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Density-Matrix Formalism for Lepton-Hadron Scattering*

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A density-matrix approach is presented for lepton-hadron reactions. We give a unified treatment of electroproduction, neutrino-induced production, decay, and colliding-beam-type processes. In this formulation, the lepton reactions are treated in a manner completely analogous to the hadronic reactions involving decaying resonances. The general content of our result for the differential cross section can be stated as a product of finite-dimensional representations of the $SO(2,1)$ group or $SO(3)$ group for the lepton pair and a hadronic density matrix. The group changes from noncompact $SO(2,1)$ to compact $SO(3)$, depending upon whether the momentum of the lepton pair involved in the interaction is spacelike or timelike. Formulas are given for general lepton-induced reactions. Specific reactions (total inelastic lepton scattering and leptonic pion, ρ, and N^* production) are discussed in detail. In this formulation, the implications of Regge-pole theory, low-energy theory, partial conservation of axial-vector current, and current algebra can be easily studied.

I. INTRODUCTION

DURING the past several years a considerable amount of effort has been devoted to the study of strong interactions at high energies using the weak and electromagnetic interactions as a probe.[1] It is commonly assumed in a high-energy scattering process of a single lepton on a hadron that only the lowest nontrivial order of the weak and electromagnetic interaction is a good approximation. Within the context of this assumption and the presently accepted theories of weak and electromagnetic interactions, it is possible to obtain direct information about the matrix elements of the electromagnetic and weak currents between physical hadron states by studying such processes. In particular, electroproduction and neutrino-induced productions at high values of total energy and virtual photon mass have occupied an important role in such investigations recently. Examples of some of the work in this field have been Regge-pole models, parton models for the deep-inelastic electron scattering, and models based upon the algebra of currents and their time derivatives.[2]

Pais and Treiman[3] have recently made a study of the implications of Regge-pole theory on some particular high-energy lepton-induced reactions. It is the purpose of this paper to present a general formulation of the density matrix that is convenient to describe lepton-induced processes or, quite generally, processes involving weak or electromagnetic currents. We give a unified treatment of electroproduction, neutrono-induced production, decay, and colliding-beam-type processes. In this formulation the lepton reactions are treated in a manner completely analogous to the hadronic reactions involving decaying resonances. The general content of our result for the differential cross section can be stated as a product of finite-dimensional representations of the $SO(2,1)$ group or $SO(3)$ group for the lepton pair and a hadronic density matrix. The group changes from noncompact $SO(2,1)$ to compact $SO(3)$ depending upon whether the total momentum of the lepton pair involved in the interaction is spacelike or timelike. Several specific examples are worked out in detail. We make extensive use of the helicity formalism and some use of group-theory techniques in the subsequent development.

The organization of the paper is as follows: Sec. II containing the general results and their derivation; some discussion of the implications of the results is also presented. Section III contains some specific examples, like total inelastic lepton scattering, leptonic pion, ρ, and N^* production, and a Regge-pole model for these high-energy weak and electromagnetic processes. Most of the emphasis in this section is directed to reactions into definite final states rather than toward total virtual photon-hadron cross sections that have been considered extensively elsewhere.[1,2] Finally, we conclude with a section on the discussion of the results and the possible directions to be taken in the future. Some technical details are left to two Appendices.

II. LEPTON-HADRON SCATTERING

A. Preliminaries

In this subsection a discussion of kinematic preliminaries and conventions is presented. The inelastic

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[1] T. D. Lee and C. N. Yang, Phys. Rev. 126, 2239 (1962); S. L. Adler, *ibid.* 135, B963 (1964); S. Drell and J. Walecka, Ann. Phys. (N. Y.) 28, 18 (1964); L. Hand, in *Proceedings of the 1967 International Symposium on Electron and Photon Interactions at High Energies* (Stanford Linear Accelerator Center, Stanford, Calif., 1967).

[2] For specific references on all of the detailed models for lepton-hadron reactions see the review talk presented by F. J. Gilman, in Proceedings of the 1969 International Symposium on Electron and Photon Interactions at High Energies, SLAC Report No. SLAC PUB-074, 1969 (unpublished).

[3] A. Pais and S. B. Treiman, Phys. Rev. D 1, 907 (1970); A. Pais, *ibid.* 1, 1349 (1970); J. D. Bjorken and E. A. Paschos, *ibid.* 1, 3151 (1970); C. F. Cho and G. J. Gounaris, Phys. Rev. 186, 1619 (1969).

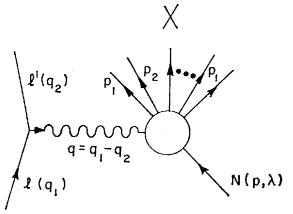

FIG. 1. Lepton scattering diagram.

amplitude for the reaction $l(q_1)+N(p,\lambda) \to l'(q_2')+X$, where l and l' are ν, e, or μ, and X is an arbitrary hadronic state, is given to lowest order in G_F or e^2 by the expression (Fig. 1)

$$M=(G_F/\sqrt{2})l_\mu(q_2,q_1)\langle X(P_x)|\mathcal{J}^\mu(0)|N(p,\lambda)\rangle. \quad (2.1)$$

In Eq. (2.1), $\mathcal{J}^\mu(0)$ is the Cabibbo current

$$\mathcal{J}_\mu(0)=\cos\theta_C\,[V_\mu{}^{I^+}(0)-A_\mu{}^{I^+}(0)] +\sin\theta_C\,[V_\mu{}^{K^+}(0)-A_\mu{}^{K^+}(0)], \quad (2.1')$$

$l_\mu(q_2,q_1)$ is the matrix element of the lepton current,

$$l_\mu(q_2,q_1)=\bar{u}(q_2)\gamma_\mu(1+\gamma_5)u(q_1)\,,\qquad q=q_1-q_2, \quad (2.2)$$

and θ_C is the Cabibbo angle; I^+ and K^+ denote the $SU(3)$ transformation properties of the weak current. Also, $G_Fm_p{}^2\simeq10^{-5}$ is the dimensionless Fermi constant.

In order to obtain the analogous expression for electron or muon production, we make the following replacements in Eq. (2.1):

$$G_F/\sqrt{2} \to e^2/q^2\,,\qquad l_\mu(q_2,q_1) \to \bar{u}(q_2)\gamma_\mu u(q_1)\,,\qquad \mathcal{J}_\mu(0) \to J_\mu{}^\gamma(0)\,. \quad (2.3)$$

For simplicity, in this part of the paper, we neglect the lepton mass. The results including the finite-mass terms are given in Sec. III and derived in Appendix B.

We now proceed with some details. The S matrix for the process $l(q_1)+N(p,\lambda) \to l'(q_2)+X$ as shown in Fig. 1 can be written

$$S_{fi}=-i(2\pi)^4\delta^4(P_f-P_i)\left(\frac{1}{EE'}\right)^{1/2}\left(\frac{1}{2\pi}\right)^3|M|^2. \quad (2.4)$$

The cross section for unpolarized incident protons and lepton spins summed can be written in the laboratory frame as

$$\frac{d\sigma}{d\nu dq^2d\psi}=\frac{1}{2E^2}\frac{1}{2\pi}Wd\Phi^{(x)}, \quad (2.5)$$

where $q^2=(q_1-q_2)^2\leq0$, $\nu=p\cdot q/m$, and ψ is defined later by Eq. (2.21), in terms of the customary laboratory quantities we have $q^2=-4EE'\times\sin^2(\frac{1}{2}\theta)$ and $\gamma=E-E'$. In Eq. (2.5), W is given by

$$W=\tfrac{1}{2}\sum_{\text{lepton spins}}|M|^2=\tfrac{1}{2}G_F{}^2\tau_{\mu\nu}W^{\mu\nu}, \quad (2.6')$$

where

$$\tau_{\mu\nu}=(q^2g_{\mu\nu}-q_\mu q_\nu+Q_\mu Q_\nu\pm i\epsilon_{\mu\nu\lambda\sigma}q^\lambda Q^\sigma)\,, \quad (2.6)$$

and

$$W_{\mu\nu}=\tfrac{1}{2}\sum_\lambda\langle X|\mathcal{J}_\mu(0)|N(p,\lambda)\rangle\langle X|\mathcal{J}_\nu(0)|N(p,\lambda)\rangle^*, \quad (2.7)$$

$$Q=q_1+q_2;$$

$d\Phi^{(x)}$ is the phase-space volume element for all of the final particles except the final lepton:

$$d\Phi^{(x)}=\prod_i\frac{d^3p_i}{E_i}\delta^4(P_x-p-q)\,. \quad (2.8)$$

In Eq. (2.8), P_x is the total momentum of the hadronic system X.

We list here some other conventions: $g_{\mu\nu}=(1,\,-1,\,-1,\,-1)$; $\bar{u}u=m$, $\bar{v}v=-m$, where u and v are the lepton spinors, and the states are normalized covariantly $\langle p'\lambda'|p\lambda\rangle=E\delta^3(\mathbf{p}-\mathbf{p}')$. Next a set of polarization vectors is given with the following properties and definitions:

$$g^{\mu\nu}=\sum_{m=-1}^{m=+1}(-)^m\epsilon_m{}^\mu(q)(\epsilon_m{}^\lambda)^*+\frac{q_\mu q_\nu}{q^2}\qquad\text{for}\quad q^2<0$$
$$=-\sum\epsilon_m{}^\mu(q)[\epsilon_m{}^\lambda(q)]^*+\frac{q_\mu q_\nu}{q^2}\qquad\text{for}\quad q^2>0, \quad (2.9)$$

where

$$\epsilon_m(q)=(-)^{m+1}(2)^{-1/2}(\hat{e}_x+im\hat{e}_y)\,,\qquad m=\pm1$$
$$\epsilon_0(q)=(-q^2)^{-1/2}(|\mathbf{q}|,q_0\hat{e}_z)\qquad\text{for}\quad q^2<0 \quad (2.10)$$
$$=(q^2)^{-1/2}(|\mathbf{q}|,q_0\hat{e}_z)\qquad\text{for}\quad q^2>0$$

in a frame where $\hat{q}=\hat{e}_z$.

The polarization vectors have the following properties:

$$\epsilon_m(q)\cdot q=0\,,$$
$$\epsilon_m{}^*(q)\epsilon_{m'}(q)=(-)^m\delta_{m,m'}\quad\text{for}\quad q^2<0 \quad (2.11)$$

and

$$\epsilon_m{}^*(q)\epsilon_m(q)=-\delta_{m,m'}\qquad\text{for}\quad q^2>0.$$

Now we insert the expansion of $g_{\mu\nu}$, Eq. (2.9), into Eq. (2.6), i.e., we project out the spin-one and spin-zero parts of the current in the helicity basis. For clarity we first neglect the leptonic mass (the detailed results including the mass term as given in Sec. III and derived in Appendix B); in this approximation the lepton current is conserved and only the spin-one part of the current services in Eq. (2.6). We obtain

$$W=\tfrac{1}{2}G_F{}^2\sum_{m,m'}(-)^{m+m'}\tau_{\mu\nu}\epsilon_m{}^\mu(\epsilon_{m'}{}^\nu)^*(\epsilon_m{}^{\mu'})^*\epsilon_{m'}{}^{\nu'}W_{\mu'\nu'}$$
$$\equiv\tfrac{1}{2}G_F{}^2\sum_{m,m'}\Phi_{mm'}\rho_{m'm}\,. \quad (2.12)$$

Here $\Phi_{m,m'}$ is the density matrix for the lepton pair and $\rho_{m'm}$ is the conventional helicity density matrix which can be expressed in terms of the helicity amplitudes describing the process $\mathcal{J}(q,m)+N(p,\lambda)\to X$ and is

$$\rho_{mm'}\equiv(-1)^{m+m'}(\epsilon_m{}^{\mu})^*\epsilon_{m'}{}^{\nu}W_{\mu\nu}. \tag{2.13}$$

All of the usual hadronic theories, i.e., Regge-pole theory, low-energy theorems, and vector dominance,[2] can be directly applied to $\rho_{mm'}$. The current is treated as a spin-one system of spacelike mass.

Notice that the normalization of the longitudinal polarization vector $\epsilon_0(q)$ for $q^2<0$ is different from that of the timelike case $q^2>0$. This question is relevant to the phase of the helicity density-matrix elements. With the phase convention of Eq. (2.10), the density matrix will have a phase given by the rules of strong-interaction dynamics. For example, in Regge-pole theory, the phase of ρ will be given by the signature factor of the exchanged Regge trajectories.

