}^{T}(a^R|P_R(i)|p_i)+\sum_{i=1\ (i\neq S)}^{T}(a^S|P_S(i)|p_i)+\sum_{i=1\ (i\neq T)}^{T}(a^T|P_T(i)|p_i)\\+(a^R|P_R(S)M\_P(R-1,\,S+1,\,R,\,S)\tilde{M}\_P_S(R)|a^S)+(a^S|P_S(T)M\_P(S+1,\,1,\,S,\,T)\tilde{M}\_P_T(S)|a^T)\\+(a^T|P_T(R)M\_P(1,\,R-1,\,T,\,R)\tilde{M}\_P_R(T)|a^R)\Big]\,,\qquad(10)\end{aligned}$$

where

$$P_S(i)=P(S,\,S+1,\,S-1,\,i)\,,\quad P_T(i)=P(T,\,1,\,T-1,\,i)\,,$$

and

$$P_R(i)=P(R,\,R-1,\,R+1,\,i)=\frac{(r_R-r_{R+1})(r_{R-1}-r_i)}{(r_{R-1}-r_{R+1})(r_R-r_i)}\,.\qquad(11)$$

It is clear that the new cross ratios occur in the same manner as before. It is also interesting to observe, from the identities given in the Appendix, that application of the twist operator, say, $\Omega^+(-p_R, a^R)$ on the $r_R$ leg, is simply equivalent to interchange of $r_{R+1}$ with $r_{R-1}$ everywhere in Eq. (10). This ensures that the cross ratio $P_R(i)$ has the same form whatever the positions of the dots.

We now proceed to write down the fourth factorization; we shall then be able to infer the general prescriptions for an arbitrary number of factorizations on the dual amplitude. Applying techniques identical to[1] those used in Eqs. (6)–(10), we obtain the result for the quadruply-factorized tree (Fig. 3):

$$\begin{aligned}\tilde{G}_{(W)}{}^{(4)}(a^R,a^S,a^T,a^U)=\int\prod dw_i\{W_U\}\exp\{&\sum_{i=1\ (i\neq U)}^{U}(a^U|P_U(i)|p_i)+\sum_{i=1\ (i\neq R)}^{U}(a^R|P_R(i)|p_i)\\&+\sum_{i=1\ (i\neq S)}^{U}(a^S|P_S(i)|p_i)+\sum_{i=1\ (i\neq T)}^{U}(a^T|P_T(i)|p_i)+(a^R|P_R(S)M\_P(R+1,S-1,R,S)\tilde{M}\_P_S(R)|a^S)\\&+(a^S|P_S(T)M\_P(S-1,T-1,S,T)\tilde{M}\_P_T(S)|a^T)+(a^T|P_T(R)M\_P(T-1,R+1,T,R)\tilde{M}\_P_R(T)|a^R)\\&+(a^R|P_R(U)M\_P(R+1,1,R,U)\tilde{M}\_P_U(R)|a^U)+(a^S|P_S(U)M\_P(S-1,1,S,U)\tilde{M}\_P_U(S)|a^U)\\&+(a^T|P_T(U)M\_P(T-1,1,T,U)\tilde{M}\_P_U(T)|a^U)\},\end{aligned}\quad(12)$$

where $w_U=\infty$, $w_1=0$, $w_{U-1}=1$, and

$$P_U(i)=P(U,1,U-1,i)\,.$$

By studying Eqs. (10) and (12) it is not hard to infer the general prescriptions for the multiply-factorized dual tree. It is also not hard to write down the explicit form of the amplitude with an arbitrary number of excited legs. We will proceed to do so in Sec. III.

Before concluding this section we discuss the factorization property of Eq. (12). We first express Eq. (12) in the $W'$ frame defined by $w_1'=\infty$, $w_2'=1$, $w_U'=0$ (Fig. 4), i.e.,

$$w_i=P(w_V',w_{V+1}',w_{V-1}',w_i')\,.$$

We then introduce $R$ and $S$ frames as before. Using the inverse relations of Eq. (8),

$$r_i=w_{M-1}'/w_i',\quad i<M-1,\qquad s_j=w_j'/w_M',\quad j>M,\qquad t=w_M'/w_{M-1}',\qquad(13)$$

and the identities in the Appendix, we can then prove that the following factorized expression is identical to the right-hand side of Eq. (12):

