Eqs. (24) and (25) give us

$$\mathfrak{L}_n(\omega)=-\frac{i}{2}\frac{2}{45}\frac{G\omega^5}{c^5}\epsilon_n{}^{ij}Q_{ip}{}^*(\omega)Q^p{}_j(\omega)\,, \tag{27}$$

in agreement with other authors.[3]

An indication that Eq. (27) is the correct angular momentum flux for a quadrupole radiator is seen by comparing the angular momentum flux with the energy flux from a slowly rotating rigid body. Since the kinetic energy of such a body equals the angular velocity times the angular momentum, the energy loss equals the angular velocity times the angular momentum loss. Equation (27) gives this result.[13]

The author would like to thank T. A. Morgan for suggesting this work and for many useful discussions.

[13] Cooperstock and Booth (Ref. 3) compare the energy and angular momentum losses for a spinning rod and find the same kind of agreement.

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Static Charged-Dust Distributions in General Relativity

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The field equations of general relativity with electromagnetic stress tensor and charged incoherent matter, along with Maxwell equations in curved space, are investigated. The metric coefficients and the electromagnetic field tensor are assumed to be independent of x^0. It is then shown that, if one assumes a functional relation between g_{00} and the electrostatic potential (which is a consequence of the Bianchi identities), one can then find a solution of the coupled Einstein-Maxwell equations in terms of a solution of the Einstein equations with incoherent matter as source.

I. INTRODUCTION

RECENTLY, an interesting theorem was established by Harrison.[1,2] It may be stated as follows: "Given any solution of the vacuum field equations of general relativity in which the metric coefficients are functions of not more than three variables, one can generate from this a solution of the coupled Einstein-Maxwell equations with a nonvanishing electromagnetic field." The theorem is a consequence of the assumption of functional dependence between one of the metric coefficients and electromagnetic potentials. Several authors have made use of this functional relationship for generating solutions of the field equations.[3] In this paper we present a theorem which may be considered, in a sense, a generalization of the above result obtained by Harrison. The field equations of general relativity with an electromagnetic field and charged incoherent matter as source, along with Maxwell equations in curved space, are investigated. The metric coefficients g_{ij}, and the electromagnetic field F_{ij} are assumed to be functions of x^1, x^2, and x^3 only. It is then shown that, if one assumes a functional relationship between g_{00} and the electrostatic potential, one can find a solution of the coupled gravitational electromagnetic equations in terms of a solution of the field equations with incoherent matter as source. In the case of vanishing matter density, our result is a slight generalization of Harrison's work[1]; he established the result by assuming a diagonal (or diagonalizable) vacuum metric. However, our method is not suitable for generating electromagnetic solutions when the metric contains an asymmetry such as rotation in the absent coordinate. Further, the result obtained here holds for the static case only.

In Sec. II we set up the field equations with electromagnetic stress tensor and charged dust as source. It is then shown that the assumption of functional relationship, referred to above, is a consequence of the Bianchi identities. Then the main result of this paper is derived. Section III contains some concluding remarks.

[1] B. K. Harrison, Phys. Rev. **138**, B488 (1965); J. Math. Phys. **9**, 1744 (1968).

[2] M. Misra and L. Radhakrishna, Proc. Natl. Inst. Sci. India **A28**, 632 (1962). These authors had already stated the theorem, but had proved it for two variables only.

[3] H. Weyl, Ann. Physik **54**, 117 (1917); S. D. Majumdar, Phys. Rev. **72**, 390 (1947); W. B. Bonnor, Proc. Phys. Soc. (London) **A66**, 145 (1953); and Refs. 1 and 2.

II. FIELD EQUATIONS AND DERIVATION OF ELECTROMAGNETIC SOLUTIONS

We work with combined Einstein-Maxwell equations which are[4]

$$G_{ij}\equiv R_{ij}-\tfrac{1}{2}g_{ij}R=-8\pi T_{ij}\,, \tag{1}$$

$$F^{ij}{}_{;j}=4\pi J^i=4\pi\sigma V^i\,, \tag{2}$$

$$F_{[ij,k]}=0\,, \tag{3}$$

[4] The range of lower case Latin indices is from 0 to 3, whereas Greek indices take the values 1, 2, and 3. The summation convention is assumed throughout. A comma denotes partial differentiation and a semicolon denotes covariant derivative. [ ] is the symbol for skew symmetrization.

with

$$T_{ij}=\rho V_iV_j+(1/4\pi)(g^{ab}F_{ia}F_{jb}-\tfrac{1}{4}g_{ij}F_{ab}F^{ab}), \quad (4)$$

where ρ and σ are matter and charge densities, respectively. V^i is the four-velocity vector and may be chosen, in the present situation, as $V^\alpha=0$, $V^0=(g_{00})^{-1/2}$. Further, we choose the metric in the form

$$ds^2=e^{2u}dt^2+e^{-2u}\gamma_{\alpha\beta}dx^\alpha dx^\beta. \quad (5)$$

This metric is the most general static line element and admits a hypersurface-orthogonal Killing vector $\xi^i=\delta_0{}^i$. We make use of a comoving coordinate system and assume that the electromagnetic field is purely electrostatic, so that $F_{\alpha\beta}=0$, $F_{0\alpha}=\phi_{,\alpha}$, where ϕ is the electrostatic potential.

