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Direct and Crossed Watson Terms in Three-Body Final-State Interactions*

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Three-body final-state interactions are considered for simple cases within the framework of the Khuri-Treiman equation. It is noted that an approximation to partial-wave amplitudes, which consists essentially of the direct- and crossed-channel Watson terms, reproduces the analytic properties of the exact amplitudes to a considerable extent, particularly in the vicinity of the physical region.

I. INTRODUCTION

SEVERAL years ago, Kacser[1] proposed a method of specifying the analytic structure of partial-wave amplitudes satisfying Khuri-Treiman-type equations. In the case of a neutral spinless state of mass m decaying into three identical neutral spinless particles of mass μ and only s-wave interactions in the final state, the relevant equation is[2] (comments to follow shortly)

$$T_0(s)=\frac{1}{\pi}\int_{4\mu^2}^{\infty} ds' f_0^*(s')T_0(s')\times\left(\frac{1}{s'-s-i\epsilon}+2G(s,s'-i\epsilon;m^2)\right). \quad (1)$$

Solution of Eq. (1) determines the transition amplitude,

$$T(s_1,s_2,s_3)=\frac{1}{\pi}\int_{4\mu^2}^{\infty} ds\frac{f_0^*(s)T_0(s)}{s-s_1-i\epsilon}+(s_1\leftrightarrow s_2)+(s_1\leftrightarrow s_3), \quad (2)$$

with $s_1=-(p_2+p_3)^2$ and cyclic permutations; p_i are the four-momenta of the final particles. Equation (1) is the s-wave projection[3] of Eq. (2) onto the two-body "channel" in which s, $s=s_1$ say, is the square of the total energy (the center-of-mass frame $\mathbf{p}_2+\mathbf{p}_3=\mathbf{0}$ is used), henceforth called the direct channel. $f_0(s)$ is the s-wave elastic scattering amplitude for $(\mu)+(\mu)\to(\mu)+(\mu)$, (μ) symbolizing a particle of mass μ. $\mu=1$ from now on. The first term in Eq. (1) represents the effects of two-body virtual intermediate states in the direct channel (direct term), whereas the second term incorporates corresponding effects in the two crossed channels (crossed term). The exact form of $G(s,s'-i\epsilon;m^2)$ is given in Refs. 3 and 2. It should be noted that a term representing decay without final-state interactions has been omitted from Eqs. (1) and (2) [the transition amplitude vanishes if $f_0(s)\equiv 0$]. This is done to avoid complications,[4] although the modification of the results for the case in which this term is a constant will be indicated later. It is assumed that the integrals in Eqs. (1) and (2) converge, but this restriction can be easily lifted.

The aim of Ref. 1 was to set up an integral equation for the partial-wave amplitude $T_0(s)$ by forming a contour integral around all its "first"-sheet singularities. The first-sheet structure of $T_0(s)$ proposed by Kacser is shown[5] in Fig. 1. It was obtained under the following assumptions about $f_0(s)$ [in addition to absence of bound-state poles, which is postulated in writing Eq. (1)]: $f_0(s)$ has a right-hand cut, $s=4$ to ∞, but contributions from inelastic intermediate states are neglected, in the spirit of the Khuri-Treiman equation[1]; $f_0(s)$ has the usual left-hand cut,[6] $s=0$ to $-\infty$; $f_0(s)$ also has poles on its second sheet, $\mathrm{Im}k<0$, where

$$k=(s-4)^{1/2}; \quad (3)$$

however, none lie in regions $U_\pm$.[5]

The case for which $f_0(s)$ has cuts as above but also a pair of second sheet poles for $k\in U_\pm$, corresponding to a resonance at $s=\alpha_0\in U_-$, was analyzed by Bonnevay.[7] Bonnevay actually set up an integral equation for the function $a(s)/k=f_0^*(s)T_0(s)/k$, after showing that this function has no right-hand cut. He also made a different choice of cuts,[8] and hence of sheets, for $T_0(s)$ from that of Ref. 1. Thus the homogeneous terms in Eq. (22) of Ref. 7 consist of two integrals running along the real axis from 0 to $-\infty$ and from $(m-1)^2$ to ∞, rather than along the left-hand cut of Fig. 1. The inhomogeneous terms contain the information that $a(s)/k$ has, on its first sheet, poles with known (given) residues at $s=\alpha_0\in U_-$ and $s=\alpha_0^*\in U_+$, due to the factor $f_0^*(s)$, and two logarithmic branch points at $s=\alpha_1\in W_-$ and at $s=\alpha_2\in V_+$ generating two distinct cuts with discon-

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[1] C. Kacser, Phys. Rev. 132, 2712 (1963).

[2] J. B. Bronzan, Phys. Rev. 134, B687 (1964). The notation of this paper is largely followed here.

[3] For subtle points concerning this projection, see especially J. B. Bronzan and C. Kacser, Phys. Rev. 132, 2703 (1963).