One can derive the leptonic density-matrix element $\Phi_{mm'}$ by calculating $\tau_{\mu\nu}\epsilon_m{}^{\mu}\epsilon_{m'}{}^{\nu*}$ directly. However, in Sec. II B we show that $\Phi_{mm'}$ is just a finite-dimensional representation of the $SO(2,1)$ group describing the decay of a particle with spin one and spacelike mass into a lepton pair. The $V-A$ form of the lepton current gives us the exact form of the leptonic density matrix. For a timelike current, $\Phi_{mm'}$ is given by a representation of $SO(3)$ describing a spin-one particle decaying into two spin-$\frac{1}{2}$ particles. The main point of this analysis is that the leptonic reactions are formulated in a manner completely analogous to the treatment of hadronic reactions.

B. Derivation and General Results

In this section we give a derivation of a general expression for the distribution function W for reactions involving currents and decaying hadronic systems. We first consider the distribution function for the lepton pair.

We will define the laboratory frame $\mathcal{L}$ for the process $\mathcal{J}(q)+N(p,\lambda)\to X$ as the frame where N is at rest and q is along the z axis. We always choose the normal to the production plane (or one of the production planes for nonplanar reactions) as the y axis. In the frame $\mathcal{L}$ we have

$$\begin{aligned} p&=m(1,0,0,0),\\ q&=(-q^2)^{1/2}(\sinh\alpha,0,0,\ \cosh\alpha), \end{aligned} \tag{2.14}$$

with

$$\alpha>0 \text{ since } P_x{}^0>[(p+q)^2]^{1/2}>0.$$

In this frame the vectors q_1 and q_2 are arbitrary, with the constraint

$$q_1-q_2=q \tag{2.15}$$

and

$$q_1{}^2=q_2{}^2=0. \tag{2.16}$$

Next we define a brick-wall frame R for spacelike q ($q^2<0$), where

$$\begin{aligned} p^R&=m(\cosh\alpha,\ 0,\ 0,\ -\sinh\alpha),\\ q^R&=(-q^2)^{1/2}(0,0,0,1). \end{aligned} \tag{2.17}$$

The frame R is reached from the laboratory $\mathcal{L}$ by a z boost of magnitude α. In the frame R the lepton momenta are

$$\begin{aligned} q_1{}^R&=(-\tfrac{1}{4}q^2)^{1/2}\\ &\quad\times(\cosh\xi,\ \sinh\xi\ \cos\psi,\ \sinh\xi\ \sin\psi,\ +1),\\ q_2{}^R&=(-\tfrac{1}{4}q^2)^{1/2}\\ &\quad\times(\cosh\xi,\ \sinh\xi\ \cos\psi,\ \sinh\xi\ \sin\psi,\ -1). \end{aligned} \tag{2.18}$$

The last statement follows from Eq. (2.16) and the fact that $(q_i{}^0)^2=(q_i{}^x)^2-(q_i{}^y)^2=(q_i{}^z)^2>0$; namely, we can reach another brick-wall frame L by an $O(2,1)$ transformation v that leaves q^R invariant and takes q_1 and q_2 to the zt plane. In the frame L, we have

$$\begin{aligned} q_1{}^L&=(-\tfrac{1}{4}q^2)^{1/2}(1,\ 0,\ 0,\ +1),\\ q_2{}^L&=(-\tfrac{1}{4}q^2)^{1/2}(1,\ 0,\ 0,\ -1),\\ q^L&=q^R. \end{aligned} \tag{2.19}$$

The $O(2,1)$ transformation v is a member of the little group of q and is parametrized by

$$v=v(\psi,\xi)=R_z(\psi)B_x(\xi). \tag{2.20}$$

In Eq. (2.20), ψ is the angle between the normal to the lepton plane and $\hat{e}_y$ (the normal to the production plane). It is given by

$$\cos\psi=(\hat{q}_1\times\hat{q}_2)\cdot\hat{e}_y. \tag{2.21}$$

Also, $\sinh\xi$ is related to the x, y components of the lepton momenta. Therefore the variable ψ and $\sinh\xi$ are invariant under z boosts. Thus $\sinh\xi$ can be evaluated in terms of the lepton momenta in any frame related to R by a z boost, for example, the s-channel c.m. frame $\mathcal{S}$ or the laboratory frame $\mathcal{L}$. In particular,

$$\begin{aligned} \sinh^2\xi&=(q_1{}^0q_2{}^0+q^2)/(q_0{}^2-q^2)\\ &\equiv q_1{}^0q_2{}^0[\cos^2(\tfrac{1}{2}\theta)](q_0{}^2-q^2)^{-1}, \end{aligned} \tag{2.22}$$

where $q_1{}^0$, $q_2{}^0$, and θ are the energy of the incoming lepton, the energy of the outgoing lepton, and the angle between the incoming and the outgoing leptons expressed in any of these frames. Finite-mass effects change Eq. (2.22); see Appendix B, Eq. (B21).

In the frame L, where the momenta q_1, q_2, and $q=q_1-q_2$ are all in the zt plane, it follows from $SO(2,1)$ (Appendix A) invariance that the lepton density matrix is

$$\Phi_{mm'}{}^L=\sum_{\sigma_1\sigma_2}a_{\sigma_1\sigma_2}{}^1(q^2)\delta_{m,\sigma_1+\sigma_2}\delta_{m',\sigma_1+\sigma_2}, \tag{2.23}$$

where m and m' are the helicities of the current as defined in Eq. (2.10), and σ_1 and σ_2 are the helicities of the incoming and outgoing leptons. The quantity $a_{\sigma_1\sigma_2}{}^1(q^2)$

is the square of the coupling of the current to the leptonic pair. The $V-A$ theory gives the values of $a_{\sigma_1\sigma_2}{}^1(q^2)$, which we give in Eq. (2.32). Next we go to the general frame R, which gives the most general distribution of the lepton pair. The transformation is just the $SO(2,1)$ transformation of Eq. (2.20). The $\Phi_{mm'}{}^L$ of Eq. (2.23) is transformed by the finite-dimensional representation of the $SO(2.1)$ group, i.e.,

$$\Phi_{m,m'}(v)=\sum_{\sigma_1\sigma_2} a_{\sigma_1\sigma_2}{}^1(q^2)\mathfrak{D}_{m,\sigma_1+\sigma_2}{}^{1*}(v)\mathfrak{D}_{m',\sigma_1+\sigma_2}{}^1(v)\,, \quad (2.24)$$

where

$$\mathfrak{D}_{m,m'}{}^1(v)=e^{i(m-m')\psi}\bar{d}_{mm'}{}^1(\xi)\,. \quad (2.25)$$

Under our convention for the $\epsilon_m{}^\mu$ of Eq. (2.10), the phases of the elements of $\bar{d}^J$ are given by

$$\bar{d}_{mm'}{}^J(\xi)=i^{|m|-|m'|}d_{mm'}{}^J(i\xi)\,, \quad (2.26)$$

where $d_{mm'}{}^J$ is the reduced-rotation matrix for the group $SO(3)$ as given by Rose.[4] Notice that $\bar{d}_{mm'}{}^J(\xi)$ is real and its symmetry properties can be derived from $d_{mm'}{}^J$. The total distribution function W is the lepton density matrix multiplied by the hadron density matrix and is given by

$$W=\tfrac{1}{2}G_F{}^2\sum_{m,m'}\Phi_{mm'}(v)\rho_{mm'}(q,p,\{p_i\})$$
$$\equiv\tfrac{1}{2}G_F{}^2\,\mathrm{Tr}[\Phi(v)\rho(q,p,\{p_i\})]\,, \quad (2.27)$$

where p_i is the momentum of the ith particle in X (Fig. 1) and $\{p_i\}$ is the set of p_i. Because of invariance of the helicity state m under z boosts, $\rho_{mm'}$ is invariant under a z boost. Therefore, the $\rho_{mm'}$ thus defined is also the s-channel density matrix, i.e., it can be expressed in terms of the s-channel helicity amplitudes.

The current J contains both vector and axial-vector current parts; therefore we can decompose the density matrix in the following way:

$$\rho_{mm'}{}^{s,+}(q,p,\{p_i\})=\sum_{\lambda,(\mu)} f_{(\mu),m'\lambda}{}^{s,V}f_{(\mu),m'\lambda}{}^{s,V*}(-1)^{m+m'}+\sum_{\lambda,(\mu)} f_{(\mu),m\lambda}{}^{s,A}f_{(\mu),m'\lambda}{}^{s,A*}(-1)^{m+m'}\,, \quad (2.28)$$

$$\rho_{mm'}{}^{s,-}(q,p,\{p_i\})=\sum_{\lambda,(\mu)} f_{(\mu),m\lambda}{}^{s,V}f_{(\mu),m'\lambda}{}^{s,A*}(-1)^{m+m'}+\sum_{\lambda,(\mu)} f_{(\mu),m\lambda}{}^{s,V*}f_{(\mu),m'\lambda}{}^{s,A}(-1)^{m+m'}\,, \quad (2.29)$$

where

$$f_{(\mu),m\lambda}{}^{sV}=\epsilon_m{}^\mu(q)\langle(X),(\mu)\,|\,V_\mu(0)\,|\,N(p,\lambda)\rangle\,,$$
$$f_{(\mu),m\lambda}{}^{sA}=\epsilon_m{}^\mu(q)\langle(X),(\mu)\,|\,A_\mu(0)\,|\,N(p,\lambda)\rangle\,,$$

and where (p_i) and (μ) denote the complete set of momentum variables and helicities for the final particles. Similarly, we can split the square of the coupling of the current to the lepton pair $a_{\sigma_1\sigma_2}{}^1(q^2)$ into two parts a^+ and a^- by

$$a_{\sigma_1\sigma_2}{}^1(q^2)=a_{\sigma_1\sigma_2}{}^{+,1}(q^2)+a_{\sigma_1\sigma_2}{}^{-,1}(q^2)\,, \quad (2.30)$$

where $a_{\sigma_1\sigma_2}{}^{+,1}(q^2)$ is given by the VV and the AA parts of the lepton current, and $a_{\sigma_1\sigma_2}{}^{-,1}(q^2)$ is given by the VA part. They satisfy the following properties under parity:

$$a_{\sigma_1\sigma_2}{}^{\pm,1}=\pm a_{-\sigma_1-\sigma_2}{}^{\pm,1}(q^2)\,. \quad (2.31)$$

In general there are four independent constants to specify a parity-nonconserving decay into a lepton pair.

So far the discussion is completely general. The specific form of the lepton current in Eq. (2.2) gives us the explicit form of the matrix $a_{\sigma_1\sigma_2}{}^{\pm,1}$. From Eqs. (2.6′) and (2.23) we obtain Φ in the frame L (see details in Appendix B):

$$\Phi_{mm'}{}^{L\pm}=\mp q^2(\delta_{m\,1}\delta_{m'\,1}\pm\delta_{m-1}\delta_{m'-1})\,, \quad (2.32)$$

where $+$ is for the $\bar{\nu}$ reaction and $-$ for the ν reaction. Therefore, we have

$$a_{\frac{1}{2},\frac{1}{2}}{}^{\pm,1}(q^2)=\mp q^2\,. \quad (2.33)$$

Similarly, $\Phi_{mm'}(v)$ in Eq. (2.24) can be separated into two parts,

$$\Phi_{mm'}{}^{\pm}(v)=\sum_{\sigma_1\sigma_2} a_{\sigma_1\sigma_2}{}^{\pm1}(q^2)\mathfrak{D}_{m,\sigma_1+\sigma_2}{}^{1*}(v)$$
$$\times\mathfrak{D}_{m',\sigma_1+\sigma_2}{}^1(v)\,. \quad (2.34)$$

For completeness we give the components of $\Phi_{mm'}{}^{\pm}$ explicitly:

Components of $(-q^2)^{-1}\Phi_{mm'}{}^{+}$ (2.35)

$m \backslash m'$	0	1	-1
0	$\sinh^2\xi$	$2^{-3/2}e^{-i\psi}\sinh 2\xi$	$-2^{-3/2}e^{i\psi}\sinh 2\xi$
1	$2^{-3/2}e^{i\psi}\sinh 2\xi$	$\frac{1}{2}(1+\cosh^2\xi)$	$-\frac{1}{2}e^{2i\psi}\sinh^2\xi$
-1	$-2^{-3/2}e^{-i\psi}\sinh 2\xi$	$-\frac{1}{2}e^{-2i\psi}\sinh^2\xi$	$\frac{1}{2}(1+\cosh^2\xi)$,

Components of $(-q^2)^{-1}\Phi_{mm'}{}^{-}$ (2.36)

$m \backslash m'$	0	1	-1
0	0	$-2^{-1/2}e^{-i\psi}\sinh\xi$	$-2^{-1/2}e^{i\psi}\sinh\xi$
1	$-2^{-1/2}e^{i\psi}\sinh\xi$	$\cosh\xi$	0
-1	$-2^{-1/2}e^{-i\psi}\sinh\xi$	0	$-\cosh\xi$

[4] M. E. Rose, *Elementary Theory of Angular Momentum* (Wiley, New York, 1957).

