refinement in the representation of the low-energy processes which seems likely to reproduce this postulated feature of the N_β Regge term. In the end this situation cannot be regarded as a limitation of the sum rules, since the high-energy data were represented exclusively by Regge poles. The inclusion of Regge cuts might sufficiently modify the u dependence of the N_β Regge term so as to displace or even eliminate the zero.

Regarding the convergence properties of the s-channel amplitudes near X_0, it should be noted that while the EXP amplitudes are to be preferred over the TH amplitudes, one could hope for better convergence properties then even the EXP amplitude possesses. However, the state of the description of the t-channel processes is so elementary that probably little can be accomplished by extending X_0 beyond the limit of available phase-shift analyses until higher t-channel states are taken into account.

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Continuous-Moment Sum Rules and Pion Photoproduction Data

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Continuous-moment sum rules with different weight functions are studied for the single-pion photoproduction process. Even granting the usual ambiguities on the high-energy side, it seems that presently available very low-energy data may be in considerable error.

A CLASS of dispersion sum rules called finite-energy sum rules[1] and their powerful generalizations, continuous-moment sum rules (CMSR),[2–5] have been studied extensively in recent years. The left-hand side of such sum rules contains integrals over purely imaginary parts or combinations of real and imaginary parts of the scattering amplitudes, multiplied by suitable weight functions. The right-hand side usually consists of contributions due to one or more Regge poles, Regge cuts, etc. The procedure that is usually adopted is to predict Regge trajectory parameters by substituting the available low-energy data. It is generally believed that the right-hand side has a number of ambiguities regarding the number of Regge poles, Regge cuts, nonleading poles, etc. Also, in the case of pion photoproduction, the energy up to which low-energy data are available may be too low for the Regge behavior to dominate. Thus it would seem that one can hardly make any statement about implications for the low-energy data from the analysis of the high-energy side. In the present note we suggest that, if one studies a number of sum rules weighting different parts of the low-energy data, one can draw some inferences about the low-energy data, even granting the ambiguities on the high-energy side. As a particular case, we have studied the single-pion photoproduction sum rules.

For the sake of definiteness, we will consider the two amplitudes

$$F_1^- = A_1^- - 2mA_4^- \quad \text{and} \quad F_2^-/(\mu^2 - t) = A_1^- + tA_2^- \tag{1}$$

in the usual notation for photoproduction processes.[6] It can be readily shown that the A_2 trajectory contributes to F_1^- and the π trajectory to F_2^-. Also the pion conspirator, if it exists, will contribute to F_1^- but the A_2 contribution will be dominant near $t=0$.

Then as shown in I and II, for both of these amplitudes one can get a CMSR of the form

$$\frac{1}{\pi}\int_{\nu_0}^{\bar\nu} d\nu(\nu^2-\nu_0^2)^{-\gamma}[\cos\pi\gamma\,\mathrm{Im}F(\nu,t) + \sin\pi\gamma\,\mathrm{Re}F(\nu,t)] + r(\nu_0^2-\nu_P^2)^{-\gamma} = -\frac{\alpha(t)\beta(t)\,\sin\pi(\frac{1}{2}\alpha-\gamma)}{2\pi\,\sin\frac{1}{2}\pi\alpha(\alpha-2\gamma)}\,\frac{(\bar\nu)^{\alpha-2\gamma}}{(\nu_1)^{\alpha-1}}. \tag{2}$$

Here we have

$$\nu=(s-u)/4m, \quad \nu_0=\mu+(t+\mu^2)/4m, \tag{3}$$

and γ is a number less than $+1$. We have assumed that

[1] Numerous references on finite-energy sum rules can be found in J. D. Jackson, Rev. Mod. Phys. **42**, 12 (1970). Application to pion photoproduction process can be found, for example, in S. Y. Chu and D. R. Roy, Phys. Rev. Letters **20**, 958 (1968); **21**, 57(E) (1968).