$$\begin{aligned}&\langle 0_M|\tilde{G}_{(R)}{}^{(3)}(a^S,a^V,a^M)D(a^{M-1\dagger}a^M,t)\tilde{G}_{(S)}{}^{(3)}(a^{M-1\dagger},a^R,a^T)|0_M\rangle\\&=\langle 0_M|\Big\{\int\prod dr_i\{R_M\}\exp\Big[\sum_{i=1\ (i\neq V)}^{M}(a^V|P_{r_V}(i)|p_i')+\sum_{i=1\ (i\neq S)}^{M}(a^S|P_{r_S}(i)|p_i')+\sum_{i=1\ (i\neq M)}^{M}(a^M|P_{r_M}(i)|p_i')\\&\quad+(a^V|P_{r_V}(M)M\_P_r(V+1,1,V,M)\tilde{M}\_P_{r_M}(V)|a^M)+(a^S|P_{r_S}(M)M\_P_r(S-1,1,S,M)\tilde{M}\_P_{r_M}(S)|a^M)\\&\quad+(a^V|P_{r_V}(S)M\_P_r(V+1,S-1,V,S)\tilde{M}\_P_{r_S}(V)|a^S)\Big]\Big\}D(a^{M-1\dagger}a^M,t)\Big\{\int\prod ds_j\{S_{U-M}\}\\&\quad\times\exp\Big[\sum_{i=M-1\ (i\neq R)}^{U}(a^R|P_{s_R}(i)|p_i'')+\sum_{i=M-1\ (i\neq T)}^{U}(a^T|P_{s_T}(i)|p_i'')+\sum_{i=M-1\ (i\neq M-1)}^{U}(a^{M-1\dagger}|P_{s_{M-1}}(i)|p_i'')\\&\quad+(a^{M-1\dagger}|P_{s_{M-1}}(R)M\_P_s(U,R+1,M-1,R)\tilde{M}\_P_{s_R}(M-1)|a^R)+(a^{M-1\dagger}|P_{s_{M-1}}(T)M\_P_s(U,T-1,M-1,T)\\&\quad\times\tilde{M}\_P_{s_T}(M-1)|a^T)+(a^T|P_{s_T}(R)M\_P_s(T-1,R+1,T,R)\tilde{M}\_P_{s_R}(T)|a^R)\Big]\Big\}|0_M\rangle,\end{aligned}\quad(14)$$

where

$$r_M=\infty\,,\quad r_{M-1}=1\,,\quad r_1=0\,,$$
$$s_{M-1}=\infty\,,\quad s_M=1\,,\quad s_U=0\,,$$

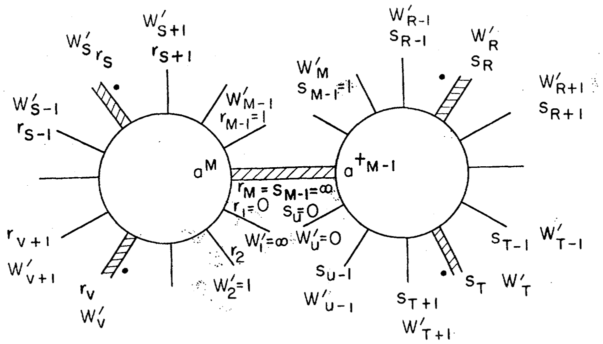


FIG. 4. Factorization of quadruply-factorized tree into two triply-factorized trees. The $r_i$ and $s_j$ refer to two new frames defined by $r_M=\infty$, $r_{M-1}=1$, $r_1=0$ and $s_{M-1}=\infty$, $s_M=1$, $s_U=0$.

and

$$
\begin{aligned}
P_{r_V}(i)&=P(r_V, r_{V+1}, r_{V-1}, r_i)\,,\\
P_r(V+1, 1, V, M)&=P(r_{V+1}, r_1, r_V, r_M)\,,\\
P_{s_T}(i)&=P(s_T, s_{T-1}, s_{T+1}, s_i)\,,\\
P_s(U, R+1, M-1, R)&=P(s_U, s_{R+1}, s_{M-1}, s_R)\,, \text{etc.}
\end{aligned}
\tag{15a}
$$

and

$$
[a_n{}^M, a_m{}^{M-1\dagger}]=\delta_{nm}\,,\quad a^M|0_M\rangle=0\,,\qquad \langle 0_M|a^{M-1\dagger}=0\,, \tag{15b}
$$

and all other commutators between $a$'s vanish; also

$$
\begin{aligned}
p_i'&=p_i\,, && i=1,2,\ldots,M-1\\
p_j''&=p_j\,, && j=M,\ldots,U\\
p_M'&=\sum_{j=M}^{U} p_j''=-\sum_{i=1}^{M-1} p_i'=-p_{M-1}''\,.
\end{aligned}
\tag{15c}
$$

From Eq. (14) it is seen that Eq. (12) does factorize into the product of the two triply-factorized trees which were given in Eq. (10) (see Fig. 4). As a consequence of the factorizability of Eq. (12), we assert that if we join any pair of the excited legs, we get the loop amplitudes. Namely, if we join the $a^S$ leg with the $a^U$ leg and the $a^R$ leg with the $a^T$ leg, we get the nonplanar double-loop diagram (Fig. 5), of which the planar double-loop diagram[3] is the special case when $S=U+1=1$, $R+1=T$. And if we join the twisted $a^R$ leg with the $a^U$ leg, the twisted $a^T$ leg with the $a^S$ leg, we then obtain the overlapping double-loop diagram (Fig. 6). Thus Eq. (12) contains all possible perturbative unitarity diagrams with three-particle intermediate states, if we join the two pairs of the excited legs.

