# Unitarized Veneziano Model with Satellites for $\pi\pi$ Scattering*

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We impose single-channel unitarity on a five-term Veneziano $\pi$-$\pi$ amplitude using the $K$-matrix approach. The violation of crossing symmetry as measured by the $s$- and $p$-wave crossing-symmetry sum rules is found to be negligible for wide ranges in the satellite coefficients. We study the conditions imposed by the non-negativity of the widths of the $\rho$, $f$, and $g$ mesons as well as all their daughters. Although these conditions still permit a continuous infinity of solutions, a limiting case is of some interest where the widths of the $\epsilon'$, $\epsilon''$, and $f'$ are made to vanish. For this special model we calculate widths, scattering lengths, and the phase shifts $\delta_0^0$, $\delta_0^2$, and $\delta_1^1$. Since the daughters are no longer degenerate with the parents, masses are also calculated.

## I. INTRODUCTION

THE single-term Veneziano representation[1] for the $\pi$-$\pi$ scattering amplitude[2] using linearly rising Regge trajectories constrained by the Adler self-consistency condition has been successful in relating to a variety of experimental data in spite of its lack of unitarity. The use of the $K$-matrix formalism as a simple way of introducing unitarity into a scattering amplitude when using the Veneziano representation was first pointed out by Lovelace.[3] However, when only a single Veneziano term is used, the model predicts unacceptable, even negative, widths for some of the resonances which occur in the theory. In addition, the crossing symmetry which is inherent in the Veneziano representation is destroyed for the unitarized scattering amplitude.

Through a study of asymptotic behavior of the Veneziano representation with complex trajectories, we have been led to consider a simple extension of the Lovelace-Veneziano model in which four satellite terms are added. Even though the Veneziano representation contains neither Regge cuts nor the Pomeranchuk phenomenon, we believe that when unitarity is imposed the model is reasonable for low energies. Thus, we include only single-channel unitarity and study only the low-energy region which includes the $\rho$, $f$, and $g$ mesons and their daughters. The Regge-trajectory parameters and the range of values for the Veneziano term coefficients which guarantee non-negative resonance widths are found.

There is an additional problem connected with the use of the $K$-matrix formalism because the relation between the $T$ matrix and the $K$ matrix is not unambiguous. When this relation is given by

$$1/T_l{}^I(s)=1/K_l{}^I(s)+R(s)\,,$$

all that is known about $R(s)$ is that it has a branch point at the elastic threshold and the discontinuity across the right-hand cut is given by phase space. A left-hand cut was included in this function by Lovelace[3] and later Lipinski added a pole on the negative real axis.[4] We have used an $R(s)$, with only a right-hand cut, that is related to the Chew-Mandelstam effective-range theory.[5] This $R(s)$ causes the resonance positions predicted by the theory to be shifted from the positions of the $K$-matrix poles by an amount related to the residue of the pole. Thus, while the Veneziano representation predicts daughters of lower spins occurring at the same mass as the resonance on the parent trajectory, this model no longer predicts parent and daughters degenerate in mass.

In the remaining sections of the paper we consider these questions. Section II describes the Veneziano representation, the $K$ matrix, and its partial-wave analysis. The questions concerning the function $R(s)$, its relation to unitarization and the mass shift effect, are discussed in Sec. III. The possible violation of crossing symmetry is investigated in Sec. IV using $s$- and $p$-wave crossing-symmetry sum rules. Section V is devoted to the choice of trajectory parameters and satellite coefficients and Sec. VI presents results and some conclusions.

## II. PARTIAL-WAVE ANALYSIS

In this section we consider the partial-wave analysis of the $K$ matrix which is defined as a Veneziano representation with any number of satellites. We begin by writing the $\pi\pi$ isospin amplitudes[2] $K^I$:

$$\begin{aligned}K^0&=\tfrac{3}{2}[F(\alpha(s),\alpha(t))+F(\alpha(s),\alpha(u))]-\tfrac{1}{2}F(\alpha(t),\alpha(u))\,,\\K^1&=F(\alpha(s),\alpha(t))-F(\alpha(s),\alpha(u))\,,\\K^2&=F(\alpha(t),\alpha(u))\,,\end{aligned}\tag{2.1}$$

where

$$F(\alpha(s),\alpha(t))=-\sum_{p=1}^{\infty}\sum_{n=0}^{p}A_{pn}\frac{\Gamma(p-\alpha(s))\Gamma(p-\alpha(t))}{\Gamma(p+n-\alpha(s)-\alpha(t))}\tag{2.2}$$

and

$$\alpha(s)=a+bs\,.\tag{2.3}$$

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[1] G. Veneziano, Nuovo Cimento **57A**, 190 (1968).
[2] C. Lovelace, Phys. Letters **28B**, 265 (1968).
[3] C. Lovelace, invited paper at Conference on $\pi\pi$ and $\pi K$ Interaction, Argonne, 1969, p. 562 (unpublished).
[4] H. M. Lipinski, University of Wisconsin report (unpublished)
[5] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960)

