Improved Upper Bound on the Imaginary Part of Elastic Scattering Amplitudes*

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The method of Singh and Roy, which obtains a bound on the imaginary part of the elastic scattering amplitude, given σ_{tot} and σ_{el}, is generalized to allow inclusion of other given data such as $d\sigma/dt$ at fixed points. A calculation is described in which one additional experimental point is given. The upper bound is compared with the experimental data for $\pi^{\pm}p$, $K^{\pm}p$, pp, and $p\bar{p}$ elastic scattering at various energies in the near-forward direction. It is found that this single extra piece of given data extends by a factor of about 3 the range for which the original bound remained close to the data and at the largest t value considered ($t\approx -0.5$ BeV^2) improves this bound by a factor of about 4.

I. INTRODUCTION

RECENTLY Singh and Roy[1] obtained an upper bound for the absorptive part of the elastic scattering amplitudes for hadronic processes, given in terms of σ_{tot} and σ_{el}. This is the best bound assuming only those data, and is valid when the forces are mainly spin-independent and the process is highly absorptive (i.e., the imaginary part of the amplitude dominates). These conditions are believed to hold in some diffraction peaks.

The Singh-Roy bound has the following interesting properties: (a) The bound is a universal function of $\rho=-4t(d\sigma/dt|_{t=0})/\sigma_{\text{el}}$, (b) the data for unpolarized, differential cross sections for the reactions $\pi^{\pm}p$, $K^{\pm}p$, pp, and $\bar{p}p$ at various high energies in the near-forward direction (ρ small) also lie on a roughly universal curve in ρ at least as far out as $\rho=15$, (c) the data are very close to the upper bound for $\rho\leqslant 3$, but (d) fall increasingly below the bound for $\rho>3$. That the experimental data are roughly universal in this region is interesting, although not unexpected, for one can show easily that, if the scattering amplitude is approximately exponential over a significant portion of its range (and has no backward peak), then $(d\sigma/dt)/(d\sigma/dt|_{t=0})$ has an $e^{-\rho/4}$ dependence at high energies in this parametrization. It turns out that the upper-bound curve starts like[2] $1-2\rho/9$ and so is close to the data for small ρ. Still, the universality of the data as a function of ρ means that, if we could improve the upper bound for large ρ, this result would be a useful improvement for all primarily absorptive reactions.

The upper bound appears as the envelope of a family of curves, each of which obtains the bound at $\rho=0$ and at only one additional point ρ_0. Each curve of the family is given by a unique set of trial partial-wave phase shifts. The variation of the phase shifts to achieve the bound for various ρ_0 is considerable. As an indication of this, we note that the curve maximizing the amplitude at $\rho_0=15$ predicts a value of $(d\sigma/dt)/(d\sigma/dt|_{t=0})$ seven times below the experimental data at $\rho=3$ (even though the bound at $\rho_0=15$ is itself ten times too high). Thus by restricting the amplitude to be given correctly at low ρ values, we reduce the variation in the phase shifts and obtain a significant improvement in the bound for higher ρ. In Sec. II we extend the Singh-Roy technique to include this additional information. Details of the calculation are given in Sec. III; the bound we obtain through this extension is a significant improvement on the Singh-Roy bound in the large-ρ region. In Sec. IV we compare the results of our calculation with experiment and present our conclusions.

II. METHOD

Let $A(t)$ denote the absorptive part of the elastic scattering amplitude for two spinless particles with fixed c.m. energy. Within the Lehmann ellipse, $A(t)$ has the partial-wave expansion

$$A(t)=\frac{s^{1/2}}{k}\sum_{l=0}^{\infty}(2l+1)\,\mathrm{Im}a_l\,P_l(\cos\theta)\,, \tag{1}$$

where k and θ are the c.m. momentum and scattering angle, respectively. Unitarity requires $0\leqslant \mathrm{Im}a_l\leqslant 1$ in this normalization, although we shall not make use of the upper bound on $\mathrm{Im}a_l$. The total and elastic cross sections are given by

$$\begin{aligned}\sigma_{\text{tot}}&=\frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\,\mathrm{Im}a_l\,,\\ \sigma_{\text{el}}&=\frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)|a_l|^2.\end{aligned} \tag{2}$$