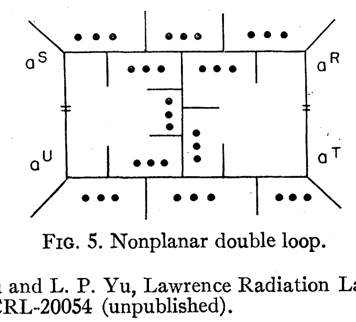


FIG. 5. Nonplanar double loop.

[3] M. Kaku and L. P. Yu, Lawrence Radiation Laboratory Report No. UCRL-20054 (unpublished).

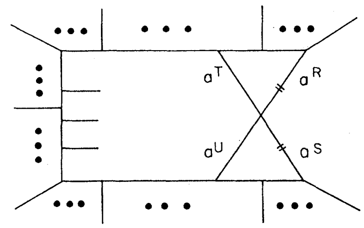


FIG. 6. Overlapping double loop.

## III. PRESCRIPTIONS OF MULTIPLY-FACTORIZED TREE AMPLITUDE

Generalizing the result of Sec. II, we now state explicitly the following set of rules in writing down the multiply-factorized tree amplitude by simple inspection of the tree diagrams.

*Rule 1.* We assign to each leg (scalar or excited) one Koba-Nielsen variable $w_i$, $i=1, 2, \ldots, M$, and an incoming four-momentum $p_i$. Corresponding to each excited leg, we assign one destruction (creation) operator $a_{\mu,n}{}^R(a_{\mu,n}{}^{R\dagger})$, where the $\mu$ are space-time indices $\mu=1, 2, 3, 4$; $n$ is the excited harmonic oscillator mode in question, $n=1, 2, \ldots, \infty$; and the superscript $R$ coincides with the labeling of the Koba-Nielsen variables, $R<M$.

*Rule. 2.* The scalar part of the multiply-factorized tree is the ordinary Koba-Nielsen representation[4] or the Donini-Sciuto representation[5] of the $M$-point dual amplitude, namely, $\int\prod dw_i\{W_M\}$.

*Rule 3.* All destruction (creation) operators $a^R$ $(a^{R\dagger})$ appear only in the exponents. The exponential factors

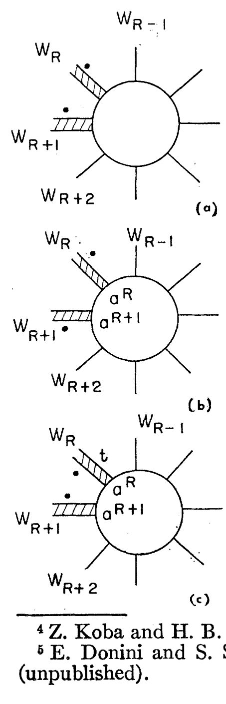


FIG. 7. (a) $w_R$ dot lies between $w_R$ and $w_{R-1}$, and the $w_{R+1}$ dot lies between $w_R$ and $w_{R+1}$. (b) $w_R$ dot lies between $w_R$ and $w_{R-1}$, and the $w_{R+1}$ dot lies between $w_{R+1}$ and $w_{R+2}$. (c) $w_R$ dot lies between $w_R$ and $w_{R+1}$, and the $w_{R+1}$ dot lies between $w_{R+1}$ and $w_R$.

[4] Z. Koba and H. B. Nielsen, Nucl. Phys. **B12**, 517 (1969).

[5] E. Donini and S. Sciuto, University of Torino report, 1969 (unpublished).