Kreps and Milgram[6] have shown that it is sufficient to consider only diagonal terms as in Eq. (2.2). Several authors have derived (2.2) from various assumptions.[7,8] There seems to be no simple way to argue whether or not an infinite number of terms are required. It is known that such a series may not even have Regge behavior.[6,8] An additional example of this problem is contained in Appendix A, where we discuss a rather physical model involving an infinite number of $A_{p0}$'s ($n=0$), elastic unitarity, and complex trajectories satisfying the proper dispersion relations. Yet we find that such a model does not have Regge behavior. In spite of these difficulties, we cannot yet conclude that an infinite number of terms is unphysical, although it seems probable. At least from a practical viewpoint, the use of an infinite number of terms is clearly unwarranted.

We apply partial-wave analysis to Eqs. (2.1) and (2.2), generalizing the method of Lipinski.[4] We may write

$$\Gamma(p-\alpha(s))=(-)^{p-n}(\alpha(s)-p+1)_{p-n}\Gamma(n-\alpha(s)),$$

where $(z)_N=\Gamma(z+N)/\Gamma(z)$ is Pochhammer's symbol. Thus

$$\frac{\Gamma(p-\alpha(s))\Gamma(p-\alpha(t))}{\Gamma(p+n-\alpha(s)-\alpha(t))}=(-)^{p-n}(\alpha(s)-p+1)_{p-n}\times\int_0^1 dz\, z^{n-\alpha(s)-1}(1-z)^{p-\alpha(t)-1}. \quad (2.4)$$

Introduce the c.m. scattering angle $\theta$ so that

$$\begin{aligned}\alpha(t)&=a+bt=a-\tfrac{1}{2}b(s-4\mu^2)+\tfrac{1}{2}b(s-4\mu^2)\cos\theta,\\ \alpha(u)&=a-\tfrac{1}{2}b(s-4\mu^2)-\tfrac{1}{2}b(s-4\mu^2)\cos\theta.\end{aligned} \quad (2.5)$$

The actual partial-wave analysis is facilitated by use of the identity[9]

$$e^{x\cos\theta}=\sum_{m=0}^{\infty}(2m+1)i_m(x)P_m(\cos\theta), \quad (2.6)$$

where

$$i_m(x)\equiv\left(\frac{\pi}{2x}\right)^{1/2}I_{m+1/2}(x)=x^m\left(\frac{1}{x}\frac{d}{dx}\right)^m\frac{\sinh x}{x}. \quad (2.7)$$

Let us define

$$U_l=\frac{1}{4}\int_{-1}^{1}P_l(\cos\theta)F(\alpha(s),\alpha(t))d(\cos\theta), \quad (2.8)$$

$$V_l=\frac{1}{4}\int_{-1}^{1}P_l(\cos\theta)F(\alpha(s),\alpha(u))d(\cos\theta), \quad (2.9)$$

$$W_l=\frac{1}{4}\int_{-1}^{1}P_l(\cos\theta)F(\alpha(t),\alpha(u))d(\cos\theta). \quad (2.10)$$

We calculate from (2.4)

$$U_l=-\tfrac{1}{2}\sum_{p=1}^{\infty}\sum_{n=0}^{p}(-)^{p-n}A_{pn}(\alpha(s)-p+1)_{p-n}\times\int_0^1 dz\, z^{n-\alpha(s)-1}(1-z)^{p-1-a+\frac{1}{2}b(s-4\mu^2)}i_l(x), \quad (2.11)$$

where

$$x=-\tfrac{1}{2}b(s-4\mu^2)\ln(1-z)$$

and

$$V_l=(-)^lU_l. \quad (2.12)$$

For $W_l$ we require

$$\begin{aligned}\frac{\Gamma(p-\alpha(u))\Gamma(p-\alpha(t))}{\Gamma(p+n-\alpha(u)-\alpha(t))}&=(n+p-3a-4b\mu^2+\alpha(s))_{p-n}\frac{\Gamma(p-\alpha(u))\Gamma(p-\alpha(t))}{\Gamma(2p-\alpha(u)-\alpha(t))}\\&=(n+p-3a-4b\mu^2+\alpha(s))_{p-n}\times\int_0^1 dz\, z^{p-\alpha(u)-1}(1-z)^{p-\alpha(t)-1}.\end{aligned} \quad (2.13)$$