Using Eq. (2.22), the *nine independent real components of $\Phi^{\pm}$ can be easily expressed in terms of the laboratory quantities*.

Since parity is not conserved at either the lepton vertex or the hadron vertex in the reaction sketched in Fig. 1, we introduce discrete indices to label this parity nonconservation, i.e.,

$$W^{\delta\delta'}=\tfrac{1}{2}G_F{}^2\sum_{mm'}\Phi_{mm'}{}^{\delta}\rho_{mm'}{}^{\delta'}\,, \tag{2.37}$$

where δ and δ' are $\pm$.

All tests of the locality of the attachment of the lepton pair are contained in Eq. (2.37) and the explicit form Φ given by Eqs. (2.35) and (2.36).

The distribution functions for definite physical processes are given by

$$\begin{aligned} W_{\bar{\nu}p}&=\sum_{\delta=\pm} W_{(-)}{}^{+,\delta'}+\sum_{\delta'} W_{(-)}{}^{-,\delta'}\,,\\ W_{\nu p}&=\sum_{\delta'=\pm} W_{(+)}{}^{+,\delta'}-\sum_{\delta'} W_{(+)}{}^{-,\delta'} \end{aligned} \tag{2.38}$$

for neutrino reactions. The subscript $(\pm)$ refers to the isospin component of the current. For unpolarized electroproduction, we have

$$W_{ep\text{ or }\mu p}=W^{++}\,,\quad \text{with } \tfrac{1}{2}G_F{}^2\to e^4/q^4\,. \tag{2.39}$$

For polarized electroproduction, we have

$$W_{ep\text{ or }\mu p}=W^{++}\pm W^{-+}\,,\quad \text{with } \tfrac{1}{2}G_F{}^2\to e^4/q^4\,, \tag{2.39$'$}$$

where the first superscript $+$ is for right-handed e or μ; the $-$ is for left-handed e or μ.

We can now easily adapt our results to the colliding-beam process $e^-(q_1)+e^+(q_1)\to X$ and the lepton-pair production processes by replacing $\Phi_{mm'}{}^{\pm}(v)$ of Eq. (2.34) by $\Phi_{mm'}{}^{+}(R)$:

$$\Phi_{mm'}{}^{+}(R)\equiv\sum_{\sigma_1\sigma_2} a_{\sigma_1\sigma_2}{}^{+,1}(q^2)\mathfrak{D}_{m,\sigma_1+\sigma_2}{}^{1*}(R)\times\mathfrak{D}_{m',\sigma_1+\sigma_2}{}^{1}(R)\,, \tag{2.40}$$

where

$$\mathfrak{D}_{m,m'}{}^{1}(R)=e^{i(m-m')\psi}d_{mm'}{}^{1}(\theta) \tag{2.40$'$}$$

is the $SO(3)$ representation. Of course the q^2 now is timelike.

All of the results can be taken over to the case of transverse polarized real photons[5] by the following replacement in Eq. (2.34), in addition to the substitution $q^2=0$:

$$\Phi_{nn'}(v)\to\Phi_{nn'}{}^{\gamma}(\psi)=e^{-i(n-n')\psi}\,,\quad n,\,n'\neq 0$$

where ψ is the angle between the transverse photon polarization and the y axis. This result is entirely to be expected, since the little group shrinks to the little group of a light-like vector which in our case is the little group of the zt plane, $e^{-i\psi J_z}$, rotations about the z axis.

[5] R. L. Thews, Phys. Rev. **157**, 1749 (1968).

So far we have taken $\mathbf{q}$ to be along the z axis in the laboratory frame (nucleon rest system). If instead we take $\mathbf{q}$ to be along the z axis in the final boson rest frame, the previous discussion will go through in exactly the same way, with the exception that m is now the t-channel helicity and $\rho_{m,m'}$ is the t-channel density matrix which is expressed in terms of t-channel helicity amplitudes. The interpretation of Eq. (2.22) is also changed accordingly.

In the next part of this section we discuss the case where the final state X is a two-particle state with the quantum numbers of a single nucleon and a current. In particular, $X=A(k,\mu)+F(p',\lambda')$, where A is an unstable boson of momentum k and helicity μ and F is an unstable fermion (isobar) with momentum p' and helicity λ'. Subsequently, we will derive a general formula where the decay distributions of A and F can be studied.

We list the usual kinematic variables for the process $\mathcal{J}(q,m)+N(p,\lambda)\to A(k,\mu)+F(p',\lambda')$:

$$\begin{gathered} s=(p+q)=(k+p')^2\,,\\ \Delta=(k-p)^2\,,\quad t=(p-p')^2=(k-q)^2\,,\\ u=(p-k)^2=(p'-q)^2\,, \end{gathered} \tag{2.41}$$

and

$$m_A{}^2=k^2\,,\quad m_F{}^2=p'^2\,.$$

In the c.m. frame of the s channel, we have the reaction illustrated in Fig. 2, where $b_1(k_1)$ and $b_2(k_2)$ are the decay products of A $[A\to b_1(k_1)+b_2(k_2)]$ and $f_1(p_1)$ and $f_2(p_2)$ are the decay products of F $[F\to f_1(p_1)+f_2(p_2)]$. For simplicity we will assume a two-body decay mode; a more complicated mode can be treated by a repetition of the analysis presented

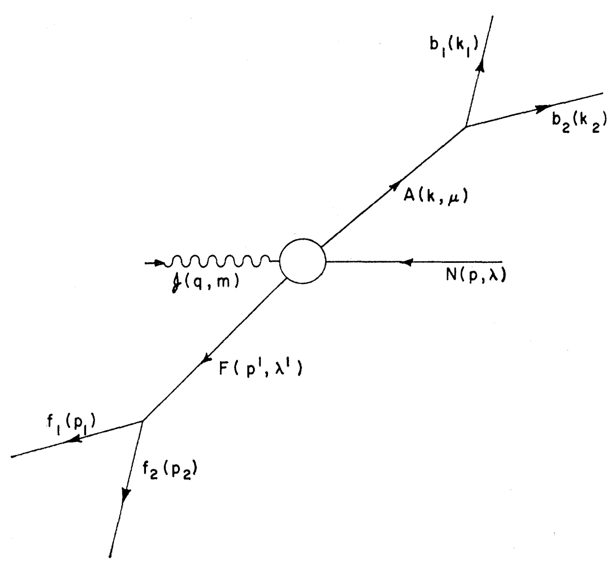

FIG. 2. Lepton scattering into a two-particle final state in the s channel.

here. Since most of this analysis is well known,[6] we present some details for completeness only.

The helicity amplitude for the process in Fig. 2 is

$$\langle b_1(k_1,\mu_1),b_2(p_2,\mu_2)| \otimes\langle f_1(p_1,\lambda_1),f_2(p_2,\lambda_2)|\mathcal{J}_\mu(0)|N(p,\lambda)\rangle\epsilon_m{}^\mu(q) = f_{m\lambda,\mu_1\mu_2\lambda_1\lambda_2}(k_1,k_2;\,p_1,p_2;\,q,p)\,, \quad (2.42)$$

where

$$q=k_1+k_2+p_1+p_2-p\,,$$
$$k=k_1+k_2\,,\quad p'=p_1+p_2\,.$$

We can specify each system of particles b_1+b_2 and f_1+f_2 by a pair of Lorentz transformations g_k and $g_{p'}$ of the form

$$g_k=B_{\mathbf{k}}(\alpha_k)u_k\,, \quad (2.43)$$

where $m_A\cosh\alpha_k$ is the energy corresponding to k, and $\hat{k}$ is a unit vector in the direction of $\mathbf{k}$ in some fixed reference system; u_k is an element of the three-parameter rotation group that defines the orientation of the momenta of b_1 and b_2 in their c.m. frame, i.e., also the rest system of A. The Lorentz transformation $B_{\mathbf{k}}(\alpha_k)$ is a pure boost along the direction of $\hat{k}$. The four-vector k is related to g_k by

$$k=g_kk_r=B_{\mathbf{k}}(\alpha_k)k_r\,, \quad (2.44)$$

with $k_r=m_A(1,0,0,0)$ and

$$|b_1(k_1),b_2(k_2)\rangle=B_{\hat{\mathbf{k}}}(\alpha_k)u_k|b(k_1{}^0),b_2(k_2{}^0)\rangle\,.$$

We choose u_k such that $k_1{}^0$ and $k_2{}^0$ are along the direction of $\hat{k}$ in the rest system of A. A completely analogous set of equations exists for the fermion with momentum P'. Next we insert a complete set of helicity states for $|A(k,\mu)\rangle$,

$$|A(k,\mu)\rangle=B_{\mathbf{k}}(\alpha_k)|A(k_r,\mu)\rangle\,, \quad (2.45)$$

where $|A(k_r,\mu)\rangle$ is quantized along the direction of $\hat{k}$ in its rest system $\mathbf{J}\cdot\hat{k}|A(k_r,\mu)\rangle=\mu|A(k_r,\mu)\rangle$. We obtain from Eq. (2.42).

$$f_{m\lambda\mu_1\mu_2\lambda_1\lambda_2}=\sum_{\lambda'\mu}\mathfrak{D}_{\mu\mu_1-\mu_2}{}^{s_A}(u_k)c_{\mu_1\mu_2}{}^{s_A}\mathfrak{D}_{\lambda'\lambda_1-\lambda_2}{}^{s_F}(u_{p'})c_{\lambda_1\lambda_2}{}^{s_F}$$
$$\times\langle B(k,\mu)|\otimes\langle F(p',\lambda')|\mathcal{J}_\alpha(0)|N(p,\lambda)\rangle\epsilon_m{}^\alpha(2)\,. \quad (2.46)$$

In Eq. (2.46), s_A and s_F are the spins of A and F, respectively, and μ_1, μ_2, and λ_1, λ_2 are the helicities of the decay products of A and F, respectively. The quantities $c_{\mu_1\mu_2}{}^{s_A}$ and $c_{\lambda_1\lambda_2}{}^{s_F}$ are the decay amplitudes for the processes $A(k_r,\mu)\to b_1(k_1{}^0)+b_2(k_2{}^0)$ and $F(p_r',\lambda')\to f_1(p_1{}^0)+f_2(p_2{}^0)$, respectively.

The distribution function W for the process in Fig. 2 then becomes, from Eqs. (2.37) and (2.46), after summing over $\mu_1\mu_2\lambda_1\lambda_2$,

$$W^{\delta\delta'}(u_k,u_p,v)=\tfrac{1}{2}G_F{}^2\sum_{\mu\mu',\lambda\lambda'}X_{\mu\mu'}(u_k)X_{\lambda\lambda'}(u_{p'}) \times\Phi_{mm'}{}^{\delta}(v)\rho_{\mu\mu',\lambda\lambda',mm'}{}^{\delta'}(q^2,s,t)$$
$$\equiv\tfrac{1}{2}G_F{}^2\,\mathrm{Tr}[X(u_k)X(u_{p'}) \times\Phi^\delta(v)\rho(q^2,s,t)]\,, \quad (2.47)$$

where

$$\rho_{\mu\mu',\lambda\lambda',mm'}(q^2,s,t) =\sum_{\lambda''}f_{\mu\lambda,m\lambda''}f_{\mu'\lambda',m'\lambda''}{}^*(-1)^{m+m'}\,, \quad (2.48)$$

and

$$X_{\mu\mu'}(u_k)=\sum_{\mu_1\mu_2}a_{\mu_1\mu_2}{}^{s_A}\mathfrak{D}_{\mu\mu_1-\mu_2}{}^{s_A}(u_k)\mathfrak{D}_{\mu'\mu_1-\mu_2}{}^{s_A*}(u_k)\,,$$
$$X_{\lambda\lambda'}(u_p)=\sum_{\lambda_1\lambda_2}a_{\lambda_1\lambda_2}{}^{s_F}\mathfrak{D}_{\lambda\lambda_1-\lambda_2}{}^{s_F}(u_{p'}) \times\mathfrak{D}_{\lambda'\lambda_1-\lambda_2}{}^{s_F}(u_{p'})\,. \quad (2.49)$$

Here $a_{\lambda_1\lambda_2}{}^{s}=|c_{\lambda_1\lambda_2}{}^{s_F}|^2$ is the decay density-matrix element and $c_{\lambda_1\lambda_2}{}^{s}$ is the decay amplitude.