The Singh-Roy bound is given in terms of known values of the above quantities. As described in the Introduction, we wish to improve this bound by in-

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† Supported in part by Royal-Society–Israel-Academy Research Fellowship.
[1] V. Singh and S. M. Roy, Phys. Rev. Letters **24**, 28 (1970); **24**, 699(E) (1970); Phys. Rev. D **1**, 2638 (1970). These two papers are referred to as Singh-Roy. See also V. S. Popov and V. D. Mur, Yadern. Fiz. **3**, 561 (1965) [Soviet J. Nucl. Phys. **3**, 406 (1966)], who obtain analogous results, but only for $t>0$ where $P_l(\cos\theta)$ does not oscillate.
[2] This dependence is obtained from squaring Eq. (1.3) of Singh-Roy, first Ref. 1.

clusion of additional information. To do this we proceed as follows.

Let a set $[b_l{}^i]$, $i=1, \ldots, n$, $l=0, 1, 2, \ldots$ of arbitrary real constants be defined so that the following functions of a set $[a_l]$,

$$T_0=\frac{k^2}{4\pi}\sum_{l=0}^{\infty}(2l+1)|a_l|^2, \tag{3}$$

$$T_i=\sum_{l=0}^{\infty}(2l+1)b_l{}^i\,\mathrm{Im}a_l, \quad i=1,\ldots,n \tag{4}$$

converge for given $[a_l]$. We wish to find the supremum of $A(t)$ over the set $[a_l]$ which reproduces given T_i, $i=0, \ldots, n$. We introduce a trial set $[a_l{}^0]$ by

$$\mathrm{Re}a_l{}^0=0,$$

$$\mathrm{Im}a_l{}^0=(\beta_0P_l(\cos\theta)+\sum_{i=1}^{n}\beta_ib_l{}^i)\times\Theta(\beta_0P_l(\cos\theta)+\sum_{i=1}^{n}\beta_ib_l{}^i), \tag{5}$$

where $\Theta(x)$ is the step function and the β's are real numbers. Let us assume that a set of β's can be found such that $a_l{}^{(0)}$ reproduce the T_i.

Theorem. The set $[a_l{}^0]$ is a least upper bound to $A(t)$ over the set of all a_l reproducing T_i, provided $\beta_0>0$.

Proof. We show that $\Delta(t)=A(t)-A^0(t)\leqslant 0$, where A is evaluated with an arbitrary set $[a_l]$ and A^0 is evaluated with $[a_l{}^{(0)}]$.

Let μ denote the set of l's for which the argument of $\Theta(x)$ is positive and ν the remaining values of l. We substitute for $A(t)$ and $A^0(t)$ using (1) and eliminate $P_l(\cos\theta)$ in the μ summation through (5). The expressions (3) and (4) for the quantities T_i, $i=0, 1, 2, \ldots, n$, evaluated with $[a_l]$ and $[a_l{}^0]$, are equated and the resulting identities substituted into the expression for Δ. After some rearrangement, we obtain

$$\Delta=-(1/2\beta_0)\sum_{\mu}(2l+1)[(\mathrm{Im}a_l-\mathrm{Im}a_l{}^{(0)})^2+(\mathrm{Re}a_l)^2]$$
$$+\sum_{\nu}(2l+1)[-\tfrac{1}{2}(\mathrm{Im}a_l)^2+\mathrm{Im}a_l(\beta_0P_l(\cos\theta)+\sum_{i=1}^{n}\beta_ib_l{}^i)]$$
$$+\left[\frac{s^{1/2}}{2k\beta_0}(T_i-T_i{}^0)-\frac{s^{1/2}}{k}\sum_{i=1}^{n}\beta_i(T_i-T_i{}^0)\right].$$