can be divided into two classes. One class involves the scalar product $a_\mu{}^R\cdot p_i{}^\mu$, the other class involves the scalar product of $a_\mu{}^R\cdot a_\mu{}^S$; $R, S\leqslant M$. The factors in the first class have the form

$$\exp\Big[\sum_{R\; R\leqslant M}\;\sum_{i=1\;(i\neq R)}^{M}(a^R|P_R(w_i)|p_i)\Big],\tag{16}$$

and those in the second class take the form

$$\exp\Big[\tfrac{1}{2}\sum_{R,S\leqslant M\;(R\neq S)}(a^R|P_R(S)M\_P(R\pm1,S\pm1,R,S)\times\tilde{M}\_P_S(R)|a^S)\Big],\tag{17}$$

where

$$\begin{aligned}P_R(i)&=P(w_R,\,w_{R-1},\,w_{R+1},\,w_i)\quad\text{if the dot of the } w_R \text{ leg lies between } w_R \text{ and } w_{R+1}\\&=P(w_R,\,w_{R+1},\,w_{R-1},\,w_i)\quad\text{if the dot of the } w_R \text{ leg lies between } w_R \text{ and } w_{R-1},\end{aligned}\tag{18a}$$

and

$$\begin{aligned}P(R\pm1,\,S\pm1,\,R,\,S)&=P(w_{R\pm1},\,w_{S\pm1},\,w_R,\,w_S)\quad\text{if the dot of the } w_R \text{ leg lies between } w_R \text{ and } w_{R\mp1}\text{, and the}\\&\qquad\text{dot of the } w_S \text{ leg lies between } w_S \text{ and } w_{S\mp1}\\&=P(w_{R\pm1},\,w_{S\mp1},\,w_R,\,w_S)\quad\text{if the dot of the } w_R \text{ leg lies between } w_R \text{ an } w_{R\mp1}\text{, and the}\\&\qquad\text{dot of the } w_S \text{ leg lies between } w_S \text{ and } w_{S\pm1}.\end{aligned}\tag{18b}$$

*Rule 4.* When two excited legs, say, $w_R$ and $w_{R+1}$, are adjacent to each other, i.e., there is no external scalar leg between them, then we must take appropriate limits of the relevant terms in Eq. (17). The limits are as follows: *Rule 4a.* When the $w_R$ dot lies between $w_{R-1}$ and $w_R$, and the $w_{R+1}$ dot lies between $w_R$ and $w_{R+1}$ [Fig. (7a)], then

$$(a^{R+1}|P_{R+1}(R)M\_P(R+2,\,R+1,\,R+1,\,R)\tilde{M}\_P_R(R+1)|a^R)\to(a^{R+1}|M\_P(R+2,\,R-1,\,R+1,\,R)|a^R).\tag{19a}$$

*Rule 4b.* When the $w_R$ dot lines betwee $w_{R-1}$ and $w_R$, and the $w_{R+1}$ dot lies between $w_{R+1}$ and $w_{R+2}$ [Fig. 7(b)], then

$$(a^{R+1}|P_{R+1}(R)M\_P(R,\,R+1,\,R+1,\,R)\tilde{M}\_P_R(R+1)|a^R)\to(a^{R+1}|P(R+2,\,R-1,\,R+1,\,R)|a^R).\tag{19b}$$

*Rule 4c.* When the $w_R$ dot and the $w_{R+1}$ dot both lie between $w_R$ and $w_{R+1}$ [Fig. 7(c)], then

$$(a^{R+1}|P_{R+1}(R)M\_P(R+2,\,R-1,\,R+1,\,R)\tilde{M}\_P_R(R+1)|a^R)\to(a^{R+1}|M\_P(R+2,\,R-1,\,R+1,\,R)\tilde{M}\_|a^R).\tag{19c}$$

However, in this case we need an asymmetrical propagator[1] for the $w_R$ leg (or $w_{R+1}$ leg),

$$D'(R_{a^R},R_{a^{R+1}},w_{R+2};\,P_R{}^2)=\int_0^1 dt\,t^{-\alpha(p_R)-1+R_R}\times(1-t)^{a-1}\left(\frac{1-t}{1-tP_R(w_{R+2})}\right)^{R_{R+1}-\alpha(p_{R+1})},\tag{20}$$

where

$$P_R(w_{R+2})=P(w_R,\,w_{R-1},\,w_{R+1},\,w_{R+2}).$$

In the cases of rules 4a and 4b, we do not modify the propagator at all. We note that rules 4a and 4b are related to rule 4c by the twisting operations.

*Rule 5.* The two amplitudes with the dot on the opposite sides of the excited leg, say, the $w_R$ leg, are related to each other by the twist operator $\Omega^+(-p_R;\,a^R)$, or, equivalently, by the interchange of $w_{R+1}$ with $w_{R-1}$ everywhere in Eqs. (16) and (17). The only additional complication to this rule is the case stated in rule 4c, where we need an asymmetrical propagator.