From Eqs. (2.46)–(2.49), ρ is the s-channel density matrix since the spin quantization axes of the boson A, the fermion F, and the current $\mathcal{J}$ are chosen to be along the direction $\hat{p}'$, $\hat{k}$, and $\hat{p}$, respectively, in their "rest" frames. The matrix ρ is the t-channel density matrix if the spin quantization axes of the boson A, the fermion F, and the current $\mathcal{J}$ are chosen to be along the directions, $\hat{q}$, $\hat{p}$, and $\hat{k}$, respectively, in their "rest" frames. Once the coordinate systems are chosen, the interpretation of u_k, $u_{p'}$, and v are defined in their respective rest frames.

The result of Eq. (2.47) can be generalized to describe reactions involving many decaying hadrons and currents.

The general expression is simply

$$W^{\delta,\delta'}(\{u\},\{v\})=\tfrac{1}{2}G_F{}^2\,\mathrm{Tr}X(\{u\})\otimes\Phi^\delta(\{v\})\rho^{\delta'}\,, \quad (2.50)$$

where

$$X(\{u\})\equiv\prod_d^{n_d}X_d(u_d)\,, \quad (2.51)$$

and

$$\Phi^\delta(\{v\})=\prod_c^{n_c}\Phi_c{}^{\delta_c}(v_c)\,,\qquad \delta=\prod_c^{n_c}\delta_c\,. \quad (2.52)$$

III. APPLICATIONS AND SPECIFIC EXAMPLES

A. Consequences of Hermiticity and Parity

In the previous section, from Eqs. (2.48)–(2.52), the total probability or distribution function W was ex-

[6] For a general discussion of the density-matrix formulation for hadronic processes, see K. Gottfried and J. D. Jackson, Nuovo Cimento **33**, 309 (1964); H. Pilkuhn and B. E. Y. Svensson, *ibid.* **38**, 518 (1965).

pressed in terms of the decay matrix X and the lepton density matrix Φ by

$$\begin{aligned} W^{\delta\delta'}(\{\varphi,\theta\};\psi,\xi) &= \tfrac{1}{2}G_F{}^2\,\mathrm{Tr}[X\otimes\Phi^\delta\cdot\rho^{\delta'}] \\ &= \tfrac{1}{2}G_F{}^2 \sum_{mm'(\mu)(\mu')} X_{(\mu)(\mu')}(\{\varphi,\theta\})\Phi_{mm'}{}^\delta(\psi,\xi)\rho_{(\mu)(\mu'),mm'}{}^{\delta'} \\ &= \tfrac{1}{2}G_F{}^2 \sum_{mm',(\mu)(\mu')} \exp i[(m-m')\psi+\sum_d(\mu_d-\mu_d')\varphi_d] \\ &\qquad \times \chi_{(\mu)(\mu')}(\{\theta\})\tau_{mm'}{}^\delta(\xi)\rho_{(\mu)(\mu'),mm'}{}^{\delta'}, \end{aligned} \tag{3.1}$$

where the group elements u and v in Eq. (2.45) have been replaced by polar and azimuthal angles through the decomposition $u=e^{-i\varphi J_z}e^{-i\theta J_y}$, and $v=e^{-i\psi J_z}e^{-i\xi K_x}$. The notation (φ,θ), (μ), etc. refers to the polar and azimuthal angles and the helicities of all of the decaying particles collectively. The τ and χ are defined as follows:

$$\chi_{(\mu)(\mu')}(\{\theta\})=\prod_d \sum_{\mu_1\mu_2} a_{\mu_1\mu_2} d_{\mu_d,\mu_1-\mu_2}{}^{s_d}(\theta_d) \times d_{\mu_d,\mu_1-\mu_2}{}^{s_d}(\theta_d), \tag{3.2}$$

$$\begin{aligned} \tau_{mm'}{}^\delta &= \sum_{\sigma_1\sigma_2} a_{\sigma_1\sigma_2}{}^{\delta,1}\bar{d}_{m\sigma_1+\sigma_2}{}^1(\xi)\bar{d}_{m',\sigma_1+\sigma_2}{}^1(\xi), \\ &\equiv e^{i(m-m')\psi}\Phi_{mm'}{}^\delta(\psi,\xi). \end{aligned} \tag{3.3}$$

In the following we shall list all the symmetry properties of X, Φ, and ρ due to parity and Hermiticity.

From Eqs. (2.34), (2.49), (3.2), and (3.3), we can easily obtain the following relations under parity:

$$\begin{aligned} \chi_{(\mu)(\mu')}(\{\theta\}) &= (-)^{\Sigma_d(\mu_d-\mu_d')}\chi_{(-\mu)(-\mu')}(\{\theta\}), \\ X_{(\mu)(\mu')}(\{\varphi,\theta\}) &= (-)^{\Sigma_d(\mu_d-\mu_d')}X_{(-\mu)(-\mu')}(\{-\varphi,\theta\}), \\ \tau_{mm'}{}^\delta(\xi) &= \delta(-)^{m-m'}\tau_{-m-m'}{}^\delta(\xi), \end{aligned} \tag{3.4}$$

and

$$\Phi_{mm'}{}^\delta(\psi,\xi)=\delta(-)^{m-m'}\Phi_{-m-m'}{}^\delta(-\psi,\xi),$$

where $\{\theta\}$ denotes the set of θ_d's. Under Hermiticity,

$$X_{(\mu)(\mu')}{}^*(\{\varphi,\theta\})=X_{(\mu')(\mu)}(\{\varphi,\theta\})$$

and (3.5)

$$\Phi_{mm'}{}^{\delta *}(\psi,\xi)=\Phi_{m'm}{}^\delta(\psi,\xi),$$

and the matrices τ and χ are real and symmetric.

Combining Eqs. (3.4) and (3.5), we obtain

$$\begin{aligned} X_{(\mu)(\mu)}(\{\theta,\varphi\}) &= \text{real}, \\ \Phi_{mm}{}^+(\psi,\xi) &= \text{real}, \end{aligned} \tag{3.6}$$

and

$$\Phi_{m-m}{}^-(\psi,\xi)=0, \qquad \text{including } m=0.$$

The parity and Hermiticity properties of the density matrix ρ are as follows: For production reactions involving a four-point function, the density matrix has the following simple property under parity:

$$\rho_{(\mu)(\mu'),mm}{}^{\delta'}=\delta'(-1)^{m-m'}(-1)^{\Sigma_d(\mu_d-\mu_d')} \times\rho_{(-\mu)(-\mu');-m-m'}{}^{\delta'}. \tag{3.7}$$

For reactions involving more than four-point functions, the reaction will not take place in a single plane. The parity rule, Eq. (3.7), is still true with the additional requirement that all angles specifying different planes in the reaction must change sign on the left-hand side of Eq. (3.7). Under Hermiticity,

$$\rho_{(\mu)(\mu'),mm'}{}^{\delta'}=(\rho_{(\mu')(\mu),m'm}{}^{\delta'})^*. \tag{3.8}$$

Combining Eqs. (3.7) and (3.8), we have

$$\begin{aligned} \rho_{(\mu)(\mu');m,-m}{}^\delta &= \text{real}, \\ \rho_{(\mu)(-\mu);m,-m}{}^+ &= \text{real, including } m=\mu=0, \\ \rho_{(\mu)(-\mu);m,-m}{}^- &= \text{imaginary}, \end{aligned} \tag{3.9}$$

and

$$\rho_{(0)(0);00}{}^-=0.$$

From these properties, the distribution functions have the property

$$W^{\delta\delta'}(\{\varphi,\theta\},\psi,\xi)=\delta\delta' W^{\delta\delta'}(\{-\varphi,\theta\};-\psi,\xi). \tag{3.10}$$

Combining this property with the reality property of W,

$$W^{\delta\delta' *}(\{\varphi,\theta\};\psi,\xi)=W^{\delta\delta'}(\{\varphi,\theta\},\psi,\xi), \tag{3.11}$$

we can easily obtain

$$\begin{aligned} W^{\delta\delta}(\{\varphi,\theta\};\psi,\xi) &= \mathrm{Tr}[\mathrm{Re}(X\otimes\Phi^\delta)\mathrm{Re}\rho^\delta] \\ &\equiv \sum_{mm'(\mu)(\mu')} \cos[\sum_d(\mu_d-\mu_d')\varphi_d+(m-m')\psi] \\ &\qquad \times\chi_{(\mu)(\mu')}(\{\theta\})\tau_{mm'}{}^\delta(\xi)\,\mathrm{Re}\rho_{(\mu)(\mu')mm'}{}^\delta \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} W^{\delta-\delta}(\{\varphi,\theta\};\psi,\xi) &= \mathrm{Tr}[\mathrm{Im}(X\otimes\Phi^\delta)\,\mathrm{Im}\rho^{-\delta}] \\ &\equiv \sum_{mm'(\mu)(\mu')} \sin[\sum_d(\mu_d-\mu_d')\varphi_d+(m-m')\psi] \\ &\qquad \times\chi_{(\mu)(\mu')}(\{\theta\})\tau_{mm'}{}^\delta\,\mathrm{Im}\rho_{(\mu)(\mu')mm'}{}^{-\delta}. \end{aligned} \tag{3.13}$$

Thus we see that the cosine always accompanies the real part of ρ and that the sine always accompanies the imaginary part of ρ.

Owing to the symmetry relation given by Eqs. (3.4)–(3.9), not all of the matrix elements of X, Φ, and ρ are independent. The matrices are folded along the two diagonals of the matrix. We shall count the number of independent matrix elements under these symmetries.

The dimensionality of ρ is given by

$$(2s+1)\equiv(2s_c+1)\prod_d(2s_d+1), \tag{3.14}$$

where s_c is the spin of the current, which is 1 in our present case.

Parity and Hermiticity give us a folding of the density matrix about the cross diagonal and main diagonal lines. For a $(2s+1)\times(2s+1)$ matrix satisfying Eqs. (3.7)–(3.9), the number of independent quantities is given by

$$\begin{aligned}
R_{\rho^+} &= \text{No. of independent components of } \mathrm{Re}\rho^+ \\
&= \tfrac{1}{4}[(2s+1)^2+2(2s+1)+\eta], \\
I_{\rho^+} &= \text{No. of independent components of } \mathrm{Im}\rho^+ \\
&= \tfrac{1}{4}[(2s+1)^2-2(2s+1)+\eta];
\end{aligned}$$

similarly, (3.15)

$$\begin{aligned}
R_{\rho^-} &= \text{No. of independent components of } \mathrm{Re}\rho^- \\
&= \tfrac{1}{4}[(2s+1)^2-\eta]; \\
I_{\rho^-} &= \text{No. of independent components of } \mathrm{Im}\rho^- \\
&= \tfrac{1}{4}[(2s+1)^2-\eta],
\end{aligned}$$

with

$$\begin{aligned}
\eta &= 0 \quad \text{if} \quad (2s+1) = \text{even integer} \\
&= 1 \quad \text{if} \quad (2s+1) = \text{odd integer}.
\end{aligned}$$

A similar discussion can be applied to $\Phi\otimes X$. We first discuss the case where several weak currents are involved.

For a $(2s+1)\times(2s+1)$ matrix with the properties given by Eqs. (3.4) and (3.6), the number of independent quantities is

$$\begin{aligned}
R_{\Phi^+} &= \text{No. of independent components of } \mathrm{Re}\Phi^+ \\
&= \tfrac{1}{4}[(2s+1)^2+2(2s+1)+\eta], \\
I_{\Phi^+} &= \text{No. of independent components of } \mathrm{Im}\Phi^+ \\
&= \tfrac{1}{4}[(2s+1)^2-\eta];
\end{aligned}$$

similarly, (3.16)

$$\begin{aligned}
R_{\Phi^-} &= \tfrac{1}{4}[(2s+1)^2-\eta], \\
I_{\Phi^-} &= \tfrac{1}{4}[(2s+1)^2-2(2s+1)+\eta].
\end{aligned}$$

Of course $\eta=1$ for currents.