[4] See R. D. Amado and J. V. Noble [Phys. Rev. 185, 1993 (1969)] for complications which can be introduced by various decay mechanisms. Considerations of simplicity also dictate restriction to spinless particles, equal masses, and s waves, although these restrictions, with the possible exception of the first one, are not crucial. Possible trouble with Eq. (2) itself is mentioned in Ref. 7.

[5] Region W_+ in Fig. 1, as well as W_-, $V_\pm$, $U_\pm$, considered later, is defined in Ref. 1; cf. also Fig. 2.

[6] It is pertinent to what follows that this cut does not affect the analysis of Ref. 1.

[7] G. Bonnevay, Nuovo Cimento 30, 1325 (1963).

[8] For a discussion see Ref. 1.

tinuities across them known in terms of $f_0(s)$ and the residue at α_0. The effect of other singularities, lying farther from the physical region on different Riemann sheets,[9] is not taken into account explicitly since the contour integrals brush by the first sheet singularities only.

In the present paper, the cases treated by Kacser[1] and Bonnevay[7] are considered simultaneously. The observation is made that under certain simplifying assumptions about $f_0(s)$ (in particular, absence of bound states, absence of inelastic thresholds on the right-hand cut, and neglect of the left-hand cut), it is possible to obtain an approximate expression for $T_0(s)$ in closed form, called $S_0(s)$, very similar to approximate expressions which have appeared in the literature,[2,10,11] with the following properties.

(1) $f_0^*(s)S_0(s)/k$ does not have a right-hand cut.[7]

(2) If $f_0(s)$ has no second-sheet poles for $k\in U_\pm$, then $S_0(s)$ can be so continued away from the physical region as to have on its first sheet precisely the Kacser cuts, Fig. 1, with discontinuities qualitatively as given in Ref. 1 for the "exact" amplitude [i.e., for the exact solution of Eq. (1) under Kacser's restrictions on $f_0(s)$]. The discontinuities are expected to be rather accurate across those portions of the cut which are close to the physical region [4, $(m-1)^2$]. $S_0(s)$ has no logarithmic singularities on the first sheet but it has some on higher sheets and in particular those of the exact amplitude which lie close to the physical region. Further, $S_0(s)$ has no poles on the first sheet, but if one crosses the right-hand cut from above one encounters poles corresponding to those of $f_0(s)$.

(3) If $f_0(s)$ has a pair of second-sheet poles for $k\in U_\pm$, corresponding to a resonance at $s=\alpha_0\in U_-$, then $S_0(s)$ has, on its first sheet, but not near the physical region, a logarithmic branch point at $s=\alpha_1\in W_-$ in addition to the cuts of Fig. 1. If one leaves the first sheet by crossing the right-hand cut from above, one encounters an identical left-hand singularity structure on the new sheet, as well as poles at $s=\alpha_0$, $s=\alpha_0^*$. The logarithmic branch point $s=\alpha_1$ on this sheet will lie close to the physical region if α_0 is close to the physical region. There are logarithmic branch points on other Riemann sheets too. The one closest to the physical region lies in V_+. It is reached from the physical region by crossing the left-hand cut to the right of $s=1+m$ and corresponds to Bonnevay's α_2. In fact, one may continue $S_0(s)$ away from the physical region in such a way as to displace part of the left-hand cut on top of the right-hand cut. Then $f_0^*(s)S_0(s)/k$ will have, on its first sheet, exactly the basic cuts and logarithmic branch points α_1, α_2 of the $a(s)/k$ of Ref. 7, with discontinuities again in qualitative agreement, particularly in the vicinity of the physical region.

Thus the analytic structure of $S_0(s)$ simulates that of the exact amplitude $T_0(s)$ in those regions of the complex plane which have the greatest influence on the behavior of the amplitude in the physical region. One may therefore speculate that $S_0(s)$ would be a good approximation to use in cases in which the restrictions on $f_0(s)$ mentioned previously can be regarded as reasonable, or at least that it would be so with some slight modification. Comparable conclusions for analogous expressions have previously been reached by various authors in various contexts, experimental and theoretical.[12] The present work is therefore of the nature of a remark on a familiar topic.

II. WATSON TERMS

Let $f_0(s)$ be parametrized by one of the usual low-energy algebraic expressions[2,10,13] which entirely neglect inelastic thresholds as well as the left-hand cut and such that $f_0(s)$ exhibits no first sheet poles,

$$f_0(s)=k/[\phi(k^2)-ik], \tag{4}$$

where ϕ is a polynomial in k^2 with real coefficients.[14] For definiteness and to achieve correspondence with Ref. 7, ϕ will be taken to be of the first degree. Then $f_0(s)$ can be rewritten as a product of two second-sheet poles,

$$f_0(s)=c_0k\prod_{j=1,2}(k-k_{0j})^{-1}, \tag{5}$$

with c_0 a constant and $k_{02}=-k_{01}^*$, $\mathrm{Im}k_{0j}<0$.