The last two terms of this expression, which are enclosed by square brackets, vanish since the same values of T are reproduced by the arbitrary set $[a_l]$ and the trial set $[a_l{}^0]$. Recalling the definition of the ν set, we see that all remaining terms are negative definite when $\beta_0>0$ and $\mathrm{Im}a_l\geqslant 0$. Hence, $\Delta\leqslant 0$. We note that, as in the Singh-Roy bound, the set of a_l which reaches the bound is the best, given only fixed T_i, and is unique if a proper set of β's be found.

We are not able to define exactly those conditions on T_i for which a set of β's can be found for arbitrary n; however, for our calculation $(n=2)$, we shall give conditions later.

For $n=1$, setting $b_l{}^{(1)}=4\pi/k^2$, we recover the Singh-Roy bound. For $n=2$, setting $b_l{}^{(1)}=4\pi/k^2$ and $b_l{}^{(2)}=(s^{1/2}/k)P_l(\cos\theta_0)$, we obtain the required generalization in which one includes the value of $d\sigma/dt$ at one additional point in the approximation that the amplitude is mainly imaginary and hence that $A(t)$ is merely the positive branch of $(d\sigma/dt)^{1/2}$.

We remark in passing that the upper-bound theorem can be extended to include arbitrary moment constraints as well.

III. CALCULATION OF UPPER BOUND

From this point on, we restrict ourselves to the case $n=2$. The argument of the previous section then enables us to set up the explicit equations relating β_0, β_1, and β_2 to T_0, T_1, and T_2. To solve them numerically, it is necessary to know $P_l(\cos\theta)$ for $\cos\theta$ near 1 and for large l. An approximation which is valid for not-too-large l is given by the first term of an asymptotic expansion for P_l, namely,

$$P_l(\cos\theta)\sim J_0(\gamma),$$
$$\gamma=(2l+1)\sin\tfrac{1}{2}\theta.$$

Its use in the exact equations relating the β's and T's is equivalent to the use of the approximate form of unitarity, $0\leqslant\mathrm{Im}\alpha(\gamma)\leqslant 1$, in the eikonal representation defined through

$$A(t)=\frac{s^{1/2}}{2k}\int_0^\infty d\gamma\,\gamma J_0(\gamma y)\,\mathrm{Im}\alpha(\gamma),$$
$$\sigma_{\mathrm{tot}}=\frac{2\pi}{k^2}\int_0^\infty d\gamma\,\gamma\,\mathrm{Im}\alpha(\gamma), \tag{6}$$
$$\sigma_{\mathrm{el}}=\frac{2\pi}{k^2}\int_0^\infty d\gamma\,\gamma|\alpha(\gamma)|^2,$$

where $y=\sin\frac{1}{2}\theta$. In this representation, trial "phase-shifts" $\alpha^0(\gamma)$, which maximize $A(t)$, are given by

$$\mathrm{Re}\alpha^0(\gamma)=0,$$
$$\mathrm{Im}\alpha^0(\gamma)=D[CJ_0(\gamma y)+(1-C)J_0(\gamma x)-E]\times\Theta(CJ_0(\gamma y)+(1-C)J_0(\gamma x)-E). \tag{7}$$

In Eq. (7), y is $\sin\frac{1}{2}\theta$ for the point at which $A(t)$ is maximized, x is $\sin\frac{1}{2}\theta$ for the point at which $A(t)$ is given, and C, D, and E are real constants replacing the β's and required to satisfy C, $D>0$ and to reproduce σ_{tot}, σ_{el}, and $A(x)$.