By using the above prescriptions and by directly inspecting the factorized tree diagram, we can write down the general formula for the multiply-factorized tree amplitude:

$$\begin{aligned}\langle 0|G_{(W)}{}^{(N)}(a^R)|\text{all }\lambda_a\rangle&=\int\prod dw_i\{W_M\}\\&\times\langle 0|\exp\Big[\sum_{R\;R\leqslant M}\;\sum_{i=1\;(i\neq R)}^{M}(a^R|P_R(i)|p_i)\\&+\tfrac{1}{2}\sum_{R,S\leqslant M\;(R\neq S)}(a^R|P_R(S)M\_P(R\pm1,S\pm1,R,S)\\&\qquad\times\tilde{M}\_P_S(R)|a^S)\Big]|\text{all }\lambda_a\rangle,\end{aligned}\tag{21}$$

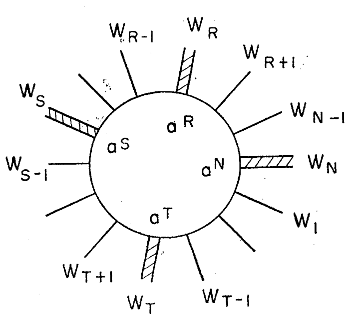


FIG. 8. *N*th-factorized tree.

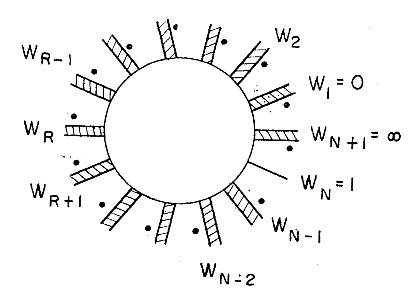


FIG. 9. *N* excited legs plus one scalar leg.

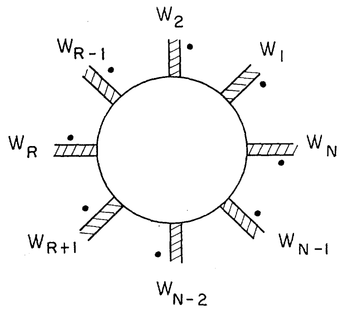


FIG. 10. Twister $w_{N-1}$ leg of the $N$-Reggeon amplitude.

where $M$ is the total number of excited legs plus the external scalar legs. Also, $w_{M+i}\equiv w_i$, $a^{M+R}\equiv a^R$. We also give an alternative recursion formula for Eq. (21), which relates the integrand of the $N$th-factorized tree amplitude to the integrand of the $(N-1)$th-factorized tree amplitude (Fig. 8)

$$\langle 0|G_{(W)}{}^{(N)}(a^R)|\text{all }\lambda_a\rangle=\int\prod dw_i\{W_M\}$$
$$\times\langle 0|\exp\Big[\sum_{i=1\ (i\neq R)}^{M}(a^R|P_R(i)|p_i)$$
$$+\sum_{S\leqslant M-1\ (S\neq R)}(a^R|P_R(S)M\_P(R\pm1,S\pm1,R,S)$$
$$\times\tilde{M}\_P_S(R)|a^S)\Big]\{\mathcal{G}_{(W)}{}^{(N-1)}(a^T;T\neq R)\}$$
$$\times|\text{all }\lambda_a\rangle. \quad (22)$$

where $\mathcal{G}_{(W)}{}^{(N-1)}(a^T;T\neq R)$ is the operator part of the $(N-1)$th-factorized tree integrand, and the index $R$ is the Koba-Nielsen label of the first-factorized excited leg. Equation (22) is useful in obtaining the $N$-Reggeon amplitude. We will proceed to do so in Sec. IV.

## IV. N-REGGEON AMPLITUDES

As we discussed in Ref. 1, to get the pure $N$-Reggeon amplitude, we should consider the case when $M=N+1$ in Eq. (21) or Eq. (22). Then we have the tree amplitude with one external scalar leg (the $w_N$ leg) together with $N$ excited legs. For simplicity we choose the dot positions of the $N$ excited legs as shown in Fig. 9. Corresponding to Fig. 9, we apply the prescriptions in Sec. III, or use Eq. (22), to get the amplitude of $N$ excited legs plus one external scalar leg (the $w_N$ leg),

$$G_{(W)}{}^{(N)}(a^1,a^2,\ldots,a^{N-1},a^{N+1})$$
$$=\int\prod_{i\neq(N+1,N,1)}dw_i\{W_{N+1}\}$$
$$\times\{\mathcal{G}_{(W)}{}^{(N)}(a^1,a^2,\ldots,a^{N-1},a^{N+1})\}$$
$$=\int\prod dw_i\{W_{N+1}\}\mathcal{G}_{(W)}{}^{(N-1)}(a^1,a^2,\ldots,a^{N-2},a^{N+1})$$
$$\times\exp\Big\{\sum_{i=1\ (i\neq N-1)}^{N+1}(a^{N-1}|P_{N-1}(i)|p_i)$$
$$+\sum_{R=1\ (R\neq N,N-1)}^{N+1}[(a^{N-1}|P_{N-1}(R)M\_P$$
$$\times P(N-2,R+1,N-1,R)\tilde{M}\_P_R(N-1)|a^R]\Big\}, \quad (23)$$