Suppose that Eq. (1) is solved by iteration according to Bronzan's method,[2] assuming the above form for $f_0(s)$. First $G(s,s'-i\epsilon;m^2)$ is set equal to zero and an exact solution is obtained,[15,2] $T_{0,0}(s)$, which is the direct Watson term (normalization adjustable)

$$T_{0,0}(s)=c_0^{-1}[\phi(k^2)-ik]^{-1}. \tag{6}$$

Substitution of $T_{0,0}(s)$ in the integrand of the complete Eq. (1) leads to an expression for the left-hand side,[2] called $T_{0,1}(s)$, which constitutes an approximation in closed form to the exact $T_0(s)$,

$$T_{0,1}(s)=T_{0,0}(s)+2W_0(s). \tag{7}$$

$W_0(s)$ is a crossed Watson term [the s-wave projection, onto the direct channel, of $T_{0,0}(s_i)$ in a crossed channel in which $s_i^{1/2}$ is the total energy in the c.m. system].

[9] All logarithmic singularities are located at α_0, α_1, α_2, or their complex conjugates. See Ref. 7. This is a feature of the equal-mass case.

[10] I. J. R. Aitchison, Nuovo Cimento **35**, 434 (1964).

[11] C. Schmid, Phys. Rev. **154**, 1363 (1967).

[12] Compare the concluding remarks of Ref. 7. Also see, e.g., Refs. 2 and 11 and I. J. R. Aitchison, Nuovo Cimento **51A**, 249 (1967); **51A**, 271 (1967). See, however, Ref. 4. For a different approach to the problem see C. Lovelace, Phys. Letters **28B**, 264 (1968), and references quoted there.

[13] G. Barton, *Dispersion Techniques in Field Theory* (Benjamin, New York, 1965), p. 167 ff.

[14] Reality follows from the fact that inelastic thresholds have been neglected, so that the phase shift δ is real for s on the right-hand cut [$k\cot\delta=\phi(k^2)$].

[15] R. Omnès, Nuovo Cimento **8**, 316 (1958). Ambiguities connected with Eq. (6) are discussed in this paper.

$W_0(s)$ is given in the physical region by Eq. (A8) of Ref. 2. Approximate expressions are often used.[16] However, these obscure or even significantly distort the analytic properties of the exact expression. As will be seen below, $W_0(s)$ can be continued into the complex plane in such a way that $T_{0,1}(s)$ has properties (2) and (3) mentioned in the Introduction. Still, $T_{0,1}(s)$ would not have property (1), since $f_0^*(s)T_{0,1}(s)/k$ has a right-hand cut.

To correct for the last-mentioned feature of $T_{0,1}$, one may consider a transformation of Eq. (1) in the manner of Omnès.[15,1,10] The transformation in this case is formal, since the singular nature of the kernel does not necessarily permit[2] all manipulations required. However, it may serve as a guide. The formal result of the transformation is

$$T_0(s)=c_0T_{0,0}(s)\times\left[C+\phi(k^2)2Y_0(s)+2\frac{\mathcal{P}}{\pi}\int_4^\infty ds'\frac{k'Y_0(s')}{s'-s}\right], \quad (8)$$

where C is an input constant, $\mathcal{P}$ denotes principal value, and the over-all normalization is arbitrary.[17] $2Y_0(s)$ denotes the crossed term in Eq. (1).

Equation (8) can now be used as a starting point for an iterative solution, just as Eq. (1) was used by Bronzan. Thus, let $T_0(s)$ be approximated by

$$S_0(s)=c_0T_{0,0}(s)[C+2\phi(k^2)W_0(s)]. \quad (9)$$

$T_{0,0}(s)$ has the right-hand cut of Fig. 1 and second-sheet poles at $s=\alpha_0$ and $s=\alpha_0^*$ ($\alpha_0=k_{01}^2+4$, $\mathrm{Im}\alpha_0<0$). Granting that $W_0(s)$ does not have the right-hand cut, it is easy to see (cf. Ref. 14) that $f_0^*(s)S_0(s)/k$ does not have this cut either and therefore possesses property (1). Furthermore, $S_0(s)$ will have properties (2) and (3) provided that $W_0(s)$ is suitably continued into the complex plane from the physical region, where it is uniquely specified.[18] The analytic continuation of $W_0(s)$ away from the physical region which enables $S_0(s)$ to have the properties mentioned in the Introduction,

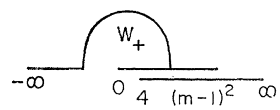

FIG. 1. Cuts of $T_0(s)$ in the s plane for the case that $f_0(s)$ has no second-sheet poles for $k\in U_\pm$, i.e., approximately, no poles in s with a real part greater than $\frac{1}{2}(m^2-1)$. The right-hand cut runs from $s=4$ to ∞. The left-hand cut starts at $s=(m-1)^2$ and includes the arc shown, which intersects the real axis at $1\pm m$. The arc is fully specified in Ref. 1, where it is called S^+. The physical region $[4,(m-1)^2]$ is located just above the right-hand cut and just below the left-hand cut. $T_0(s)$ has no poles on the first sheet.