The presence of the Θ function in Eq. (7) and the fact that J_0 approaches zero for large argument means that for positive E the integrands in Eq. (6) contribute only over a finite range of γ. This range consists of a set of intervals with end points

$$0=\gamma_0\leqslant\gamma_1\leqslant\gamma_2\cdots\leqslant\gamma_{2n+1},$$

defined by the solutions to the equation in γ,

$$CJ_0(\gamma y)+(1-C)J_0(\gamma x)-E=0.$$

The integrals in Eq. (6) can be evaluated for $\alpha(\gamma)=\alpha^0(\gamma)$ using[3]

$$\int_0^r dz\, zJ_0(az)J_0(bz)=\frac{r}{a^2-b^2}[aJ_1(ar)J_0(br)-bJ_1(br)J_0(ar)],$$

$$\int_0^r dz\, zJ_0{}^2(ar)=\tfrac{1}{2}r^2[J_0{}^2(ar)+J_1{}^2(ar)].$$

We obtain

$$\sigma_{\rm tot}=\frac{2\pi D}{k^2}\sum_i(-)^i\gamma_i{}^2\left[\frac{C}{\gamma_i y}J_1(\gamma_i y)+\frac{(1-C)}{\gamma_i x}J_1(\gamma_i x)-\tfrac{1}{2}E\right],\quad\text{(8a)}$$

$$\begin{aligned}\sigma_{\rm el}=\frac{2\pi D^2}{k^2}\sum_i(-)^i\gamma_i{}^2\Big[&\tfrac{1}{2}C^2(J_0{}^2(\gamma_i y)+J_1{}^2(\gamma_i y))+\tfrac{1}{2}(1-C)^2[J_0{}^2(\gamma_i x)+J_1{}^2(\gamma_i x)]+\tfrac{1}{2}E^2\\&-\frac{2EC}{\gamma_i y}J_1(\gamma_i y)-\frac{2E(1-C)}{\gamma_i x}J_1(\gamma_i x)+\frac{2C(1-C)}{\gamma_i{}^2x^2-\gamma_i{}^2y^2}\\&\times[\gamma_i xJ_1(\gamma_i x)J_0(\gamma_i y)-\gamma_i yJ_1(\gamma_i y)J_0(\gamma_i x)]\Big],\quad\text{(8b)}\end{aligned}$$

$$\begin{aligned}A(x)=\frac{s^{1/2}D}{2k}\sum_i(-)^i\gamma_i{}^2\Big[&\frac{C}{\gamma_i{}^2x^2-\gamma_i{}^2y^2}[\gamma_i xJ_1(\gamma_i x)J_0(\gamma_i y)\\&-\gamma_i yJ_1(\gamma_i y)J_0(\gamma_i x)]+\tfrac{1}{2}(1-C)\\&\times[J_0{}^2(\gamma_i x)+J_1{}^2(\gamma_i y)]-\frac{E}{\gamma_i x}J_1(\gamma_i x)\Big],\quad\text{(8c)}\end{aligned}$$

$$\begin{aligned}A(y)=\frac{s^{1/2}D}{2k}\sum_i(-)^i\gamma_i{}^2\Big[&\frac{(1-C)}{\gamma_i{}^2x^2-\gamma_i{}^2y^2}\\&\times[\gamma_i xJ_1(\gamma_i x)J_0(\gamma_i y)-\gamma_i yJ_1(\gamma_i y)J_0(\gamma_i x)]\\&+\tfrac{1}{2}C[J_0{}^2(\gamma_i y)+J_1{}^2(\gamma_i y)]-\frac{E}{\gamma_i y}J_1(\gamma_i y)\Big].\quad\text{(8d)}\end{aligned}$$

We replace the argument θ of A by

$$\rho_{\sin\frac{1}{2}\theta}=-k^2\sigma_{\rm tot}{}^2\sin^2\tfrac{1}{2}\theta/\pi\sigma_{\rm el}.$$

Because of the form of Eqs. (8), C and E can be determined in terms of ρ_x, ρ_y using (8a)–(8c); then $A(\rho_y)/A(0)$ as determined from Eq. (8d) can be given in terms of ρ_x and ρ_y only.