where $P_{N-1}(i)=P(N-1,N-2,N,i)$ and we choose the frame

$$w_{N+1}=\infty,\quad w_N=1,\quad w_1=0.$$

Now analogously to Ref. 1, we consider the scalar $w_N$ leg attached to the excited $w_{N+1}$ leg. In other words, we want to write Eq. (23) in the form $W^{(N)}(a^R)D^{(N)}\times V(p_N)$ and then to remove $V(p_N)=\exp(a^{N+1\dagger}|p_N)\times\exp(a^{N+1}|p_N)$ by letting $p_N\to 0$ and modifying the spectrum of the relevant trajectories, as we did[6] in Ref. 1. Guided by the tricks used in Eqs. (33)–(35) of Ref. 1, where we have taken down the common factor $[(1-w_3)/(1-w_2)]^{R_c}(w_3)^{R_d}$ from the $\exp\{\cdots\}$ of Eq. (33) to the propagator $D'$ of Eq. (35), we now observe, from the factors $P_{N-1}(w_i)$ and $P_{N+1}(w_i)$ in Eq. (23), that we can bring down the similar common factor

$$[(w_N-w_{N-1})/(w_N-w_{N-2})]^{R_{N-1}}(w_{N-1})^{R_{N+1}}$$

from the $\exp\{\cdots\}$ of Eq. (23) to the propagator $D^{(N)}$. It is this common factor that causes the factorization of $w_{N-2}$ and $t$ to be unfeasible, and presumably it is closely connected with the linear-dependence problem.[7]

After setting $p_N\to 0$ and redefining $a^{N+1}=a^N$, $p_{N+1}=p_N$ and reintroducing the Koba-Nielsen variables $w_N=\infty$, $w_{N-1}=1$, $w_1=0$ in Eq. (23), we find the modified propagator[8] of the new $w_N$ leg,

$$D^{(N)}(a^N,a^{N-1},p_N)=\int_0^1 dt\,t^{-\alpha(p_N)-1+R_N}(1-t)^{a-1}$$
$$\times\left(\frac{1-t}{1-tP(w_N,w_1,w_{N-1},w_{N-2})}\right)^{R_{N-1}-\alpha(p_{N-1})}, \quad (24)$$

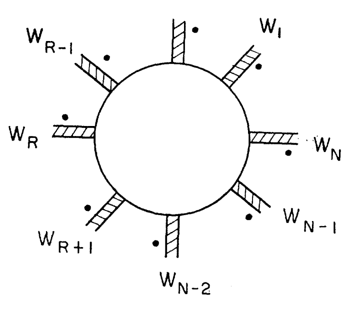


FIG. 11. Symmetrical $N$-Reggeon amplitude.

[6] We emphasize that this procedure is not a necessary one since we can obtain the same conclusion by directly factoring this unequal-mass case.

[7] M. Kaku, C. B. Thorn, and J. H. Weis (private communications).

[8] This propagator is of course consistent with rule 4c in Sec. III.

and the $N$-Reggeon amplitude (Fig. 10)

$$W^{(N)}(a^1,a^2,\ldots,a^N)$$

$$=\int \prod dw_i\{W_N\} \exp\{ \sum_{R=1\ (R\neq N-1)}^{N} \sum_{i=1\ (i\neq R)}^{N} (a^R|P_R(i)|p_i)+\sum_{i=1\ (i\neq N-1)}^{N} (a^{N-1}|P(N-1, N-2, N, i)|p_i)$$

$$+\sum_{R=1\ (R\neq N-1,N)}^{N} [(a^R|M\_P(R+1, R-2, R, R-1)|a^{R-1})+(a^{N-1}|P(N, N-3, N-1, N-2)|a^{N-2})$$

$$+(a^N|M\_P(1, N-2, N, N-1)\tilde{M}\_|a^{N-1})]+\sum_{R=1\ (R\neq N-1,1)}^{N} [(a^R|P_R(R-2)M\_P(R+1, R-1, R, R-2)$$

$$\times\tilde{M}\_P_{R-2}(R)|a^{R-2})+(a^{N-1}|P(N-1, N-2, N, N-3)M\_P(N-2, N-2, N-1, N-3)$$

$$\times\tilde{M}\_P(N-3, N-2, N-4, N-1)|a^{N-3})+(a^1|P(1, 2, N, N-1)M\_P(2, N-2, 1, N-1)$$

$$\times\tilde{M}\_P(N-1, N-2, N, 1)|a^{N-1})]+\cdots+\sum_{R=1\ (R\neq N-1,K-1)}^{N} [(a^R|P_R(R-K)M\_P(R+1, R-K+1, R, R-K)$$