For a reaction with a single current, $s=s_c=1$. Equation (3.16) also applies for the case where Φ is the leptonic density matrix for more than one current, all of which do not conserve parity; then $2s+1\equiv\prod_c(2s_c+1)$. We see that

$$R_{\Phi^+}=R_{\rho^+},\ I_{\Phi^-}=I_{\rho^+},\ R_{\Phi^-}=R_{\rho^-},\ \text{and}\ I_{\Phi^+}=I_{\rho^-}, \tag{3.17}$$

i.e., the number of independent density-matrix ρ elements matches the number of independent lepton density-matrix elements. In this case, from Eqs. (3.12) and (3.13), all of the independent elements of ρ can be measured by untangling the independent kinematic factors in the distribution function W.

For the case of one current without the measurement of the decay of a final hadronic system, we have

$$R_{\Phi^+}=R_{\rho^+}=4,\quad I_{\Phi^-}=I_{\rho^+}=2,\quad R_{\Phi^-}=R_{\rho^-}=1,$$

and (3.18)

$$I_{\Phi^+}=I_{\rho^-}=2.$$

Therefore, all of the nine independent elements of $\rho_{mm'}{}^{\pm}$ can be measured.

Next we discuss the counting for the parity-conserving decay of strongly interacting systems. For each parity-conserving decay, the number of independent quantities contained in the decay matrix X_d is given by

$$\begin{aligned}
R_{X_d} &= \text{No. of independent components of } \mathrm{Re}X_d \\
&= \tfrac{1}{4}[(2s_d+1)^2+2(2s_d+1)+\eta_d], \\
I_{X_d} &= \text{No. of independent components of } \mathrm{Im}X_d \\
&= \tfrac{1}{4}[(2s_d+1)^2-\eta_d].
\end{aligned} \tag{3.19}$$

Here $\eta_d=1$ for $2s_d+1$ odd, and $\eta_d=0$ for $2s+1$ even. For more than one unstable system,

$$R_X=\sum_{\text{all } n''\text{ even}}\Big(\prod_d^{n'}R_{X_d}\Big)\Big(\prod_d^{n''}I_{X_d}\Big); \tag{3.20}$$

here $\sum_{\text{all even } n''}$ means the sum of all terms with n'' = even integer, $n'+n''$ = total number of strong decays studied, and

$$I_X=\sum_{\text{all odd } n''}\Big(\prod_d^{n'}R_{X_d}\Big)\Big(\prod_d^{n''}I_{X_d}\Big). \tag{3.20$'$}$$

Notice that there are no X^- terms; therefore, from Eqs. (3.12) and (3.13), no imaginary parts of the density matrix ρ can be measured if all of the decays conserve parity. Also notice that for more than one strong decay, not even all of the real parts of the density matrix can be measured since $R_\rho>R_X$ in Eqs. (3.15) and (3.20) with $s_c=0$.

Now for the most general case where both parity-conserving and parity-nonconserving decays are present, we can combine the results of Eqs. (3.16), (3.19), (3.20), and (3.20′) and obtain

$$\begin{aligned}
R_{X\otimes\Phi^\pm} &\equiv \text{No. of independent components of } \mathrm{Re}(X\otimes\Phi^\pm) \\
&= \Big(\sum_{\text{all even } n''}\prod_d^{n'}R_{X_d}\prod_{d'}^{n''}I_{X_{d'}}\Big)R_{\Phi^\pm} \\
&\quad +\Big(\sum_{\text{all odd } n''}\prod_d^{n'}R_{X_d}\prod_{d'}^{n''}I_{X_{d'}}\Big)I_{\Phi^\pm},
\end{aligned} \tag{3.21}$$

where $n'+n''$ = total number of parity-conserving decays;

$$\begin{aligned}
I_{X\otimes\Phi^\pm} &= \text{No. of independent components of } \mathrm{Im}(X\otimes\Phi^\pm) \\
&= \Big(\sum_{\text{all even } n''}\prod_d^{n'}R_{X_d}\prod_{d'}^{n''}I_{X_{d'}}\Big)I_{\Phi^\pm} \\
&\quad +\sum_{\text{all odd } n''}\Big(\prod_d^{n'}R_{X_d}\prod_{d'}I_{X_{d'}}\Big)R_{\Phi^\pm}.
\end{aligned} \tag{3.21$'$}$$

However, if the polarizations of the decay products are observed, an effect equivalent to parity nonconservation is introduced and all of the independent elements of ρ can again be obtained.

Finally, from the use of parity and Hermiticity, we can recast Eqs. (3.12) amd (3.13) into the form where only the independent elements of ρ appear:

$$W^{\delta\pm\delta}(\{\varphi,\theta\},\psi,\xi)=\sum_{\mu'\geq|\mu|;m'\geq|m|}N_{(\mu)(\mu')mm'}{}^{\delta\pm\delta}$$

$$\times\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}\Big[\sum_d(\mu_d-\mu_d')\varphi_d+(m-m')\psi\Big]$$

$$\times\chi_{(\mu)(\mu')}(\{\theta\})\tau_{mm'}{}^{\delta}\rho_{(\mu)(\mu')mm'}{}^{\pm\delta}. \quad (3.22)$$

In Eq. (3.22), all m and μ are summed over; the summation over m' and μ' is restricted as we have indicated. The quantities N are weighting factors due to the folding of the density matrix and are given by

$$\begin{aligned}
N_{(\mu)(\mu')mm'}{}^{++}&=2\,,\quad \mu=\mu'\,,\quad m=m'\,,\quad \mu=\mu'=m=m'=0\,,\\
&=2\,,\quad \mu=-\mu'\,,\quad m=-m'\,,\\
&=4\,,\quad \text{otherwise;}\\
N_{(\mu)(\mu')mm'}{}^{-+}&=0\,,\quad \mu=\pm\mu'\,,\quad m=\pm m'\,,\\
&=4\,,\quad \text{otherwise;}\\
N_{(\mu)(\mu')mm'}{}^{--}&=2\,,\quad \mu=\mu'\,,\quad m=m'\,,\\
&=0\,,\quad \mu=-\mu'\,,\quad m=-m'\,, \qquad (3.22')\\
&\qquad\qquad \mu=\mu'=m=m'=0\\
&=4\,,\quad \text{otherwise;}\\
N_{(\mu)(\mu')mm'}{}^{+-}&=0\,,\quad \mu=\mu'\,,\quad m=m\,,\quad \text{and}\\
&\qquad\qquad \mu=\mu'=m=m'=0\,,\\
&=2\,,\quad \mu=-\mu'\,,\quad m=-m'\,,\\
&=4\,,\quad \text{otherwise.}
\end{aligned}$$

In Eq. (3.22) the number of density-matrix elements has been reduced from Eqs. (3.12) and (3.13).

B. Specific Reactions

In this subsection we apply our general results given in the last section to a few specific reactions. We also include the terms due to finite lepton mass explicitly. If finite-mass terms are explicitly retained, the lepton current is no longer conserved and the lepton density matrix has both spin-one and spin-zero parts. The calculation of the terms due to finite mass is straightforward and we leave the details for Appendix B.

1. $l(q_1)+N(p,\lambda)\to l'(q_2)+X(P_x)$, total cross section

For this type of reactions only the momenta of the initial and the final leptons are measured. Therefore, the formula for these reactions is given by Eq. (3.22) with all of the variables integrated over except θ.

(a) *Inelastic unpolarized electroproduction or muon-induced production,*

$$l=l'=e \text{ or } \mu\,,$$
$$m_l=m_{l'}=m=m_e \text{ or } m_\mu\,.$$

For these reactions only the vector current is involved; therefore, the only nonvanishing contribution to W is W^{++}:

$$W(\xi,q^2,s)\equiv W^{++}(\xi,q^2,s)$$
$$=\frac{e^4}{q^4}\sum_m\tau_{mm}{}^{+}(\xi)\rho_{mm}{}^{+}(q^2,s)\,. \quad (3.23)$$

From Eqs. (B11) and (B12) in Appendix B, we have

$$\tau_{mm}{}^{+}(\xi)=4m^2\bar{d}_{m0}{}^{1}(\xi)\bar{d}_{m0}{}^{1}(\xi)$$
$$-q^2[\bar{d}_{m1}{}^{1}(\xi)+\bar{d}_{m-1}{}^{1}(\xi)\bar{d}_{m-1}{}^{1}(\xi)]\,, \quad (3.24)$$

where $\bar{d}^1$ is given by Eq. (2.26).

Using the explicit values of the reduced rotation matrices, we obtain

$$W^{++}(\xi,q^2,s)=(e^4/q^4)[2m^2\sinh^2\xi-q^2(1+\cosh^2\xi)]$$
$$\times\rho_{11}{}^{+}(q^2,s)+(e^4/q^4)(4m^2\cosh^2\xi-q^2\sinh^2\xi)$$
$$\times\rho_{00}{}^{+}(q^2,s)\,, \quad (3.25)$$

where the ρ is the s-channel helicity density matrix, the variable ξ can be expressed in terms of the laboratory quantities by Eq. (2.22).

The helicity projections ρ_{11} and ρ_{00} are related to the conventional structure functions W_1 and W_2 by

$$W_1=\rho_{11}{}^{+}\,,\quad W_2=(-q^2/|\mathbf{q}|^2)(\rho_{11}{}^{+}+\rho_{00}{}^{+})\,, \quad (3.26)$$

with $|\mathbf{q}|^2=-q^2+q_0{}^2$.

(b) *Total inelastic neutrino production,* i.e.,

$$l=\nu \text{ or } \bar{\nu}\,,\quad l'=e \text{ or } \mu\,,$$
$$m_l=0\,,\quad m_{l'}=m=m_e \text{ or } m_\mu\,, \quad (3.27)$$
$$W_{(\pm)}(\xi,q^2,s)=W_{(\pm)}{}^{++}(\xi,q^2,s)\pm W_{(\pm)}{}^{--}(\xi,q^2,s)\,.$$

Here the $(\pm)$ is the isospin index for the current. The $(+)$ refers to the $\bar{\nu}$ reactions, and the $(-)$ to the ν reactions. In general, we shall drop this subscript for clarity of notation. From the general result of Eq. (3.22),

$$W^{++}(\xi,q^2,s)=\tfrac{1}{2}G_F{}^2\Big[\sum_m\tau_{mm}{}^{+}(\xi)\rho_{mm}{}^{+}(q^2,s)$$
$$+\tau_{00}{}^{10}\,\mathrm{Re}\rho_{00}{}^{+10}(q^2,s)$$
$$+\tau_{00}{}^{00}\rho^{+00}(q^2,s)\Big]\,, \quad (3.28)$$

$$W^{--}(\xi,q^2,s)=\tfrac{1}{2}G_F{}^2\sum\tau_{mm}{}^{-}(\xi)\rho_{mm}{}^{-}(q^2,s)\,. \quad (3.29)$$

With finite-mass terms retained, there are spin-zero and spin-one parts in the lepton current and the production amplitude.[7] From Appendix B we have

$$\tau_{mm'}{}^{+}=(m^2-q^2)[\bar{d}_{m,1}{}^1(\xi)\bar{d}_{m',1}{}^1(\xi)+\bar{d}_{m,-1}{}^1(\xi)\bar{d}_{m,-1}{}^1(\xi)]+\frac{m^2(m^2-q^2)}{-q^2}d_{m,0}{}^1(\xi)d_{m',0}{}^1(\xi)\,,$$

$$\tau_{mm'}{}^{-}=-(m^2-q^2)[\bar{d}_{m,1}{}^1(\xi)\bar{d}_{m',1}{}^1(\xi)-\bar{d}_{m,-1}{}^1(\xi)\bar{d}_{m',-1}{}^1(\xi)]\,, \quad (3.30)$$

$$\tau_{00}{}^{10}=\frac{m^2(m^2-q^2)}{-q^2}\bar{d}_{00}{}^1(\xi)\equiv\frac{m^2(m^2-q^2)}{-q^2}\cosh\xi\,,$$

$$\tau_{00}{}^{00}=\frac{m^2(m^2-q^2)}{-q^2}\,.$$

The quantities $\rho_{00}{}^{10}$ and ρ^{00} are two additional terms that arise strictly out of the finite-mass terms; they are given by

$$\rho_{00}{}^{+,10}(q^2,s)=\frac{q^\mu}{(-q^2)^{1/2}}\epsilon_0{}^\nu(q)\sum_\lambda\int d^4x\, e^{-iqx}\times\langle N(p,\lambda)|\mathcal{J}_\mu(x)\mathcal{J}_\nu(0)|N(p,\lambda)\rangle\,, \quad (3.31)$$

$$\rho^{+,00}(q^2,s)=\frac{q^\mu q^\nu}{q^2}\sum_\lambda\int d^4x\, e^{-iqx}\times\langle N(p,\lambda)|\mathcal{J}_\mu(x)\mathcal{J}_\nu(0)|N(p,\lambda)\rangle\,.$$