with Kacser's choice of cuts, is

$$W_0^{\mathrm{I}}(s)=\sum_{j=1}^{2}A_j\frac{s}{[U(s)]^{1/2}}\ln\frac{k_{0j}^*\pm k_+(s)}{k_{0j}^*+k_-(s)}, \quad (10)$$

where the upper sign holds in all regions except W_+ and the lower sign holds in W_+.[19] The superscript I denotes the first sheet. A_j is a constant[20] depending on the parametrization of $f_0(s)$. $U(s)$ is a polynomial,[2]

$$U(s)=s(s-4)[s-(m-1)^2][s-(m+1)^2]. \quad (11)$$

The branch of the logarithm is chosen[2] to make $W_0(s)$ finite at $s=4$. Further,

$$k_\pm(s)=[s_\pm(s)-4]^{1/2}. \quad (12)$$

The functions $s_\pm(s)$, which have the same cuts as $U(s)^{1/2}$, are defined in Refs. 1 and 2 (they are denoted by $s^\mp$ in Ref. 7). They have the property that $s_\pm\to s_\mp$ when $U^{1/2}\to -U^{1/2}$, which property is not shared by $k_\pm$, as will be seen shortly. It is helpful to specify the principal branch of k_+ and k_- as functions of s by means of schematic drawings (Figs. 2–4) showing a path in s and the corresponding paths traced by $k_+(s)$ and $k_-(s)$ in the k_+ and k_- planes, respectively.[21] This is the branch used in Eq. (10) and in what follows. The principal branch of $U^{1/2}$ is given in Fig. 3 of Ref. 1. It is such that for $4<s<(m-1)^2$, $U(s-i\epsilon)^{1/2}=|U(s)^{1/2}|$, and for $(m-1)^2<s<(m+1)^2$, $U(s-i\epsilon)^{1/2}=i|U(s)^{1/2}|$ ($\epsilon>0$, infinitesimal).

With reference to Figs. 2–4 and Eq. (10), it can readily be verified that $W_0^{\mathrm{I}}(s)$, and hence $S_0^{\mathrm{I}}(s)$, possesses exactly the left-hand cut of Fig. 1. The discontinuities are not, of course, exactly the same as those of $T_0(s)$. Apart from an over-all correction factor $c_0T_{0,0}(s)\phi(k^2)$, the discontinuities of $S_0^{\mathrm{I}}(s)$ are the same as those which would result from the integral expressions Eqs. (3.19a)–(3.19f) of Ref. 1 if one used $T_{0,0}(s')$ to approximate $T_0(s')$ in the integrands. However, such an approximation often works quite well, even as a final answer for $T_0(s)$ itself, in the physical region.[22]

[16] Eq. (3.5) of Ref. 2; Eq. (25) of Ref. 10; also Eq. (28) of Ref. 10; Eq. (24) of Ref. 7; Eq. (25) of Ref 11. In all these cases $f_0(s)$ has a pole at $s=\alpha_0$ corresponding to a resonance ($\mathrm{Re}\alpha_0>4$, $|\mathrm{Im}\alpha_0|$ small). The latter three expressions treat this resonance as a particle of complex mass, with the result that the analytic structure of $W_0(s)$ is much simpler than the one considered here and cannot satisfy the general analyticity requirements of $T_0(s)$. This is a consequence of the omission of the right-hand cut in $f_0(s)$.

[17] At this point it is convenient to consider the effects of including a constant decay vertex in Eq. (1). This may be thought of as fixing the normalization of $T_0(s)$ at some point $s=s_0$, so that a subtraction can be made at this point. The only effect of this subtraction in the present case of convergent integrals (other than fixing the normalization) is to replace $Y_0(z)$ in Eq. (8) by $Y_0(z)-Y_0(s_0)$ (cf. Ref. 1). Thus the subsequent discussion will still apply.

[18] Equation (10) coincides with Eq. (A8) of Ref. 2 in the physical region. It is only differently written. The physical limit is $W_0^{\mathrm{I}}(s-i\epsilon)$ (cf. Fig. 1).

[19] A uniform + sign would lead to a cut on the real axis from $-\infty$ to $(m-1)^2$ for $W_0^{\mathrm{I}}(s)$, as originally in Ref. 7 (before part of this cut is placed there on the positive real axis—see below).

[20] $A_j=2c_0(k_{0j}^*)^2(\mathrm{Im}k_{0j})^{-1}(k_{0j}^*-k_{0\lambda}^*)^{-1}(k_{0j}^*-k_{0\lambda})^{-1}$, $\lambda\neq j$.

[21] Compare also Eqs. (3.3) and (3.4) of Ref. 1, which give the mappings $s\to s_\pm(s)$.

[22] See, e.g., the papers by Aitchison quoted in Ref. 12.