To discuss the range of input data for which Eq. (8) can be solved to give positive C and E, we first describe the Singh-Roy[1] problem. Setting $C=1$ and using only $\sigma_{\rm tot}{}^2/\sigma_{\rm el}$ from (8a) and (8b), we obtain the Singh-Roy bound $U^{\rm SR}(\rho_y)=A(\rho_y)/A(0)$. In this solution E is independent of ρ_x, and $A(\rho_x)/A(0)$ is simply the equation of the curve in ρ_x which maximizes $A(\rho_y)/A(0)$ at ρ_y. We call this function $f_{\rho_y}(\rho_x)$. Singh and Roy[1] have shown the existence of f_{ρ_y} for all ratios $k^2\sigma_{\rm tot}{}^2/4\pi\sigma_{\rm el}>1$. From its behavior we may infer the domain of solubility of the general problem. We use the function f_{ρ_y} to obtain a value ρ_0 such that, for $\rho>\rho_0$, the solution to the complete problem exists with C and E between zero and 1 and is an improvement on the Singh-Roy bound. The function $f_{\rho_y}(\rho)$ is plotted in Fig. 1 for various values of ρ_y. Relative to $U^{\rm SR}(\rho)$, $f_{\rho_y}(\rho)$ behaves as follows: it equals $U^{\rm SR}$ at $\rho=0$, ρ_y and is strictly below it at all other points. For large ρ it behaves asymptotically as

$$f_{\rho_y}(\rho)\underset{\rho\to\infty}{\sim}\frac{2\rho_y{}^2J_1(\gamma_1 y)}{\gamma_1 yJ_2(\gamma_1 y)}\frac{J_0((\gamma_1 y/\rho_y)\rho)}{\rho^2}.\quad(9)$$

Since $U^{\rm SR}(\rho_x)$ is a bound on the data, an input value of $A(\rho_x)/A(0)>U^{\rm SR}(\rho_x)$ has no solution with real C and E. An input value $A(\rho_x)/A(0)=U^{\rm SR}(\rho_x)$ has a solution with $C=0$ and the new upper bound given by $U^{\rm new}(\rho)=f_{\rho_y}(\rho)$. There is also a minimum value of $A(\rho_x)$ that can be given. For instance, Singh and Roy[1] obtain an absolute lower bound on $A(\rho)$ for fixed $\sigma_{\rm tot}$ and $\sigma_{\rm el}$; in addition, when the eikonal amplitude $\alpha^0(\gamma)$ is given by Eq. (7), this lower bound is raised. However, setting $r=0$. Analogous formulas exist for the integrals of higher-order Bessel functions.

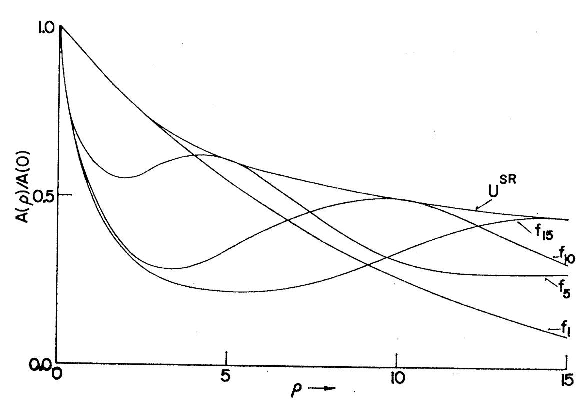

FIG. 1. Graph of the function $f_{\rho_y}(\rho)$ for various values of ρ_y, plotted for comparison with the Singh-Roy bound $U^{\rm SR}$.

[3] These integral identities can be proven directly by differentiation with respect to r. The constant of integration is found by setting $r=0$. Analogous formulas exist for the integrals of higher-order Bessel functions.

this does not affect our calculation since this minimum is well below the data (see, for instance, Fig. 1).