Combining Eqs. (3.28)–(3.31), we obtain

$$\begin{aligned}W(\xi,q^2,s)=&-\tfrac{1}{2}G_F{}^2(q^2-m^2)\\&\times\{[1+\cosh^2\xi+(m^2/q^2)\sinh^2\xi]\rho_{11}{}^{+}(q^2,s)\\&+[\sinh^2\xi+(m^2/-q^2)\cosh^2\xi]\rho_{00}{}^{+}\\&\mp2\cosh\xi\,\rho_{11}{}^{-}(q^2,s)+(m^2/-q^2)\cosh\xi\\&\times\mathrm{Re}\rho_{00}{}^{10}(q^2,s)+(m^2/-q^2)\rho^{+,00}(q^2,s)\}\,. \quad (3.32)\end{aligned}$$

At high s, all elements of ρ excepting $\rho_{11}{}^{-}$ in Eq. (3.32) will be dominated by vacuum trajectory exchange. We assume that the usual rules of strong interactions at high energies apply here. The $\rho_{11}{}^{-}$ is an interference term between the axial-vector and vector currents. If there are only currents of the first kind,[8] and the G parity is given by $G_A=(-)^I$, $G_V=(-)^I$ for the axial-vector current and for the vector-current, respectively, the Pomeranchuk trajectory cannot contribute to $\rho_{11}{}^{-}$. However, if there are currents of the second kind, the above classification of the G parity of the current is irrelevant and the Pomeranchuk trajectory can contribute to $\rho_{11}{}^{-}$.

In the difference of cross sections $\sigma_{\nu p}-\sigma_{\bar{\nu}p}$ given by $W^{(+)}-W^{(-)}$, ρ^{+} contains the leading $I=1$ exchange and $\rho_{11}{}^{-}$ contains the leading $I=0$ exchange of odd G parity, if there are only currents of the first kind. Therefore, the Pomeranchuk trajectory cannot contribute to $\sigma_{\nu p}-\sigma_{\bar{\nu}p}$ unless there are second-kind currents. For the cross sections $\sigma_{\nu p}-\sigma_{\bar{\nu}n}$ or $\sigma_{\bar{\nu}p}-\sigma_{\nu n}$, only the $\rho_{11}{}^{-}$ term contributes, and the existence of second-kind currents can also be checked here.

Adler[1] originally noticed that for $m_X\neq m_N$ and forward lepton scattering, the pure spin-0 term dominates as the lepton mass is neglected. We can see this in the following way: Under the condition that $q^2\to m^2\to0$ at high-energy and vanishing lepton mass, we observe from Eqs. (3.31) and (3.32) that the effective factor for the pure spin-0 part is $m^2(m^2-q^2)/q^4$, which goes to a constant in this limit. All of the other terms go to zero. Consequently, at large q^2, the pure spin-one part dominates. Notice that the interference term of spin-one and spin-zero parts survives only at intermediate small q^2 values. Thus it may be difficult to observe.

2. *Pion Production* $l(q_1)+N(p,\lambda)\to l'(q_2)+F(p',\lambda')+\pi(k)$

(a) *Spin-one part.* The independent components of W are given by

$$W^{++}=\tfrac{1}{2}G_F{}^2\sum_{m'\geq|m|}N_{mm'}{}^{++}\cos(m-m')\psi\,\tau_{mm'}{}^{+}(\xi)\times\mathrm{Re}\rho_{mm'}{}^{+}(q^2,s,t)\,. \quad (3.33)$$

All of the τ's here and later on are the same as in Eq. (3.30). There are four independent components of $\mathrm{Re}\Phi_{mm'}{}^{+}$ and four of $\mathrm{Re}\rho_{m,m'}{}^{+}$, and $N_{mm'}{}^{++}$ is given in Eq. (3.22′);

$$\begin{aligned}W^{-+}&=\tfrac{1}{2}G_F{}^2\sum N_{mm'}{}^{-+}\sin(m-m')\psi\,\tau_{mm'}{}^{-}(\xi)\times\mathrm{Im}\rho_{mm'}{}^{+}(q^2,s,t)\\&=\tfrac{1}{2}G_F{}^2N_{10}{}^{-+}\sin\psi\,\tau_{10}{}^{-}\,\mathrm{Im}\rho_{10}{}^{+}(q^2,s,t)\\&\equiv-\tfrac{1}{2}G_F{}^2(m^2-q^2)2^{3/2}\sin\psi\sinh\xi\times\mathrm{Im}\rho_{10}{}^{+}(q^2,s,t)\,. \quad (3.34)\end{aligned}$$

Here there is one independent $\mathrm{Im}\rho_{10}$ and one $\mathrm{Im}\phi_{10}{}^{-}$,

$$\begin{aligned}W^{+-}&=\tfrac{1}{2}G_F{}^2\sum_{mm'}N_{mm'}{}^{+-}\tau_{mm'}{}^{+}(\xi)\sin(m-m')\psi\times\mathrm{Im}\rho_{mm'}{}^{-}(q^2,s,t)\\&\equiv\tfrac{1}{2}G_F{}^2(m^2-q^2)\sin^2\psi\sinh^2\xi\,\mathrm{Im}\rho_{1-1}{}^{-}(q^2,s,t)\\&\quad+2^{1/2}\sin\psi\sinh2\xi\,\mathrm{Im}\rho_{10}{}'(q^2,s,t)\,. \quad (3.35)\end{aligned}$$

There are two independent components of $\mathrm{Im}\rho_{mm'}{}^{-}$ and two of $\mathrm{Im}\Phi_{mm'}{}^{+}$ in this term.

$$\begin{aligned}W^{--}&=\tfrac{1}{2}G_F{}^2\sum_{mm'}N_{mm'}{}^{-}\tau_{mm'}{}^{--}(\xi)\cos(m-m')\psi\times\mathrm{Re}\rho_{mm'}{}^{-}(q^2,s,t)\\&=-\tfrac{1}{2}G_F{}^2(m^2-q^2)[(1+\cosh^2\xi)\rho_{11}{}^{-}+2^{3/2}\cos\psi\sinh\xi\,\mathrm{Re}\rho_{10}{}^{-}]\,. \quad (3.35')\end{aligned}$$

Notice that the number of independent components of

[7] The spin-zero terms, and subsequently the spin-one–spin-zero terms, provide a framework to check partial conservation of axial-vector current. See S. L. Adler in Ref. 1.

[8] S. Weinberg, Phys. Rev. **112**, 1375 (1958).

ρ is just matched by the number of independent components of Φ, and all of the real and imaginary parts of ρ can be measured.

(b) *Spin-zero part.* For the spin-zero–spin-zero noninterference term, we have

$$W^{++}=\tfrac{1}{2}G_F{}^2\tau^{00}\rho^{00}(q^2,s,t)$$
$$\equiv\tfrac{1}{2}G_F{}^2[m^2(m^2-q^2)/(-q^2)]\rho^{00}(q^2,s,t)\,, \qquad (3.36)$$

where

$$\rho^{00}(q^2,s,t)=\frac{q^\mu q^\nu}{-q^2}\frac{1}{2}\sum_\lambda \langle F(p',\lambda'),\pi(k)|A_\mu(0)|N(p,\lambda)\rangle$$
$$\times\langle F(p',\lambda'),\pi(k)|A_\nu(0)|N(p,\lambda)\rangle^*. \qquad (3.37)$$

(c) *Spin-one–spin-zero interference term.*

$$W^{++}=\tfrac{1}{2}G_F{}^2\sum_m N_{m,0}{}^{++}\cos m\psi\,\tau_{m0}{}^{+10}\,\mathrm{Re}\rho_{m0}{}^{+10}(q^2,s,t)\,.$$

$$W^{++}=\tfrac{1}{2}G_F{}^2[m^2(m^2-q^2)/(-q^2)][\cosh\xi\,\mathrm{Re}\rho_{00}{}^{+10}(q^2,s,t)$$
$$-2\sqrt{2}\cos\psi\sinh\xi\,\mathrm{Re}\rho_{10}{}^{+10}(q^2,s,t)]\,, \qquad (3.38)$$

$$W^{-+}=W^{--}=0\,,$$

and the components of $\mathrm{Re}\rho_{m0}{}^{-10}$ and $\mathrm{Im}\rho_{m0}{}^{+10}$ cannot be measured in this type of experiment;

$$W^{+-}=\tfrac{1}{2}G_F{}^2\sum_m N_{m0}{}^{+-}\sin m\psi\,\tau_{m0}{}^{+}\,\mathrm{Im}\rho_{m0}{}^{-10}$$
$$\times(q^2,s,t)(-1)^m$$
$$\equiv-(2)^{1/2}\sin\psi\sinh\xi\,\mathrm{Im}\rho_{10}{}^{-10}(q^2,s,t)\,, \qquad (3.39)$$

$$\rho_{m0}{}^{\pm1,0}=\epsilon_m{}^\mu(q)\frac{q^\nu}{(-q^2)^{1/2}}\frac{1}{2}$$
$$\times\sum_{\lambda\lambda'}\langle F(p',\lambda'),\pi(k)|\begin{pmatrix}A_\mu(0)\\ V_\mu(0)\end{pmatrix}|N(p,\lambda)\rangle$$
$$\times\langle F(p',\lambda'),\pi(k)|A_\nu(0)|N(p,\lambda)\rangle^*. \qquad (3.40)$$

Note that $F(p',\lambda')$ can be a single particle or a system of particles in any of the equations here as long as the helicity λ' is summed. If all of the phase space of the system $F(p',\lambda')$ is not integrated out, ρ depends upon more variables.

Note. There are a total of nine independent real kinematic factors contained in the lepton density matrix $\Phi_{mm'}$. They are generated entirely by the spin-one terms alone. The kinematic factors contained in the spin-zero and spin-one–spin-zero terms are redundant. We list all of the terms separately, in case they receive independent emphasis in applications.

3. Production of Vector System: $l(q_1)+N(p,\lambda)\to l'(q_\gamma)+\rho(k)+F$

Here we include the angular distribution of the decaying vector system. The case we consider in detail is the ρ meson. From Eq. (3.22) we have the following:

(a) *Spin-one part.*

$$W^{++}=\tfrac{1}{2}G_F{}^2\sum_{m'\geq|m|;\,\mu'\geq|\mu|}N_{\mu\mu'mm'}{}^{++}$$
$$\times\cos[(\mu-\mu')\varphi+(m-m')\psi]b_{00}{}^{+}d_{\mu0}{}^{1}(\theta)d_{\mu'0}{}^{1}(\theta)$$
$$\times\tau_{mm'}{}^{+}(\xi)\,\mathrm{Re}\rho_{\mu\mu'mm'}{}^{+}\,, \qquad (3.41)$$

where the number of independent components of $\mathrm{Re}\rho_{\mu\mu'mm'}{}^{+}$ is 25 and the number of independent components of $\mathrm{Re}(X\otimes\Phi^+)$ is 20;

$$W^{-+}=\tfrac{1}{2}G_F{}^2\sum N_{\mu\mu'mm'}{}^{-+}\sin[(\mu-\mu')\varphi+(m-m')\psi]$$
$$\times b_{00}{}^{+}d_{\mu0}{}^{1}(\theta)d_{\mu'0}{}^{1}(\theta)\tau_{mm'}{}^{-}(\xi)\,\mathrm{Im}\rho_{\mu\mu'mm'}{}^{+}\,, \qquad (3.42)$$

where the number of independent components of $\mathrm{Im}\rho^+$ is 16, and the number of independent components of $\mathrm{Im}X\otimes\Phi^-$ is 8;

$$W^{--}=\tfrac{1}{2}G_F{}^2\sum N_{\mu\mu'mm'}{}^{--}\cos[(\mu-\mu')\varphi+(m-m')\psi]$$
$$\times b_{00}{}^{+}d_{\mu0}{}^{1}(\theta)d_{\mu'0}{}^{1}(\theta)\tau_{mm'}{}^{-}(\xi)\,\mathrm{Re}\rho_{\mu\mu'mm'}{}^{-}\,, \qquad (3.43)$$

where the number of independent components of $\mathrm{Re}\rho^-$ is 20, and the number of independent components of $\mathrm{Re}X\otimes\Phi^-$ is 10;

$$W^{+-}=\tfrac{1}{2}G_F{}^2\sum N_{\mu\mu'mm'}{}^{+-}\sin[(\mu-\mu')\varphi+(m-m')\psi]$$
$$\times b_{00}{}^{+}d_{\mu0}{}^{1}(\theta)d_{\mu'0}{}^{1}(\theta)\tau_{mm'}{}^{+}\,\mathrm{Im}\rho_{\mu\mu'mm'}{}^{-}\,, \qquad (3.44)$$

where the number of independent components of $\mathrm{Im}\rho^-$ is 20 and the number of independent components of $\mathrm{Im}(X\otimes\Phi^+)$ is 16.