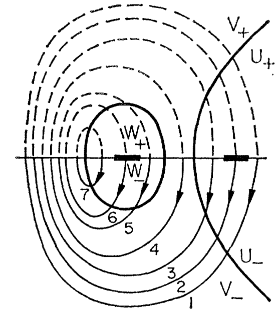

FIG. 2. Paths of s used in Figs. 3 and 4. All paths originate just below the real axis and follow the arrows. Regions $W_\pm$, $V_\pm$, and $U_\pm$ are defined in Ref. 1. The two heavily drawn line segments on the real axis represent the cuts of $U(s)^{1/2}$, namely, $[0,4]$ and $[(m-1)^2, (m+1)^2]$. The common boundary of V_+, W_+ (or V_-, W_-), called S^+ (S^-) in Ref. 1, intersects the real axis at $1-m$ and $1+m$. The common boundary of V_+, U_+ (or V_-, U_-), called H^+ (H^-) in Ref. 1, intersects the real axis at $\frac{1}{2}(m^2-1)$.

Since the integrals which give the discontinuities of $T_0(s)$ across $[4,(m-1)^2]$ involve in their integrands $T_0(s')$ almost exclusively for s' in the physical region,[23] the approximation should work quite well in obtaining these discontinuities. The discontinuities across the rest of the left-hand cut may not be as accurate, but these cuts are further away from the physical region. The situation with the right-hand cut of $S_0(s)$ is similar. If $k_{0j}\notin U_\pm$, there are no other cuts on the first sheet. In particular, there are no logarithmic cuts since it is only for $k_+\in U_\pm$ that $\mathrm{Im}k_+<0$ ($\mathrm{Im}k_-$ is always positive). Thus the first-sheet structure of $T_0(s)$ is qualitatively reproduced in this case by using only direct and crossed Watson terms[24] as modified in Eq. (9).

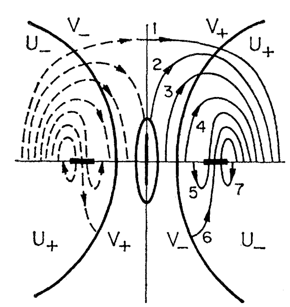

FIG. 3. Paths of the principal branch of $k_+(s)$ in the k_+ plane corresponding to the paths in Fig. 2. The k_+ plane is divided into regions which are mappings of the regions in Fig. 2; e.g., for $s\in U_-$, $s_+(s)\in V_+$ so that $k_+(s)\in V_+$, etc. Here only $U_\pm$ and $V_\pm$ are labeled. Owing to the fact that the three final particles have the same mass, the regions in the k_+ plane obtained from the mapping $s\rightarrow k_+(s)$ may also be obtained (although not in the same order) from the simpler mapping $s\rightarrow k(s)$; the labeling has been chosen to correspond to the latter mapping. Similar comments apply to Fig. 4.

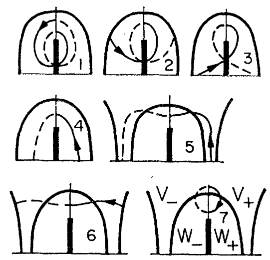

FIG. 4. Paths of the principal branch of $k_-(s)$ in the k_- plane corresponding to the paths of Fig. 2; cf. caption of Fig. 3. Only upper half-plane regions are relevant (W_+ and W_- are shown in the first four drawings, W_+, W_-, V_+, and V_- in the last three). The heavily drawn vertical line is $0\lesssim -ik_-\lesssim 2$, k_- imaginary.

It is also of interest to examine higher Riemann sheets of $W_0(s)$ since some singularities on them may lie near the physical region. $W_0(s)$ possesses four "primary" sheets, i.e., sheets which connect among themselves through cuts which are independent of k_{0j}. On each of the sheets there can be a certain number of logarithmic branch points, generally not located on any of the basic cuts, whose position depends on k_{0j}. Each of these points generates an infinite number of "secondary" sheets. Equation (10) describes the first (primary) sheet, chosen according to Kacser's specifications. The second, third, and fourth can be chosen as follows:

$$W_0{}^{\mathrm{II}}(s)=\sum_{j=1}^{2} A_j\frac{s}{[U(s)]^{1/2}}\ln\frac{k_{0j}{}^*\mp k_+(s)}{k_{0j}{}^*-k_-(s)},$$

$$W_0{}^{\mathrm{III}}(s)=\sum_{j=1}^{2} A_j\frac{s}{[U(s)]^{1/2}}\ln\frac{k_{0j}{}^*\pm k_+(s)}{k_{0j}{}^*-k_-(s)},\qquad(13)$$

$$W_0{}^{\mathrm{IV}}(s)=\sum_{j=1}^{2} A_j\frac{s}{[U(s)]^{1/2}}\ln\frac{k_{0j}{}^*\mp k_+(s)}{k_{0j}{}^*+k_-(s)},$$

where the lower sign again is for $s\in W_+$. All sheets have exactly the left-hand cut of Fig. 1, except for the third and fourth which have an additional cut $[(m-1)^2, (m+1)^2]$. These sheets can be reached from the first one as follows: $W_0{}^{\mathrm{II}}(s)$ by crossing the cut $[1+m, (m-1)^2]$ from below or above; $W_0{}^{\mathrm{III}}(s)$ by crossing $[0, 1+m]$ from below[25]; $W_0{}^{\mathrm{IV}}(s)$ by crossing from below or above the complex cut which starts at $1+m$ and goes to $-\infty$. The remaining connections among the sheets can be easily established.