[2] Y. Liu and S. Okubo, Phys. Rev. Letters **19**, 190 (1967); M. G. Olsson, Phys. Letters **20B**, 310 (1960).

[3] K. V. Vasavada and K. Raman, Phys. Rev. Letters **21**, 577 (1968); K. Raman and K. V. Vasavada, Phys. Rev. **175**, 2191 (1968). These will be referred to as I and II, respectively, in the text.

[4] P. Di Vecchia *et al.*, Phys. Letters **27B**, 296 (1968); **27B**, 521 (1968); Nuovo Cimento **58A**, 532 (1969).

[5] Y. Liu and J. J. McGee, Phys. Rev. D **1**, 3123 (1970).

[6] J. S. Ball, Phys. Rev. **124**, 2014 (1961).

TABLE I. CMSR and MCMSR results for the F_1^- amplitude (A_2 trajectory). Natural units ($\hbar=c=\mu=1$) are used. The numbers correspond to the input data of Walker (Ref. 9). Changes due to the use of BDW data are discussed in the text.

	CMSR			MCMSR		
γ	Integral	Born term	$\alpha_{A_2}(0)$	Integral	Born term	$\alpha_{A_2}(0)$
0.75	2.80×10^{-1}	-6.78×10^{-1}	>1	0.58×10^{-2}	-2.53×10^{-2}	0.10
0.5	5.35×10^{-1}	-6.90×10^{-1}	>1	2.62×10^{-2}	-7.71×10^{-2}	0.27
0.0	3.28×10^{-1}	-7.15×10^{-1}	0.44	3.29×10^{-1}	-7.15×10^{-1}	0.42
-0.5	-3.70×10^{-1}	-7.41×10^{-1}	0.52	3.42	-6.63	0.36
-1.0	5.44	-7.68×10^{-1}	0.54	3.34×10^{1}	-6.15×10^{1}	0.23
-1.5	1.66×10^{1}	-7.96×10^{-1}	0.77	3.15×10^{2}	-5.71×10^{2}	0.08
-2.0	-3.75×10^{2}	-8.25×10^{-1}	0.71	2.92×10^{3}	-5.29×10^{3}	-0.05
-2.5	-1.76×10^{3}	-8.56×10^{-1}	0.62	2.67×10^{4}	-4.9×10^{4}	$\ll 0$
-3.0	$+2.10\times10^{4}$	-8.87×10^{-1}	0.60	2.44×10^{5}	-4.55×10^{5}	$\ll 0$

TABLE II. Same as in Table I for the F_2^- amplitude (π trajectory).

	CMSR			MCMSR		
γ	Integral	Born term	$\alpha_\pi(0)$	Integral	Born term	$\alpha_\pi(0)$
0.75	0.55×10^{-1}	-1.45×10^{-1}	>1	-0.73×10^{-3}	-5.40×10^{-3}	-0.01
0.5	0.84×10^{-1}	-1.47×10^{-1}	>1	-0.26×10^{-2}	-1.64×10^{-2}	-0.02
0.0	-0.26×10^{-1}	-1.53×10^{-1}	0.37	-0.27×10^{-1}	-1.52×10^{-1}	-0.02
-0.5	-8.03×10^{-1}	-1.58×10^{-1}	0.00	-0.26	-1.41	$+0.01$
-1.0	-0.84×10^{-1}	-1.64×10^{-1}	-0.04	-0.23×10^{1}	-1.31×10^{1}	$+0.015$
-1.5	$+2.55\times10^{1}$	-1.70×10^{-1}	-0.03	-0.22×10^{2}	-1.21×10^{2}	$+0.030$
-2.0	1.23×10^{1}	-1.76×10^{-1}	-0.045	-0.20×10^{3}	-1.12×10^{3}	$+0.045$
-2.5	-1.24×10^{3}	-1.82×10^{-1}	-0.040	-0.17×10^{4}	-1.05×10^{4}	$+0.050$
-3.0	-7.74×10^{2}	-1.89×10^{-1}	-0.055	-0.16×10^{5}	-9.71×10^{4}	$+0.070$