Let us consider an input value $A(\rho_x)/A(0)$ between $U^{\mathrm{SR}}(\rho_x)$ and this minimum. As the maximization point ρ_y increases beyond ρ_x, the value of $f_{\rho_y}(\rho)$ at $\rho=\rho_x$ drops below $U^{\mathrm{SR}}(\rho_x)$ and in general takes on the input value $A(\rho_x)/A(0)$ several times. Let ρ_0 be the largest value of ρ_y for which $f_{\rho_y}(\rho_x)=A(\rho_x)/A(0)$. We show that for all $\rho>\rho_0$, the upper bound $U^{\mathrm{new}}(\rho)$ exists and is between $f_{\rho_0}(\rho)$ and $U^{\mathrm{SR}}(\rho)$. By choice of ρ_0, it follows that ρ_0 increases monotonically as the input value of $A(\rho_x)/A(0)$ is decreased. Thus for $\rho>\rho_0$, $f_\rho(\rho_x)<A(\rho_x)/A(0)$. When C lies between zero and 1, the bound $|J_0(x)|\leqslant 1$, $x\geqslant 0$ requires E also to lie between zero and 1. For this range

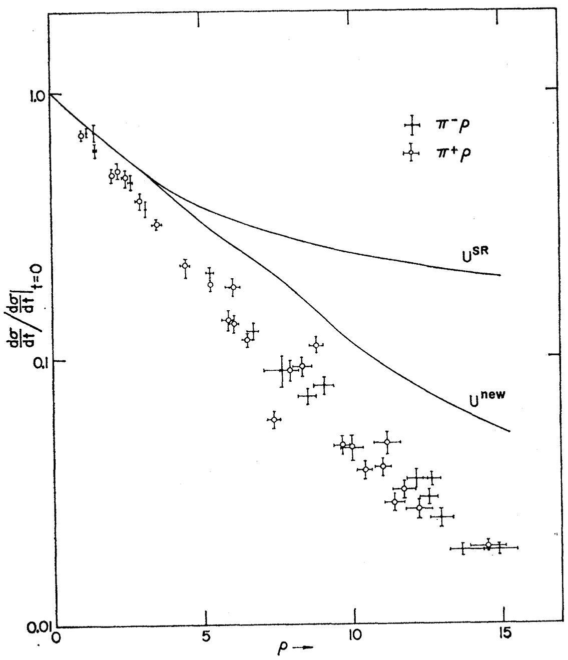

FIG. 2. Comparison of experimental data for $\pi^\pm p$ at various energies with the theoretical bounds U^{SR} and U^{new} described in the text.

of C and E the quantities σ_{tot}, σ_{el}, and $A(x)$ depend continuously on C and E. Thus, as C is changed from zero to 1, the predicted $A(\rho_x)/A(0)$ drops from $U^{\mathrm{SR}}(\rho_x)$ to $f_\rho(\rho_x)$ continuously, passing through the input value of $A(\rho_x)/A(0)$. Hence the solution exists. The solution is necessarily bounded above by $U^{\mathrm{SR}}(\rho)$, and since $f_{\rho_0}(\rho)$ satisfies all of the input data, it must bound $U^{\mathrm{new}}(\rho)$ from below.

IV. COMPARISON WITH DATA AND CONCLUSIONS

In the comparison of our bound with experiment, we use the elastic scattering data from six different reac-