Now the field equations can easily be set up; one obtains

$$\Delta_2(U)=-e^{-2U}[4\pi\rho+\Delta_1(\phi)], \quad (6)$$

$$P_{\alpha\beta}=-2[U_{,\alpha}U_{,\beta}+e^{-2U}\phi_{,\alpha}\phi_{,\beta}], \quad (7)$$

where the differential parameters of first and second order are defined as[5]

$$\Delta_1(U)=\gamma^{\alpha\beta}U_{,\alpha}U_{,\beta}, \quad (8)$$

$$\Delta_1(U,V)=\gamma^{\alpha\beta}U_{,\alpha}V_{,\beta}, \quad (9)$$

$$\Delta_2(U)=\gamma^{\alpha\beta}U_{;\alpha\beta}=\gamma^{\alpha\beta}(U_{,\alpha\beta}-\Sigma_{\alpha\beta}{}^\lambda U_{,\lambda}). \quad (10)$$

$\Sigma_{\alpha\beta}{}^\lambda$ and $P_{\alpha\beta}$ are the Christoffel symbols and contracted curvature tensor, respectively, with respect to the three-space metric $\gamma_{\alpha\beta}$, satisfying

$$\gamma^{\alpha\beta}\gamma_{\alpha\lambda}=\delta_\lambda{}^\beta. \quad (11)$$

Further, the only nontrivial Maxwell equation in (2) and (3) yields

$$\Delta_2(\phi)-2\Delta_1(U,\phi)=4\pi\sigma e^{-U}. \quad (12)$$

One also obtains, by taking the divergence of (1) and making use of Eqs. (2)–(4),

$$V_{i;j}V^j=(\sigma/\rho)F_{il}V^l. \quad (13)$$

Equation (13) gives, in view of our assumptions regarding V^i and F_{ij},

$$e^U U_{,\alpha}=-(\sigma/\rho)\phi_{,\alpha}. \quad (14)$$

The above equation shows that U (or g_{00}), ϕ, and ρ/σ are functionally related. We shall make use of this fact to derive our desired result. We may proceed in two different ways; either we may write

$$U=U(\phi), \quad (15)$$

or we may assume the functional dependence between U and ϕ in the form

$$U=U(\theta), \quad \phi=\phi(\theta). \quad (16)$$

However, a more manageable form of the field equations is obtained if we write

$$V=e^U \quad (17)$$

and use the following form:

$$V=V(\theta), \quad \phi=\phi(\theta), \quad (18)$$

where $\theta=\theta(x^1,x^2,x^3)$. Now if one stipulates a functional relation of the form (15), one is led to the equality of charge and matter densities,

$$\sigma^2=\rho^2, \quad (19)$$

as a consequence of Eqs. (6) and (12).[6,7] However, we use Eqs. (18) along with (17) and obtain by substitution in Eqs. (6), (12), and (14)

$$V'\Delta_2(\theta)+\Delta_1(\theta)[V''-(V'^2-\phi'^2)/V]=-4\pi\rho/V, \quad (20)$$

$$\phi'\Delta_2(\theta)+\Delta_1(\theta)(\phi''-2V'\phi'/V)=4\pi\sigma/V, \quad (21)$$

$$V'=-(\sigma/\rho)\phi', \quad (22)$$

where the prime denotes differentiation with respect to θ. If we now assume that $\Delta_1(\theta)\neq0$ and make use of Eqs. (19) and (22) in Eqs. (20) and (21), we observe the interesting fact that the two equations become identical, provided that

$$V''=(V'^2-\phi'^2)/V, \quad (23)$$

$$\phi''=2V'\phi'/V. \quad (24)$$

The solution of these equations may be taken as

$$V=\lambda\ \mathrm{sech}\theta, \quad (25)$$

$$\phi=\lambda\ \tanh\theta, \quad (26)$$

where λ is a constant. Thus, in this case we have a nonlinear differential equation for θ, so that we may readily obtain a solution for Eqs. (6) and (12) with two arbitrary functions. We may mention the explicit solutions for the spherically symmetric case obtained by Bonnor.[8]

If we substitute Eqs. (25) and (26) into Eq. (7), we find, in view of Eq. (17), the equation

$$P_{\alpha\beta}=-2\theta_{,\alpha}\theta_{,\beta}. \quad (27)$$

But if we took the field equations with incoherent matter only, i.e., $\phi=0$, and put $V=e^\theta$, we would obtain Eq. (6) with the last term absent and Eq. (28). [Of course in (6) U is replaced by θ.] Thus, we have established the following result.