With the help of Figs. 2–4 it can be seen that, for $k_{0j}\in W_\pm$, there are eight logarithmic branch points of $W_0(s)$, distributed as follows (notation chosen to corre-

[23] In all integrals s' is in the physical region except in the second integral of Eq. (3.19d) of Ref. 1, where $s_-(s_1)$ must be taken to be above the cut $[4, 1+m]$ rather than below; cf. Fig. 7 of Ref. 1.

[24] Attention should, perhaps, be drawn to the terminology. The direct Watson term alone will, upon substitution into Eq. (2), give what is generally called the Watson approximation. Since it is Eq. (2) that is in practice obtained from (1), not the reverse, consideration of the crossed Watson terms in $T_0(s)$ goes one step further than the usual Watson terms in approximating $T(s_1,s_2,s_3)$, since it represents a particular "triangle" type of rescattering process; see, e.g., Fig. 12, Ref. 7. The "triangle amplitude" for $T_0(s)$ itself is only partially included (cf. Ref. 28).

[25] Crossing $[4, 1+m]$ from above also leads to the third sheet, but crossing $[0,4]$ from above leads to the fourth.

spond to Ref. 7):

Sheets I and IV: none;

Sheet II: $\alpha_1 \in V_+$, $\alpha_2^* \in U_+$, $\alpha_2 \in U_-$, $\alpha_1^* \in V_-$;

Sheet III: $\alpha_4 \in V_+$, $\alpha_5^* \in U_+$, $\alpha_5 \in U_-$, $\alpha_4^* \in V_-$.

If α_0 is a resonance near the physical region, i.e., approximately $4 < \mathrm{Re}\alpha_0 < 1+m$, $|\mathrm{Im}\alpha_0|$ small, then α_1 and α_2^* will be very close to the physical region since they are encountered immediately upon crossing $[1+m,(m-1)^2]$ from below (recall that the physical region is below the left-hand cut).

If now $k_{0j} \in V_\pm$, some of the above singularities cross a basic cut of $W_0(s)$ and appear on a different sheet:

Sheet I: none;

Sheet II: $\alpha_4 \in W_+$, $\alpha_2 \in U_+$, $\alpha_2^* \in U_-$, $\alpha_1^* \in W_-$;

Sheet III: $\alpha_1 \in W_+$, $\alpha_4^* \in W_-$;

Sheet IV: $\alpha_5 \in U_+$, $\alpha_5^* \in U_-$.

In this case, α_1 and α_2 are close to the physical region if α_0 is below $[1+m, \frac{1}{2}(m^2-1)]$, close to the real axis.

Finally, when $k_{0j} \in U_\pm$, the logarithmic branch points are as follows:

Sheet I: $\alpha_1 \in W_-$;

Sheet II: $\alpha_4^* \in W_+$, $\alpha_2 \in V_+$, $\alpha_2^* \in V_-$;

Sheet III: $\alpha_4 \in W_-$;

Sheet IV: $\alpha_1^* \in W_+$, $\alpha_5 \in V_+$, $\alpha_5^* \in V_-$.

This is the case studied in Ref. 7. If α_0 is just below $[\frac{1}{2}(m^2-1), (m-1)^2]$, then α_1 will be close to the physical region, below $[4, 1+m]$ (cf. Refs. 2 and 11). The singularity at α_2 will also be very close, since it is just above the cut $[1+m, \frac{1}{2}(m^2-1)]$. Thus α_1 and α_2 are the two most important logarithmic branch points in this case, as in Ref. 7. The reason for the appearance of α_2 here is that $s_+(\alpha_2)=\alpha_0$. The other "end point" singularity[7] at α_2, due to $s_-(\alpha_2)=\alpha_1$, is not taken into account by Eq. (13) because the iteration procedure for finding $T_0(s)$ has stopped at the second cycle. Thus the second term in Bonnevay's Eq. (21) and hence the third term in Eq. (22) have no correspondence here (which is tantamount to neglecting his cut α_2 to ∞). However, all other terms are qualitatively represented. This can be seen, perhaps, most easily if part of the cut of $W_0^{\mathrm{I}}(s)$, attached to the singularity at $(m-1)^2$, is pushed from below onto the positive real axis, or rather infinitesimally above the right-hand cut of $T_0(s)$, as in Ref. 7. The analytic continuation of $W_0(s)$ which can accomplish this is not unique, but the one specified by the following definition of primary sheets is a possible choice:

$$\begin{aligned}
W_0^{\mathrm{I}}(s) &= \sum_{j=1}^{2} A_j \frac{s}{[U(s)]^{1/2}} \ln\frac{k_{0j}^* \pm k_+(s)}{k_{0j}^* \pm k_-(s)}, \\
W_0^{\mathrm{II}}(s) &= \sum_{j=1}^{2} A_j \frac{s}{[U(s)]^{1/2}} \ln\frac{k_{0j}^* \mp k_+(s)}{k_{0j}^* \mp k_-(s)}, \\
W_0^{\mathrm{III}}(s) &= \sum_{j=1}^{2} A_j \frac{s}{[U(s)]^{1/2}} \ln\frac{k_{0j}^* - k_+(s)}{k_{0j}^* + k_-(s)}, \\
W_0^{\mathrm{IV}}(s) &= \sum_{j=1}^{2} A_j \frac{s}{[U(s)]^{1/2}} \ln\frac{k_{0j}^* + k_+(s)}{k_{0j}^* - k_-(s)},
\end{aligned} \tag{14}$$

where the upper sign holds in W_-, V_-, U_- and the lower sign in W_+, V_+, U_+. The first sheet now has the basic cuts of the exact $a(s)/k$ of Ref. 7, i.e., $-\infty$ to 0 and $(m-1)^2$ to ∞. The discontinuities of $f_0^*(s)S_0(s)/k$ are the same as those obtained for $a(s)/k$ from the exact equations (17b) and (23) of Ref. 7, apart from an over-all correction factor[26] $c_0\phi(k^2)T_{0,0}(s)$, if one approximates $T_0(s')$ by $T_{0,0}(s')$ in the integrands. This time the values of $T_{0,0}(s')$ required do not belong to the physical region, but on a limited portion of the cut to the right of $s=(m-1)^2$, s' in the integrand is not too far from the physical region since the limits of integration lie on the boundary of (W_+,V_+) and (W_-,V_-). Thus, the discontinuities across a portion of the cut lying closest to the physical region should be fairly accurately given by Eq. (14). In crossing the cut $(m-1)^2$ to ∞ from either below or above, one enters the second sheet. This has the same cuts as the first, while the last two have cuts $-\infty$ to 0 and $[4, (m+1)^2]$. The logarithmic branch points found previously are now partially redistributed among the various sheets. In particular, for $k_{0j} \in W_\pm$ one has, on the first sheet, $\alpha_1 \in V_+$, $\alpha_2^* \in U_+$. For $k_{0j} \in V_\pm$ one has $\alpha_1 \in W_+$, $\alpha_2 \in U_+$. Finally, for $k_{0j} \in U_\pm$ one has $\alpha_1 \in W_-$, $\alpha_2 \in V_+$. That is, both α_1 and α_2 are now on the first sheet of $a(s)/k$ [or $W_0(s)$], as in Ref. 7. With the present choice of cuts it is even easier to see that these (first sheet) logarithmic singularities will be the ones that are very close to the physical region if α_0 is close to the physical region. What is equally important to note is that none of those singularities of the exact amplitude which lie near the physical region have been omitted, as can be seen by following Bonnevay's iteration procedure.[27]

In conclusion, it may be conjectured that, in all cases in which a low-energy parametrization of the

[26] This factor also multiplies the discontinuity across the logarithmic cut $[\alpha_1,\alpha_2]$, for which similar comments apply (the residue at the pole α_0 is, however, taken to be adjustable here). See also Ref. 28.

[27] For instance, Bonnevay's α_3 and α_3^*, which do not appear here at all, are further from the physical region than either α_1 or α_2. This is true even if the resonance is in V_- or W_-. If α_0 is not near the physical region (e.g., if $|\mathrm{Im}\alpha_0|$ is very large), none of the logarithmic branch points will be very close to the physical region, so that a detailed discussion of their location does not appear worthwhile in this case.

elastic scattering amplitude $f_0(a)$ as previously described is adequate, $S_0(s)$ possesses enough structure[28] to be a useful explicit approximation to the partial-wave amplitude $T_0(s)$ in the physical region.