$$\times\tilde{M}\_P_{R-K}(R)|a^{R-K})+(a^{N-1}|P(N-1, N-2, N, N-K-1)M\_P(N-2, N-K, N-1, N-K-1)$$

$$\times\tilde{M}\_P(N-K-1, N-K, N-K-2, N-1)|a^{N-K-1})+(a^{K-1}|P_K(N-1)M\_P(K, N-2, K-1, N-1)$$

$$\times\tilde{M}\_P(N-1, N-2, N, K-1)|a^{N-1})]+\cdots+\sum_{R=1\ (R\neq N-1,\frac{1}{2}N-1)}^{N} [\tfrac{1}{2}(a^R|P_R(R-\tfrac{1}{2}N)$$

$$\times M\_P(R+1, R-\tfrac{1}{2}N+1, R, R-\tfrac{1}{2}N)\tilde{M}\_P_{R-\frac{1}{2}N}(R)|a^{R-\frac{1}{2}N})+(a^{N-1}|P(N-1, N-2, N, \tfrac{1}{2}N-1)$$

$$\times M\_P(N-2, \tfrac{1}{2}N, N-1, \tfrac{1}{2}N-\tfrac{1}{2})\tilde{M}\_P_{\frac{1}{2}N-1}(N-1)|a^{\frac{1}{2}N-1})]\}\,, \quad (25)$$

where $a^{N+i}\equiv a^i$, $P_R(S)\equiv P(w_R, w_{R+1}, w_{R-1}, w_S)$, and we have assumed $N=$even integer. For $N=$odd integer, we then replace $\frac{1}{2}N$ by $\frac{1}{2}(N-1)$ and drop the factor $\frac{1}{2}$ in the last two terms of Eq. (25). We now also can release the frame-dependent constraint, since all terms in Eq. (25) are projectively invariant cross ratios. Hence our result of $N$-Reggeon amplitude is manifestly dual.[9]

We can further obtain the completely symmetrical $N$-Reggeon amplitude by applying the twist operator $\Omega^+(-p_{N-1}, a^{N-1})$ to the right of Eq. (25) times Eq. (24), and passing it to the left of the propagator $D^{(N)}(a^N,a^{N-1},p_N)$. Using the exact arguments given by Caneschi and Schwimmer,[10] we simply drop the factor

$$\left(\frac{1-t}{1-tP(N, 1, N-1, N-2)}\right)^{a^{R_{N-1}}-\alpha(p_{N-1})}$$

in Eq. (24), and apply the twist operator $\Omega^+(-p_{N-1},a^{N-1})$ directly to Eq. (25). We finally get the completely symmetrical $N$-Reggeon amplitude (Fig. 11):

$$W_{\text{sym}}^{(N)}(a^1,a^2,\ldots,a^N)=\int \prod dw_i\{W_N\} \exp\{ \sum_{R=1}^{N} \sum_{i=1\ (i\neq R)}^{N} (a^R|P_R(i)|p_i)$$

$$+\sum_{R=1}^{N} (a^R|M\_P(R+1, R-2, R, R-1)|a^{R-1})+\sum_{R=1}^{N} (a^R|P_R(R-2)M\_P(R+1, R-1, R, R-2)\tilde{M}\_P_{R-2}(R)|a^{R-2})$$

$$+\cdots+\sum_{R=1}^{N} (a^R|P_R(R-K)M\_P(R+1, R-K+1, R, R-K)\tilde{M}\_P_{R-K}(R)|a^{R-K})$$

$$+\cdots+\tfrac{1}{2}\sum_{R=1}^{N} (a^R|P_R(R-\tfrac{1}{2}N)M\_P(R+1, R-\tfrac{1}{2}N+1, R, R-\tfrac{1}{2}N)\tilde{M}\_P_{R-\frac{1}{2}N}(R)|a^{R-\frac{1}{2}N})\}\,, \quad (26)$$

[9] This is true up to the asymmetrical propagator, which destroys the exact duality.
[10] L. Caneschi and A. Schwimmer, Nuovo Cimento Letters **3**, 213 (1970).

where $a^{N+R}\equiv a^R$, $P_R(S)\equiv P(R,R+1,R-1,S)$, etc., and $N=$ even integer. For $N=$ odd integer, we replace $\frac{1}{2}N$ by $\frac{1}{2}(N-1)$ and drop the factor $\frac{1}{2}$ in the last term of Eq. (26). And the propagator of the $w_N$ leg, $D^{(N)}(\cdots)$, becomes the ordinary one.[11]