For every solution $V=e^\theta$ and $\gamma_{\alpha\beta}$ of the field equations with incoherent matter as source, we may also form a solution of the coupled Einstein-Maxwell equations with a nonvanishing electromagnetic field with the same $\gamma_{\alpha\beta}$ and the functions $V=\lambda\ \mathrm{sech}\theta$, $\phi=\lambda\ \tanh\theta$.

This result may be stated in an alternative simple

[5] L. P. Eisenhart, *Riemannian Geometry* (Princeton U. P., Princeton, N. J., 1949).

[6] A. Das, Proc. Roy. Soc. (London) **A267**, 1 (1962).

[7] U. K. De and A. K. Raychaudhuri, Proc. Roy. Soc. (London) **A303**, 97 (1968).

[8] W. B. Bonnor, Z. Physik **160**, 59 (1960); Monthly Notices Roy. Astron. Soc. **129**, 443 (1965).

form. Suppose that the metric

$$ds^2 = V^2 dt^2 + V^{-2}\gamma_{\alpha\beta}dx^\alpha dx^\beta, \tag{28}$$

with V and $\gamma_{\alpha\beta}$ functions of x^1, x^2, and x^3, satisfies the field equations with incoherent matter as source. Then the metric

$$ds^2 = \frac{4\lambda^2 V^2}{(V^2+1)^2}dt^2 + \frac{(V^2+1)^2}{4\lambda^2 V^2}\gamma_{\alpha\beta}dx^\alpha dx^\beta \tag{29}$$

and the potential

$$\phi = \lambda(V^2-1)/(V^2+1) \tag{30}$$

satisfy the Einstein-Maxwell equations.

III. CONCLUDING REMARKS

The immediate use of the present result is in the generation of electromagnetic metrics from known solutions of the field equations corresponding to nonempty spaces filled with dust. Further, from the correspondence of the metrics (28) and (29), we note that any singularity in $\gamma_{\alpha\beta}$ of one metric is also a singularity of the second metric. Also, if $V^2 \ll 1$, we observe that the singularities $V=0$ in Eq. (28) give the same singularities as in (29). However, if V is large, $V^2/(V^2+1)^2$ tends to zero and the two metrics behave differently with regard to their singularities. Thus, there is a one-to-one correspondence between the singularities of both metrics except when $|V|\to\infty$ in the nonempty space. This correspondence between the singularities of metrics (28) and (29) may have some relevance in the study of the problem of gravitational collapse.

In conclusion, we hope that the results of the present paper will lead to deeper understanding of gravito-electrodynamics in nonempty spaces.

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Relativistic Center-of-Mass Motion and the Electromagnetic Interaction of Systems of Charged Particles

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Effects arising from relativistic corrections to the c.m. motion for particles interacting with an external electromagnetic field and also for their interparticle interaction are considered. The Hamiltonian discribing the electromagnetic interaction to order v^2/c^2, including Coulomb, magnetic, and spin-orbit effects, between two slowly moving charged particles is derived and the correct choice of c.m. dynamical variables is shown to be different from the usual nonrelativistic form. With the modified treatment of the c.m. motion introduced by considerations of relativistic invariance, the Hamiltonian is put in a form which exhibits, except for one term, a clean separation of over-all c.m. motion and internal dynamics and, but for this term, is of the required relativistic form to order v^2/c^2. The extra term, which spoils the above results, is demonstrated to be removed by correct treatment of the Thomas precession of the internal orbital angular momentum. Alternatively, the desired c.m. separation can be achieved by a modification of the free c.m. relativistic variables induced by the interaction. The relativistic variables for an N-body system with arbitrary interaction are briefly considered.

I. INTRODUCTION

In order to discuss the dynamics of composite systems, it is essential to distinguish the kinematics of the motion of the system as a whole from the dynamics arising from forces between the constituent particles. Nonrelativistically, this is just the problem of the separation of c.m. motion, which is well understood for particles interacting through local potentials.

For slowly moving charged particles, the forces arising from electromagnetic interaction can be distinguished according to the order in v^2/c^2 (where v is the velocity of the moving particles) in which they arise. After the Coulomb forces the next most important effect, of first order in v^2/c^2, is the interaction due to the magnetic fields of each particle acting on the others.[1] While

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[1] A lucid discussion of the magnetic interaction between two spinless particles is given in E. Breitenberger, Am. J. Phys. **36**, 505 (1968). This paper also provides a useful selection of background references on many aspects on electromagnetic forces and slowly moving particles. For readers who appreciate a rather more discursive approach, we would strongly recommend digesting this paper first. The present article has been written so as to avoid unnecessary overlap.