[28] The so-called triangle amplitude for $T_0(s)$, not $T(s_1,s_2,s_3)$ [cf. Refs. 10 and 11 and I. J. R. Aitchison and C. Kacser, Phys. Rev. **173**, 1700 (1968)], is partially represented by $S_0(s)$, since its δ-function part has been added to $T_{0,1}(s)$: $c_0T_{0,0}(s)\phi(k^2)=1+if_0(s)$. Its principal-value part is the third term in Eq. (8) with $Y_0(s)$ set equal to $W_0{}^{\mathrm{I}}(s)$. Now $\phi(k^2)$ vanishes on the right-hand cut when the phase shift δ goes through $\frac{1}{2}\pi$, but it does not vanish at the exact (complex) position of the resonance, $s=\alpha_0$. Hence it is not obvious how much interference there is between the integrals of the direct and crossed Watson terms in $T(s_1,s_2,s_3)$ when s_1, say, is near α_0. As far as the logarithmic singularity of $T_0(s)$ at $a_1\in W_-$ is concerned, the discontinuity across the cut which it generates is proportional to $c_0T_{0,0}(s)\phi(k^2)$. Addition of a further $if_0(s)$ to $c_0T_{0,0}(s)\phi(k^2)$, to represent the contribution from the principal part of the triangle amplitude near $s=\alpha_1$, would result in a factor $\exp(2i\delta)$. This may be more appropriate than $c_0T_{0,0}(s)\times\phi(k^2)$ for s near α_1, as shown in Ref. 11 and the paper by Aitchison and Kacser cited above, which study in detail the triangle singularity at $s=\alpha_1$ and its effect on the singularity of the crossed Watson term at the same point. However, $f_0{}^*(s)\times\exp(2i\delta)/k$ does have an undesirable right-hand cut starting at $k=0$. It is not quite obvious which of the two effects, the one at $s=4$ or the one at $s=\alpha_1$, will be more important. $s=4$ is the physical threshold. α_1 is also close to the physical region, since one meets it by crossing from above the right-hand cut of $S_0(s)$ [remaining in the same sheet of $W_0(s)$] or the right-hand cut of $T_{0,1}(s)+$triangle amplitude, as the case may be. However, for a resonance of finite width, α_1 will not be exactly in the physical region but some distance away. It is for this reason and the fact that, even though α_1 is a fairly strong (logarithmic) singularity, no spectacular effects seem to be associated with it, whether the contribution of the triangle amplitude is considered or not, (cf., e.g., Refs. 2, 4, 10, and 11) that the approximation to $T_0(s)$ has been chosen so as to satisfy property (1) of the Introduction. As a compromise one could, of course, replace $\phi(k^2)$ in Eq. (9) by some polynomial in k^2 to be determined from experiment. This is a more realistic procedure in any case, but it has the disadvantage of introducing additional unknown parameters into $S_0(s)$.

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Analytic Hard-Pion Methods: The $A_1\rho\pi$ System*

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We use current algebra and analyticity to study vertex functions occurring in the $A_1\rho\pi$ system. Employing the conserved vector current and partially conserved axial-vector current relations, and the $SU(2)\times SU(2)$ algebra of currents, we generate Ward identities which relate two- and three-point functions of vector and axial-vector currents. Extracting the pion poles from these vertex functions and exposing their isospin content, we define form factors whose analytic properties may readily be studied and, in particular, deduce from the Ward identities a relation involving the pion form factor. With suitable low-energy approximations which maintain the correct cut structure, we use this relation to calculate an effective-range formula for the pion form factor, and consequently from unitarity, the p-wave $\pi\pi$ phase shift. Our results are generally in agreement with experiment. Using A_1 dominance, we are able to obtain analytic effective-range formulas for the form factors appearing in the vector-current matrix element of π, A_1 mesons. From these form factors, measurable in the reaction $e^+e^-\to\pi A_1$, we calculate the $A_1\to\rho\pi$ width and the A_1-ρ spin correlation. Finally, we extend our methods to encompass both $\pi\pi$ and πA_1 cut contributions, and derive a set of coupled integral equations which we solve approximately for the $\pi\pi$ and πA_1 form factors. We conclude with a general observation on the complementary roles played by current algebra and by unitarity.

INTRODUCTION

HARD-PION methods refer to the procedure whereby hadronic matrix elements of physical interest can be extrapolated to off-mass-shell values of the particle momenta. The foundations[1] of the procedure originate in the conserved vector-current (CVC) theory, the partially conserved axial-vector current (PCAC) hypothesis, and current algebra. Earlier approaches involved zero four-momentum limits and provided such exact statements about extrapolated amplitudes as the Adler consistency relation[2] and the soft-pion theorems.[3] The hard-pion techniques[4] pertain to arbitrary four-momenta, thus extending the range of utility of these off-shell methods. Physical mesonic matrix elements are expressed, in their extrapolated form, in terms of vacuum expectation values of products of local operators which can be identified with hadronic vector and axial-vector currents satisfying the algebra of currents. As Schnitzer and Weinberg[4] have shown, the content of this method is summarized in a set of Ward identities, which give constraints among the N-point functions of the theory. Further dynamical structure must be added to this system of constraints in order to obtain detailed knowledge of the matrix elements in question. In this

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[1] For general background, see S. L. Adler and R. F. Dashen, *Current Algebras* (Benjamin, New York, 1968).
[2] S. L. Adler, Phys. Rev. **139**, B1638 (1965).
[3] See, e.g., S. Weinberg, Phys. Rev. Letters **17**, 616 (1966).
[4] H. J. Schnitzer and S. Weinberg, Phys. Rev. **164**, 1828 (1967), hereafter referred to as SW. See also S. G. Brown and G. B. West, *ibid.* **168**, 1605 (1968); T. Das, V. S. Mathur, and S. Okubo, Phys. Rev. Letters **19**, 1067 (1967), and Refs. 9 and 12 below.