only one Regge pole contributes to the right-hand side. $\alpha(t)$ and $\beta(t)$ are the Regge-pole parameters. ν_1 is a scale factor chosen to be 1 GeV. $\bar{\nu}$ is the upper limit up to which the low-energy data are available. This corresponds to a photon lab momentum of about 1.2 GeV/c. The momentum transfer t is chosen to be zero throughout.[7] The r's represent the residues of the Born poles at $\nu=\nu_P=(-\mu^2+t)/4m$ and are given by

$$r_1=-(eG/4m)[1+2m(\mu_p-\mu_n)] \tag{4}$$

and

$$r_2=-eG/4m \tag{5}$$

for $t=0$. If I and $J\beta(t)$ stand, respectively, for the left- and the right-hand sides of Eq. (2), then as explained in II, one varies $\alpha(t)$ till $(\partial I/\partial\gamma)/I$ and $(\partial J/\partial\gamma)/J$ become equal. The value of $\beta(t)$ is then obtained from I/J. This procedure can be repeated for different values of γ. If the low-energy data are good and the high-energy one-pole approximation is good, the various values of $\alpha(t)$ and $\beta(t)$ should agree at least approximately. In II such a procedure was attempted for F_2^- and consistent results were obtained for the pion trajectory functions for $-2.5\leq\gamma\leq-1$. Note that higher values of γ (i.e., $-1<\gamma<1$) emphasize the low-energy part, whereas for $\gamma<-1$ the high-energy part of the integral is dominant. Thus by varying γ, various energy regions become dominant. At $\gamma=1$ the integral diverges.

[7] The point $t=0$ is slightly in the unphysical region but we have checked that the continuation does not cause any problems. In general, the $t=0$ sum rules are "cleaner" than the $t\neq0$ sum rules. Hence in the present work, we have considered just these sum rules.

In the above CMSR, the weight function multiplying the amplitude was $e^{i\pi\gamma}/(\nu^2-\nu_0^2)^\gamma$. Large number of sum rules can be written by varying this function. We will just consider one such sum rule, first discussed by Chan and Chavda.[8] This modified sum rule (MCMSR) employs the weight function $1/(\bar{\nu}^2-\nu^2)^\gamma$. The new sum rule then reads

$$\frac{1}{\pi}\int_{\nu_0}^{\bar{\nu}}\frac{\operatorname{Im}F(\nu)}{(\bar{\nu}^2-\nu^2)^\gamma}d\nu+\frac{r}{(\bar{\nu}^2-\nu_p^2)^\gamma} =-\frac{\beta(t)\alpha(t)(\bar{\nu})^{\alpha-2\gamma}}{4\pi(\nu_1)^{\alpha-1}}\frac{\Gamma(\frac{1}{2}\alpha)\Gamma(1-\gamma)}{\Gamma(\frac{1}{2}\alpha+1-\gamma)}. \tag{6}$$

This sum rule contains only the imaginary parts of the amplitude, which are usually better known than the real parts. Now, however, negative values of γ emphasize the low-energy part and the positive values emphasize the high-energy part of the integral. This is completely opposite to the situation encountered in the above case of CMSR. We have studied the two sum rules given by Eqs. (2) and (6) for the photoproduction amplitudes F_1^- and F_2^- in the one-Regge-pole approximation. The data used are taken from two sources. Multipoles up to photon lab energy of about 1.2 BeV are available from the phenomenological fit of Walker,[9] using Born terms, resonant terms, and certain free adjustable parameters. Berends, Donnachie, and Weaver (BDW)[10] have in-

[8] C. H. Chan and L. K. Chavda, Phys. Rev. Letters **22**, 1228 (1969).

[9] R. L. Walker, Phys. Rev. **124**, 2016 (1961).