It is also interesting to see that the symmetrical $N$-Reggeon amplitude, Eq. (26), can be obtained directly by using the prescriptions of Sec. III and letting all the Koba-Nielsen variables associated with the external scalar legs vanish. One could also similarly use this illegitimate method to recover Eq. (25). However, we then lose the asymmetrical behavior of the propagator in the $w_N$ leg. This again indicates the complications raised by the problem of linear dependence.[12,13]

## V. CONCLUSION

We see that the multiple factorizations on the dual amplitude can be carried out in a rather neat and compact way. We also see that the $N$-Reggeon amplitude takes a fairly simple form, which is manifestly dual. However, the intrusive problem of linear dependence still remains unsolved. In fact, it is clear that the $N$-Reggeon amplitude itself should be factorizable. But in directly proving the factorization of Eq. (25) or (26), we encounter the unequal-mass problem, for which we do not have the bootstrap condition $a+bp_i^2=0$. Therefore the degeneracy of the spectrum of the trajectories appears to increase, though we know that it cannot do so. In order to decrease the degeneracy of the spectrum of the trajectories and at the same time prove the factorization of the $N$-Reggeon amplitude, we shall have to take into account the existence of linear dependences among the excited states in the dual amplitude. Once we achieve the proof of factorization of the $N$-Reggeon amplitude, we may be able to construct Nambu's[14] field-theoretical-type dynamical equation of motion.

## ACKNOWLEDGMENTS

I would like to thank Professor Stanley Mandelstam and Professor Korkut Bardakci for many helpful discussions and guidance. I would also like to thank M. Kaku, C. B. Thorn, and J. H. Weis for discussions on the linear-dependence problem.

[11] The symmetrical $N$-Reggeon amplitude, Eq. (26), is manifestly exactly dual.

[12] K. Bardakci and S. Mandelstam, Phys. Rev. **184**, 1640 (1969); S. Fubini and G. Veneziano, Nuovo Cimento **64A**, 811 (1969).

[13] C. B. Chiu, S. Matsuda, and C. Rebbi, Phys. Rev. Letters **23**, 1526 (1969); C. B. Thorn, Phys. Rev. D **1**, 1693 (1970); F. Gliozzi, Nuovo Cimento Letters **2**, 846 (1969); R. C. Brower and J. H. Weis, *ibid.* **3**, 285 (1970).

[14] Y. Nambu, University of Chicago report, 1969 (unpublished).

## APPENDIX

The notations in the following equations have been explained in Eq. (1). The function

$$\frac{B_{nm}(x)}{(nm)^{1/2}}=\sum_{i=0}^{m}\binom{n}{i}\binom{-n}{m-i}(-1)^m m x^i$$

defined in Ref. 1 is equal to $[M_-(1-x)\tilde{M}_-]_{nm}$. We further denote

$$(xM_\pm yM_\pm z)_{nm}=x^n(M_\pm)_{ni}y^i(M_\pm)_{im}z^m \quad (\text{sum over } i);$$

then we have the following set of identities:

$$M_-\tilde{M}_-=M_+, \tag{A1}$$

$$M_+\tilde{M}_-=M_-, \tag{A2}$$

$$M_-M_-=M_0, \tag{A3}$$

$$M_+xM_-=\frac{1}{1-x}M_+\frac{1}{1-1/x}, \tag{A4}$$

$$M_-xM_-=(1-x)M_-\frac{1}{1-1/x}, \tag{A5}$$

$$M_+(1-x)\tilde{M}_-=M_-(1-1/x)\tilde{M}_-x, \tag{A6}$$

$$xM_+(1-x)\tilde{M}_-=xM_-(1-1/x)\tilde{M}_-x =M_-(1-x)M_+x. \tag{A7}$$

The identities involving the twist operator are

$$(a|M_-=(\bar{a}|, \tag{A8}$$

$$\tilde{M}_-|a)=|\bar{a}), \tag{A9}$$

$$(p|a)=-(p|\bar{a}), \tag{A10}$$

$$(p|x|a)=-(p|\bar{a})+(p|(1-x)|\bar{a}), \tag{A11}$$

$$\bar{a}=(T_a{}^\dagger)^{-1}aT_a{}^\dagger, \quad T_a=:\exp(a^\dagger|M_-|a):. \tag{A12}$$

The identities involving the cross ratios are

$$P(x,y,z,w)=\frac{(x-z)(y-w)}{(x-w)(y-z)}, \tag{A13}$$

$$1-P(x,y,z,w)=P(x,z,y,w), \tag{A14}$$

$$1/P(x,y,z,w)=P(x,y,w,z), \tag{A15}$$

$$P(x,y,z,w)=P(y,x,w,z)=P(w,z,y,x) =P(z,w,x,y), \tag{A16}$$

$$P(x,y,z,w)P(x,y,u,z)=P(x,y,u,w). \tag{A17}$$