In Eqs. (3.41)–(3.44), the folding number $N_{\mu\mu'mm'}{}^{\delta\delta'}$ is given by Eq. (3.22′) and $b_{00}{}^{+}$ is the square of the $\rho\to2\pi$ decay amplitude in the configuration where the π's are along the direction of the ρ.

We see here explicitly that not all of the independent density-matrix elements can be measured.

(b) *Spin-zero part.*

$$W=W^{++}=\tfrac{1}{2}G_F{}^2\sum_{\mu'\geq\mu;\text{ all }\mu}N_{\mu\mu'}{}^{++}\cos(\mu-\mu')\varphi b_{00}{}^{+}$$
$$\times d_{\mu0}{}^{1}(\theta)d_{\mu'0}{}^{+}(\theta)\tau_{00}{}^{+00}\,\mathrm{Re}\rho_{\mu\mu'}{}^{+00}. \qquad (3.45)$$

(c) *Spin-one–spin-zero interference part.*

$$W^{++}=\sum N_{\mu m\mu'0}{}^{+}\cos[(\mu-\mu')\varphi+m\psi]b_{00}{}^{+}d_{\mu0}{}^{1}(\theta)d_{\mu'0}{}^{1}(\theta)$$
$$\times\tau_{m0}{}^{+10}(\xi)\,\mathrm{Re}\rho_{\mu m\mu'0}{}^{+10}\,,$$
$$\tau_{m0}{}^{+10}=[m^2(m^2-q^2)/-q^2]\bar{d}_{m0}{}^{10}(\xi) \qquad (3.46)$$
$$\equiv[m^2(m^2-q^2)/-q^2](\sinh\xi m)/\sqrt{2}\,,\quad m=\pm1\,,$$

where there are 14 independent components of $\mathrm{Re}\rho_{\mu m\mu'0}{}^{+}$ and $\mathrm{Re}X\otimes\Phi^+$, and $N_{\mu m\mu'0}{}^{++}$ is equal to 1 for $\mu=m=\mu'=0$, and equal to 2 otherwise.

$$W^{-+}=W^{--}=0\quad\text{since}\quad\mathrm{Im}X^-=0\quad\text{and}\quad\mathrm{Re}X^-=0\,,$$
$$W^{+-}=\tfrac{1}{2}G_F{}^2\sum N_{\mu m\mu'0}{}^{+-}\sin[(\mu-\mu')\varphi+m\psi] \qquad (3.47)$$
$$\times b_{00}{}^{+}d_{\mu0}{}^{1}(\theta)d_{\mu'0}{}^{1}(\theta)\tau_{m0}{}^{+10}\,\mathrm{Im}\rho_{\mu m\mu'0}{}^{-10};$$

the number of independent components of $\mathrm{Im}\rho^-$ and of

$\mathrm{Im}X\otimes\Phi^+$ is 13. The definition of $\rho_{\mu m\mu'0}{}^{\pm 10}$ is similar to Eq. (3.40) with $|\pi(k)\rangle$ replaced by $|\rho(k,\mu)\rangle$ in the appropriate places.

For electroproduction, $W^{--}=W^{+-}=0$ in the spin-one part; all other parts involving the spin-zero part of the current are zero. In addition, $\tau_{mm'}{}^{\pm}$ is given by Eq. (3.30).

We close this subsection with the following remark: We have presented two examples in some detail. However, any other applications, like $l+N\to l'+\rho+N^*$, can be easily extracted from Eq. (3.22). The finite-mass corrections can be easily included in all applications by consulting Appendix B. With a little modification, Eq. (3.22) can also be used for reactions involving more than one current.

C. Definite Spin Parity Exchange

The constraints due to the exchange of a Regge trajectory with definite spin parity, i.e., $P(-1)^J=\eta=\pm 1$, are well known for two-body–to–two-body hadronic reactions. Since we have formulated the lepton reactions in the same language, all of the known analysis can easily be applied here. We summarize the essential features of the results here. For the reaction

$$\mathcal{J}(q,m)+A(k,\mu)\to N(p,\lambda)+\bar{F}(p',\lambda')$$

described by the t-channel helicity amplitude $f_{m\mu,\lambda\lambda'}{}^t$, we have the first result

$$f_{m\mu\lambda\lambda'}{}^t\cong\eta'(-1)^{m-\mu}f_{-m-\mu\lambda\lambda'}{}^t \quad \text{for} \quad s\to\infty\,, \tag{3.48}$$

where $\eta'=\eta_E\eta_A\eta_{\mathcal{J}}$, where η_E, η_A, and $\eta_{\mathcal{J}}$ are the spin parities of the exchanged system, the particle A, and the current, respectively. Equation (3.48) holds asymptotically to the leading order in s; thus we have the second result: if $m-\mu=0$, then

$$f_{m\mu\lambda\lambda'}{}^t=\eta' f_{-m-\mu\lambda\lambda'}{}^t\,, \tag{3.49}$$

and Eq. (3.48) becomes an equality.

There are also relations for the s-channel helicity amplitudes[9]:

$$f_{\lambda'\mu\lambda m}{}^s\cong(-1)^{m-\mu}\eta' f_{\lambda'-\mu\lambda-m}{}^s \quad \text{for} \quad s\to\infty\,. \tag{3.50}$$

Now we can apply all of these properties to the density matrix:

$$\rho_{\mu\mu'mm'}{}^t=\eta'\rho_{-\mu\mu'-mm'}{}^t \qquad \text{for} \quad m-\mu=0 \tag{3.51}$$

$$\cong\eta'(-1)^{m-\mu}\rho_{-\mu\mu'-mm'}{}^t \quad \text{for} \quad m-\mu\neq 0\,, \tag{3.52}$$

and correspondingly for the s-channel density-matrix element. Here

$$\rho_{\mu\mu'mm'}=\tfrac{1}{2}\sum_{\lambda\lambda'} f_{m\mu\lambda\lambda'}f_{m'\mu'\lambda\lambda'}{}^*\,.$$

In addition to parity and Hermiticity, which have folded the density matrix along the diagonal and the antidiagonal, spin parity gives us another folding of the density matrix with respect to the line $m=\mu=0$.

If only one helicity in the t channel is studied, i.e.,

$$\rho_{mm'}=\sum_{\mu\lambda\lambda'} f_{m\mu\lambda\lambda'}f_{m'\mu\lambda\lambda'}{}^*\,, \tag{3.53}$$

the symmetry

$$f_{m\mu'\lambda\lambda}{}^t\cong\eta'(-1)^{m-\mu}f_{-m-\mu\lambda\lambda'}{}^t \tag{3.54}$$

does not bring about obvious symmetry properties of $\rho_{mm'}$; usually this relation leads to more complicated relations among the density-matrix elements.[10]

D. Single Trajectory Dominance

In the kinematic region $s\gg -q^2$ and $s\gg M^2$, where M is any other hadronic mass, if a single trajectory dominates the scattering, then all helicity amplitudes have the same phase and $\mathrm{Im}\rho^{\pm}$ is equal to zero. Therefore all of the coefficients of

$$\sin[\textstyle\sum_d(\mu_d-\mu_d')\varphi_d+(m-m')\psi]$$

in all applications are zero.

At special kinematic points, i.e., $t\to 0$ as $s\to\infty$, there are restrictions due to the conservation of the M quantum number as defined by Toller or Freedman and Wang.[11] In particular, the elements of the density matrix behave like

$$\rho_{mm';\mu\mu'} \underset{s\to\infty}{\sim} (s^{1/2})^{-|M-|m-\mu||}(s^{1/2})^{-|M-|m'-\mu'||}\,. \tag{3.55}$$

One of the consequences of this statement is that all helicity-flip amplitudes vanish for $t\to 0$ if $M=0$. This statement is true for both the s- and t-channel helicity density matrix. Since there are no accepted candidates for trajectories with $M\neq 0$, there are no analogous statements for the helicity nonflip amplitude.

In general, factorization gives us the additional statement that the determinant of the density matrix and the determinants of all of its minors vanish.

IV. CONCLUSIONS

In this paper we have developed a formalism to analyze specific final states in electroproduction and neutrino-induced reactions. Particular emphasis has been given to the correlation between the lepton distribution and the decay distribution of the various unstable systems produced in the reaction. With the advent of large bubble chambers, it is possible that some experiments of this type can be done in future years.

In most of this work we have assumed that the usual form of the current×current interaction is valid for semileptonic weak processes, and that the quantum

[9] J. P. Ader, M. Capdeville, G. Cohen-Tannoudji, and P. H. Salin, Nuovo Cimento **56**, 952 (1968).

[10] G. A. Ringland and R. L. Thews, Phys. Rev. **170**, 1569 (1968).

[11] A. Sciarrino and M. Toller, J. Math. Phys. **7**, 1670 (1967); D. Freedman and J. M. Wang, Phys. Rev. **160**, 1560 (1967).

numbers of the current are given by the usual Cabibbo form. It will be interesting to relax this assumption also, together with time-reversal invariance, and explore even more possibilities in the future.

APPENDIX A

In this appendix, we want to prove Eq. (2.34) explicitly. Of course we could evaluate it from Eq. (2.2) directly. However, it probably is more instructive to prove (2.34) in a group-theoretical manner which can easily be generalized to arbitrary helicities and spins. The coupling of a current to the lepton pair

$$c_{m\sigma_2\sigma_1}(v)=c_{m\sigma_2\sigma_1}(qq_2q_1)=\langle q_2\sigma_a|l_\mu|q_1\sigma_1\rangle\epsilon_m{}^\mu(q)\,,\qquad\text{(A1)}$$

where l_μ is a leptonic current. The Lorentz transformation property of $c_{m\sigma_2\sigma_1}(v)$ is as follows:

$$c_{m\sigma_2\sigma_1}(qq_2q_1)=\sum_{m'}c_{m'\sigma_2\sigma_1}(\Lambda q,\Lambda q_2,\Lambda q_1)\mathfrak{D}_{m'm}{}^1\times(L^{-1}(\Lambda q)\Lambda L(q))\qquad\text{(A2)}$$

for any complex Lorentz transformation Λ. Equation (A2) follows from the fact that the leptons are massless and therefore their helicities do not change under Lorentz transformation. $R_W(\Lambda,q)=L^{-1}(\Lambda q)\Lambda L(q)$ is the well-known Wigner rotation. Incidentally, Eq. (A2) is equivalent to the following relation for $e_m(q)$:

$$\Lambda_\nu{}^\mu\epsilon_m{}^\nu(q)=\sum_{m'=0,\pm1}\mathfrak{D}_{m'm}{}^1(R_W(\Lambda,q))\epsilon_{m'}{}^\mu(\Lambda q)\,.\qquad\text{(A3)}$$

Let us now take $\Lambda=v^{-1}=e^{i\xi K_x}e^{i\psi J_z}$. Since

$$q^J=q^R=(0,0,0,1)(-q^2)^{1/2}=(0,0,0,-i)(q^2)^{1/2}\,,$$

we have

$$L(q)=e^{-iK_z(-i\pi/2)}=e^{i(iK_z)\pi/2}.$$

Also $L(\Lambda q)=L(q)$. Therefore

$$R_W(\Lambda,a)=e^{-i(iK_z)\pi/2}e^{i\xi K_x}e^{i\psi J_z}e^{i(iK_z)\pi/2}.\qquad\text{(A4)}$$

Now $[J_z,K_z]=0$. Further from the fact that iK_x, J_y, and iK_z satisfy the same commutation relations as J_x, J_y, and J_z, we see very easily that

$$e^{-i(iK_z)\pi/2}(iK_x)e^{i(iK_z)\pi/2}=J_y\,.\qquad\text{(A5)}$$

We see, therefore, that

$$R_W(\Lambda,q)=e^{\xi J_y}e^{i\psi J_z}\,,\qquad\text{(A6)}$$

and

$$\mathfrak{D}_{m'm}{}^1(L^{-1}(\Lambda q)\Lambda L(q))=\tilde{d}_{m'm}{}^1(\xi)e^{-im\psi}=\mathfrak{D}_{m'm}{}^1(v)\,.\qquad\text{(A7)}$$

However,

$$c_{m'\sigma_2\sigma_1}(v^{-1}q,v^{-1}q_2,v^{-1}q_1)=\delta_{m',\sigma_1+\sigma_2}\delta_{\sigma_1\sigma_2}\,,\qquad\text{(A8)}$$

$$c_{m\sigma_2\sigma_1}(v)=k\mathfrak{D}_{\sigma_1+\sigma_2,m}{}^1(v)\,,\qquad\text{(A9)}$$

where k is a constant.