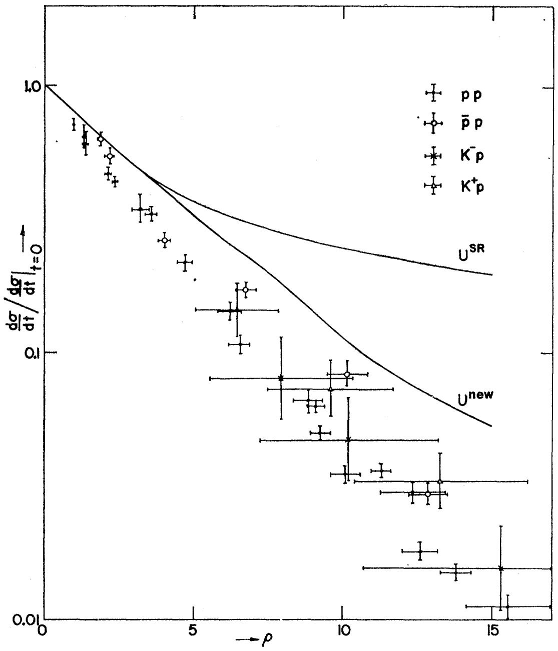

FIG. 3. Comparison of experimental data for pp, $\bar{p}p$, $K^\pm p$ at various energies with the theoretical bounds U^{SR} and U^{new} described in the text.

tions at various laboratory momenta. These are: $\pi^\pm p$ [4] between 6.8 and 18.9 BeV/c, pp [4] between 6.8 and 19.6 BeV/c, K^+p [5] at 6.8 and 14.8 BeV/c, K^-p [5] at 9 BeV/c, and $\bar{p}p$ [5] at 8.9 and 12.0 BeV/c. The first two of these are plotted in Fig. 2, the remaining four in Fig. 3. Not all of the data at these energies have been plotted in order to avoid overcrowding; however, inclusion of the missing data or data from other high energies does not affect the character of the figures. The information on α, the ratio of real to imaginary part of the scattering amplitude, is as follows: For π^+p, α is known to be less than 0.2 in magnitude in the energy range stated above.[6] A Regge-pole fit to the pion-nucleon and nucleon-nucleon data by Rarita *et al.*[6] also predicts a value of α less than 0.2 in magnitude. The π^-p data vary between -0.4 and -0.1 with the Regge fit bounded by about -0.25 in this energy range.[6] The smaller magnitudes of α are generally at higher energies. The pp data are around -0.3, with the Regge fit agreeing at lower energies and predicting a value closer to zero at higher energies.[6] α for K^+p is about -0.5 or -0.6 over most of the energy range 3–15 BeV/c, while the α for K^-p is practically zero, lying between 0.0 and 0.1 over this same energy range.[7] Lastly, although there we could

[4] K. J. Foley, S. J. Lindenbaum, W. A. Love, S. Ozaki, J. J. Russell, and L. C. L. Yuan, Phys. Rev. Letters **11**, 425 (1963).
[5] K. J. Foley *et al.*, Phys. Rev. Letters **11**, 503 (1963).
[6] W. Rarita, R. J. Riddell, C. B. Chiu, and R. J. N. Phillips, Phys. Rev. **165**, 1615 (1968).
[7] L. de Lella, in *Proceedings of the Heidelberg Conference on Elementary Particles*, edited by H. Filthuth (North-Holland, Amsterdam, 1968), p. 159.

find no experimental measurements of α for $\bar{p}p$, the Regge fit of Rarita *et al.*[6] predicts a very small ratio, between 0.0 and -0.1, over a wide range of energies for this reaction.

Because we plot the ratio $(d\sigma/dt)/(d\sigma/dt|_0)$, errors due to α enter the vertical scale only as variations of α^2 at various t values (although ρ goes from zero to 15, the typical t variation is from 0.0 to -0.5 BeV^2). The error in the ρ coordinate depends only quadratically on α. For the reactions we have plotted, we can then estimate the error in the applicability of the theory to be about 4% in π^+p, 15% in π^-p, 10% in pp, 30–40% in K^+p, 1% in K^-p, and 1% in $\bar{p}p$. The π^+p, pp, and K^+p reactions have been included with the other more relevant reactions because of the observation that they appear not to change the approximate universality of the figures and to indicate the general applicability of the method.