[10] F. A. Berends, A. Donnachie, and D. L. Weaver, Nucl. Phys. **B4**, 1 (1968).

corporated constraints given by dispersion relations and give multipoles up to a photon lab energy of about 500 MeV. There are clear discrepancies between the two sets.[11] In general, however, one may hope that the imaginary parts are better determined than the real parts. The sum rule (6) has the advantage that it contains only the imaginary parts.

Using the method discussed above, we have determined values of the α's and β's for different values of the parameters γ.[12] The results are shown in Tables I and II. For comparison we also give values of the integral and the Born terms in each case.

First, we discuss the difference between the two sets (Walker and BDW). For the CMSR the two sets give practically identical results except for positive values of γ. This happens because in such cases the major contributions to the integrals come from the region above 500 MeV. For the case of MCMSR, the values of the integrals are noticeably different, the difference being of the order of about 5%. This changes the trajectory parameters by about 5–10%.

Now in the case of CMSR, fairly consistent results are obtained for both the pion and the A_2 trajectories for γ lying between -0.5 and -3. Fluctuation in α_{A_2} near $\gamma\approx-1.5$ may be due to inaccuracies in the real part of the amplitude. It is still not entirely unreasonable. In the case of F_2^-, the Born term dominates the integral for $-1<\gamma<0.5$. It has been suggested previously that, because of this fact, the pion sum rules cannot give any information about low-energy data.[13] This is, however, no longer true for $\gamma<-1$. For the case of F_1^-, the integral terms are comparable to or even dominate the Born terms for the range of γ. For $\gamma>-0.5$, the sum rules lead to rather poor values of the trajectory parameters. In such cases the very low-energy region (say less than 500 MeV) would make a very large contribution. This leads us to suspect that perhaps the data are not accurate in this region. This is confirmed by the results of the MCMSR.

As mentioned above, in the case of MCMSR, the situation is opposite to that of CMSR as regards weighting of the contributions from the various energy regions. For negative values of γ, the contributions from the very low-energy region are enhanced. For the pion case (F_2^-), the Born term dominates the integral for the entire range of γ. So the results for α_π are still not unreasonable. For the case of A_2 (F_1^-) this is no longer the case and the results become very poor as soon as one goes to values of $\gamma<-1$. These results along with those of CMSR suggest that perhaps even the imaginary parts of the very low-energy region have considerable errors. Unfortunately, unlike the case of π-N scattering, a detailed error analysis is not possible at present. It should be noted that our amplitudes are obtained by summing a number of multipole amplitudes. The errors on each of these have not been well determined yet. We note, however, that multipoles other than M_{1+} do make a substantial contribution to the integrals. These multipoles may be in error. For the case of CMSR, in spite of the fact that the real parts were needed, the results were reasonable. This can be understood from the fact that the very low-energy part of the integral did not make a very significant contribution.

It should be emphasized that the above conclusions are drawn regardless of the ambiguities of the right-hand side of the sum rules. As has been pointed out, the right-hand side can be fitted equally well by including the cut[14] or conspirator contributions. Near $t=0$ one can expect the A_2 to dominate over cuts or conspirator contributions. But whatever the case may be, the point here is not the prediction of a precise value of $\alpha(t)$ or $\beta(t)$. These may be just some effective trajectory parameters. Small variations in these may not be significant. The main point is the comparison of the results of different sum rules with different weight functions. If one set of sum rules gives reasonable results and the other equally valid set does not give reasonable results, then one can suspect the low-energy data. Such a comparison of different sum rules can then lead to a meaningful inferences.

[11] See, for example, Ref. 5, and G. C. Fox and D. Z. Freedman, Phys. Rev. **182**, 1628 (1969).

[12] In I, $\alpha_{A_2}(t)$ was obtained for a range of t by looking at the zeros of the integral.

[13] See, for example, Harari's remark quoted in Ref. 14.

[14] J. D. Jackson and C. Quigg, Phys. Letters **29B**, 236 (1969).