This completes the proof.

APPENDIX B

The $O(2,1)$ expression for the lepton-pair distribution for electromagnetic and weak currents.

1. Electromagnetic Current

For any reaction involving an electron or muon pair, to the lowest order in e^2 the total probability density is

$$W=(e^4/q^4)\tau_{\mu\nu}W_{\mu\nu};\qquad\text{(B1)}$$

here

$$\tau_{\mu\nu}=\sum_{\sigma_1\sigma_2}\bar{u}_{\sigma_2}(q_2)\gamma^\mu u_{\sigma_1}(q_1)[\bar{u}_{\sigma_2}(q_2)\gamma^\nu u_{\sigma_1}(q_1)]^*=(q^2g_{\mu\nu}-q_\nu q_\mu+Q_\mu Q_\nu)\,,\qquad\text{(B2)}$$

where $Q=q_1+q_2$ and $q=(q_1-q_2)$.

Now let us express $\tau_{\mu\nu}$ in terms of the spin polarization vectors $\epsilon_\mu{}^m$ and q_μ in the frame L, where, for a spacelike current,

$$\begin{aligned}&q^L=(0,\,0,\,0,\,(-q^2)^{1/2})\,,\\&q_1{}^L=((m_l{}^2-\tfrac{1}{4}q^2)^{1/2},\,0,\,0,\,\tfrac{1}{2}(-q^2)^{1/2})\,,\\&q_2{}^L=((m_l{}^2-\tfrac{1}{4}q^2)^{1/2},\,0,\,0,\,-\tfrac{1}{2}(-q^2)^{1/2})\,,\\&Q^L=(2(m_l{}^2-\tfrac{1}{4}q^2)^{1/2},\,0,\,0,\,0)\,;\end{aligned}\qquad\text{(B3)}$$

$$\begin{aligned}&\epsilon_\mu{}^1=-(1/\sqrt{2})(\hat{e}_x+i\hat{e}_y)\,,\\&\epsilon_\mu{}^{-1}=(1/\sqrt{2})(\hat{e}_x-i\hat{e}_y)\,,\\&\epsilon_\mu{}^0=[1/(-q^2)^{1/2}](|\mathbf{q}|,0,0,q_0)^L\equiv(1,0,0,0);\end{aligned}\qquad\text{(B4)}$$

$$\begin{aligned}&g_{\mu\nu}=\sum_m(-)^m\epsilon_\mu{}^m\epsilon_\nu{}^m+q_\mu q_\nu/q^2\,,\\&Q_\mu{}^LQ_\nu{}^L=(4m_l{}^2-q^2)\epsilon_\mu{}^0\epsilon_\nu{}^0.\end{aligned}\qquad\text{(B5)}$$

Substituting Eq. (B5) into (B2), we obtain

$$\tau_{\mu\nu}=4m_l{}^2\epsilon_\mu{}^0\epsilon_\nu{}^0-q^2\epsilon_\mu{}^1\epsilon_\nu{}^1-q^2\epsilon_\mu{}^{-1}\epsilon_\nu{}^{-1}.\qquad\text{(B6)}$$

Now we express

$$W=\tau_{\mu\nu}g_{\mu\mu'}g_{\nu\nu'}W_{\mu'\nu'}\,,\qquad\text{(B7)}$$

and substituting Eq. (B5) into (B7),

$$W=(e^4/q^4)[4m_e{}^2\delta_{m0}\delta_{m'0}-q^2(\delta_{m1}\delta_{m'1}+\delta_{m-1}\delta_{m'1})]\rho_{mm'}\equiv\Phi_{mm'}{}^L\rho_{mm'}\,.\qquad\text{(B8)}$$

This is the lepton distribution in frame L. The general frame for the lepton pair is related to the L frame by the $SO(2,1)$ transformation $e^{-i\psi J_z}e^{-i\xi K_x}$. In this general frame R,

$$\begin{aligned}&q^R=(0,\,0,\,0,\,(-q^2)^{1/2})\,,\\&q_1{}^R=\tfrac{1}{2}(-q^2)^{1/2}(\cosh\xi,\,\sinh\xi\,\sin\psi,\,\sinh\xi\,\cos\psi,\,1)\,,\\&q_2{}^R=\tfrac{1}{2}(-q^2)^{1/2}(\cosh\xi,\,\sinh\xi\,\sin\psi,\,\sinh\xi\,\cos\psi,\,-1)\,.\end{aligned}\qquad\text{(B9)}$$

The representation of the $SO(2,1)$ transformation is related to the lepton density matrix by

$$\Phi_{mm'}(v)=\mathfrak{D}_{m,n}{}^1(\psi,\xi)\mathfrak{D}_{m',n'}{}^{1*}(\phi,\xi)\Phi_{nn'}{}^L$$
$$=e^4/q^4\{4m_l{}^2\mathfrak{D}_{m,0}{}^1(\psi,\xi)\mathfrak{D}_{m',0}{}^1(\psi,\xi)$$
$$-q^2[\mathfrak{D}_{m',1}{}^1(\psi,\xi)\mathfrak{D}_{m',1}{}^1(\psi,\xi)+\mathfrak{D}_{m,-1}{}^1(\psi,\xi)$$
$$\times\mathfrak{D}_{m',-1}{}^1(\psi,\xi)]\} \quad \text{(B10)}$$
$$=\{4m_l{}^2e^{i(m-m')\psi}\bar{d}_{m,0}{}^1(\xi)\bar{d}_{m',0}{}^1(\xi)$$
$$-q^2e^{i(m-m')\psi}[\bar{d}_{m,1}{}^1(\xi)\bar{d}_{m,1}{}^1(\xi)$$
$$+\bar{d}_{m,-1}{}^1(\xi)\bar{d}_{m'-1}{}^1(\xi)]\}. \quad \text{(B11)}$$

As expected, the current contains only a spin-one part, since the current is a conserved current. This gives us the coefficients a in Eqs. (2.30),

$$a_{\frac{1}{2},-\frac{1}{2}}{}^{+}=4m_l{}^2,\quad a_{\frac{1}{2},\frac{1}{2}}{}^{+}=-q^2.$$

For a timelike current, as is well known, the lepton distribution is specified by the $SO(3)$ representation. We simply replace $\bar{d}^1(\xi)$ by $d^1(\theta)$ in Eq. (B11). Now θ and φ are the polar and the azimuthal angles of the momentum of the two leptons in the c.m. frame of the current, i.e., $q=((q^2)^{1/2},0,0,0)$.

2. Weak Current

The derivation here is the same as in the previous case:

$$\tau_{\mu\nu}=\tfrac{1}{2}G^2\sum_{\sigma_1,\sigma_2}\bar{u}_{\sigma 2}(q_2)(1\pm\gamma_5)\gamma^\mu u_{\sigma 1}(q_1)$$
$$\times[\bar{u}_{\sigma 2}(q_2)(1\pm\gamma_5)\gamma^\mu u_{\sigma 1}(q_1)]^*$$
$$=\tfrac{1}{2}G^2[(q^2-m_l{}^2)g_{\mu\nu}-q_\mu q_\nu+Q_\mu Q_\nu\pm i\epsilon_{\mu\nu\lambda\sigma}Q^\lambda q^\sigma]. \quad \text{(B12)}$$

The $-$ sign is for the neutrino reaction and the $+$ sign is for the antineutrino reaction.

The lepton momenta in the frame L are

$$q^L=(0,\,0,\,0,\,(-q^2)^{1/2}),$$
$$q_1{}^L=((m_l{}^2+|\mathbf{q}_2|^2)^{1/2},\,0,\,0,\,(m_l{}^2+|\mathbf{q}_2|^2)^{1/2}),$$
$$q_2{}^L=((m_l{}^2+|\mathbf{q}_2|)^{1/2},0,0,-|\mathbf{q}_2|), \quad \text{(B13)}$$
$$Q^L=(2(m_l{}^2+|\mathbf{q}_2|^2)^{1/2},0,0,-|\mathbf{q}_2|+(m_l{}^2+|\mathbf{q}_2|^2)^{1/2}),$$

and

$$|\mathbf{q}_2|=-(m_l{}^2+q^2)/2(-q^2)^{1/2}.$$

Here q_1 is the neutrino momentum and q_2 is the lepton momentum. Going through the same procedure as for the electromagnetic current, we obtain

$$\Phi_{mm'}{}^{+}(v)=(m_l{}^2-q^2)e^{i(m-m')\varphi}[\bar{d}_{m,1}{}^1(\xi)\bar{d}_{m',1}{}^1(\xi)$$
$$+\bar{d}_{m,-1}{}^1(\xi)\bar{d}_{m',-1}{}^1(\xi)]+[m_l{}^2(m_l{}^2-q^2)/-q^2]$$
$$\times e^{i(m-m')\varphi}\bar{d}_{m,0}{}^1(\xi)\bar{d}_{m',0}{}^1(\xi), \quad \text{(B14)}$$

$$\Phi_{mm'}{}^{-}(v)=-(m_l{}^2-q^2)e^{i(m-m')\varphi}[d_{m,1}{}^1(\xi)d_{m',1}{}^1(\xi)$$
$$-d_{m,-1}{}^1(\xi)d_{m',-1}{}^1(\xi)] \quad \text{(B15)}$$

for the spin-one part of the current,

$$\Phi_{00}{}^{+00}=m_l{}^2(m_l{}^2-q^2)/(-q^2) \quad \text{(B16)}$$

for the spin-zero part, and

$$\Phi_{m0}{}^{+10}=[m_l{}^2(m_l{}^2-q^2)/(-q^2)]e^{im\varphi}\bar{d}_{m,0}{}^1(\xi) \quad \text{(B17)}$$

for the spin-one and spin-zero mixed part. $\Phi_{00}{}^{-00}$ and $\Phi_{m0}{}^{-10}$ are zero. Therefore the a's in Eqs. (2.30) and (2.33) are

$$a_{\frac{1}{2},\frac{1}{2}}{}^{+,1}=(m_l{}^2-q^2),$$
$$a_{\frac{1}{2},-\frac{1}{2}}{}^{+,1}=m_l{}^2(m_l{}^2-q^2)/(-q^2), \quad \text{(B18)}$$

and

$$a_{\frac{1}{2},\frac{1}{2}}{}^{-,1}=-(m_l{}^2-q^2)$$

for the spin-one part;

$$a_{\frac{1}{2},\frac{1}{2}}{}^{+0}=m_l{}^2(m_l{}^2-q^2)/(-q^2) \quad \text{(B19)}$$

for the spin-zero part; and

$$a_{\frac{1}{2},\frac{1}{2}}{}^{+10}=m_l{}^2(m_l{}^2-q^2)/(-q^2) \quad \text{(B20)}$$

for the spin-one and spin-zero mixed term.

For a timelike weak current, the distribution functions are just Eqs. (B13)–(B17) with $\bar{d}(\xi)$ replaced by $d(\theta)$. When finite-mass terms are included, the relation Eq. (2.22) is to be replaced by

$$\sin^2\xi=|\mathbf{q}_1|\,|\mathbf{q}_2|\sin\theta\,[|\mathbf{q}|\,(m^2-q^2)/2(-q^2)^{1/2}]^{-1}. \quad \text{(B21)}$$