It can be seen from Figs. 2 and 3 that the data are reasonably well bounded by U^{SR} in the region ρ between zero and 3. Beyond ρ equals 3 or 4, the data continue to decrease roughly exponentially while the bound U^{SR} decreases much more slowly. As we have shown in Sec. III, improvement of the bound for $\rho>4$ through knowledge of $A(\rho_x)/A(0)$ at one fixed value of ρ_x requires $\rho_0<4$. ρ_0 can be made smaller by decreasing ρ_x or by choosing $A(\rho_x)/A(0)$ closer to $U^{\mathrm{SR}}(\rho_x)$ for fixed ρ_x. Since the data become closer to the Singh-Roy bound for smaller ρ_x, both effects are in the same direction, and we need just decrease ρ_x until $\rho_0<4$. On the other hand, we cannot make ρ_x too small, both because we do not have data at very small values of ρ_x and because the new bound at large ρ becomes very sensitive to the actual input value at small ρ_x.

We have chosen $\rho_x=1$ and set $A(\rho_x)/A(0)=0.85$, a value which is a rough average of the data at that point. The calculation was done by computer, and the results are shown in Figs. 2 and 3. The new bound $(U^{\mathrm{new}})^2$ is plotted only for $\rho>\rho_0\sim3.5$.

A measure of the sensitivity of this bound to the input value can be given through the quantity

$$\mathcal{S}=\frac{d\ln[U^{\mathrm{new}}(15)]^2}{d\ln[A(1)/A(0)]^2}.$$

At a value of $A(1)/A(0)=U^{\mathrm{SR}}(1)$, $\mathcal{S}=-\infty$ so that a choice of input value very close to U^{SR} will not give a reliable bound. [This is not surprising since if $A(\rho_0)/A(0)$ achieve $U^{\mathrm{SR}}(\rho_0)$, A is known for all ρ and is given by $f_{\rho_0}(\rho)$.] Fortunately the sensitivity falls off sharply as $A(1)/A(0)$ is lowered, and $\mathcal{S}\sim-6$ around the input value we have used. This means that the variation of $A(1)/A(0)$ by ±0.025 (which are roughly the error bounds on the data at that point) changes $U^{\mathrm{new}}(15)$ by about ∓0.011, which is quite acceptable. We remark that raising $A(1)/A(0)$ sufficiently above the data causes U^{new} to fall below the data in the region $\rho\sim10$ to 15, and thus the large ρ data give a bound on $A(1)/A(0)$. However, this bound does not turn out to be useful because it lies just below $U^{\mathrm{SR}}(1)$.

There is an upper limit to the values of ρ for which we expect U^{new} to be useful. From Eq. (9) we see that $f_1(\rho)$, which is always below $U^{\mathrm{new}}(\rho)$, $\rho>3.5$, decreases only like $15\rho^{-5/2}$. Since the data continue to drop exponentially even to relatively large ρ values, U^{new} cannot be expected to bound the data tightly out to very large ρ. We have therefore chosen to evaluate the bound only out to $\rho=15$.

The $\pi^\pm p$ data are relatively uniform in falling on an approximately universal curve. The scatter of the points increases as ρ increases, indicating that the curve is not really universal. The points do lie, however, within a wedge-shaped region which starts at $\rho=0$ at 1 and by $\rho=15$ has a width on a logarithmic scale corresponding to a spread of a factor of 2 in the data. As we have mentioned above, inclusion of the π^+p data in Fig. 2 and the pp and K^+p data in Fig. 3 does not particularly change either the character or approximate universality of the graphs. The pp data are, perhaps, a bit lower than the other data but, again, all of the data on both figures fall within the same wedge-shaped region. The upper bound obtained from our calculation with

$$[d\sigma(1)/dt]/[d\sigma(0)/dt]=0.7225$$

is shown on the figures. It can be seen that the new bound is as much as a factor of 4 better than the Singh-Roy bound. With a spread in the data of about a factor of 2, we should not expect the new bound, which is a universal function of ρ, to be exceptionally close to the data. It does bound the wedge, however, by about a factor of 2.

In summary, this method provides a procedure for obtaining upper bounds for the imaginary part of the elastic scattering amplitudes which are relatively good bounds for the extreme forward direction out to values of $\rho\sim15$.