Isolating the exponential in $\cos\theta$ from $\alpha(u)$ and $\alpha(t)$, we find with the aid of (2.6)

$$W_l=-\tfrac{1}{2}\sum_{p=1}^{\infty}\sum_{n=0}^{p}A_{pn}(n+p-3a-4b\mu^2+\alpha(s))_{p-n}\times\int_0^1 dz[z(1-z)]^{p-1-a+\frac{1}{2}b(s-4\mu^2)}i_l(y), \quad (2.14)$$

where $y=-\frac{1}{2}b(s-4\mu^2)\ln[(1-z)/z]$.

The partial-wave decompositions of the isospin amplitudes are therefore given by

$$K_l^0=3U_l-\tfrac{1}{2}W_l \quad \text{for even } l, \quad (2.15)$$

$$K_l^1=2U_l \quad \text{for odd } l, \quad (2.16)$$

$$K_l^2=W_l \quad \text{for even } l. \quad (2.17)$$

As one expects from generalized Bose statistics, $K_l^0$ and $K_l^2$ vanish for odd $l$ and $K_l^1$ vanishes for even $l$.

At the position of a pole in $K_l^I$, we have approximately

$$K_l^I=-\frac{H(j,l)}{2b}\frac{\mathcal{I}(l)}{s-M_j^2}, \quad (2.18)$$

where

$$H(j,l)=\int_{-1}^{1}d(\cos\theta)P_l(\cos\theta)R^j(\alpha(t)) \quad (2.19)$$

and

$$R^j(\alpha(t))=\sum_{p=1}^{j}\sum_{n=0}^{p}\frac{(-)^{p-n+1}A_{pn}}{\Gamma(j-p+1)}(\alpha(t)-p+1)_{j-n}, \quad (2.20)$$

$$\begin{aligned}\mathcal{I}(l)&=\tfrac{3}{2} \quad \text{for even } l\\ &=1 \quad \text{for odd } l.\end{aligned} \quad (2.21)$$

[6] R. E. Kreps and M. S. Milgram, Phys. Rev. D **1**, 2271 (1970).
[7] C. Schmid, Phys. Letters **28B**, 348 (1968); N. N. Khuri, Phys. Rev. **185**, 1875 (1969); G. B. West, *ibid.* **185**, 1927 (1969).
[8] S. Matsuda, Phys. Rev. **185**, 1811 (1969); C. Tiktopoulos, Phys. Letters **31B**, 138 (1970).
[9] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions*, (Dover, New York, 1964), p. 445 (see Eqs. 10.2.36 and 10.2.24).

In the ordinary Veneziano model, $H(j,l)$ is related to the decay rate of the resonance of spin $l$ belonging to the parent of spin $j$, $\Gamma(j,l)$, by[10]

$$\Gamma(j,l)=[\mathcal{J}(l)/(2bM_j)][1-4\mu^2/M_j^2]^{1/2}H(j,l). \quad (2.22)$$

In the case $j=l$, $H$ is easily computed and is given by Eq. (A3) of Appendix A. A list of $H(j,l)$ for $j\leq 3$ is given in Appendix B.

Although the analysis up to this point is quite general, we will henceforth specialize to the five-term case considered previously by Antoniou *et al.*[11] and by Kreps and Milgram.[6] The nonzero $A_{pn}$'s are $A_{10}$, $A_{11}$, $A_{20}$, $A_{21}$, and $A_{22}$. As usual, we impose the Adler condition, which is represented by

$$F(\alpha(\mu^2),\alpha(\mu^2))=0. \quad (2.23)$$

Thus we require

$$\tfrac{1}{2}=a+b\mu^2 \quad (2.24)$$

and

$$A_{22}=-2(A_{20}+A_{21})-8A_{11}, \quad (2.25)$$

in order to ensure Eq. (2.23). As will be indicated in Sec. III, the Adler condition will persist after unitarization.

### III. K-MATRIX UNITARIZATION

In order to unitarize the partial-wave Veneziano amplitude, we regard $K_l^I$ of Eqs. (2.15)–(2.17) as the $K$ matrix of the unitarized amplitude in the $s$-channel[3]

$$T_l^I=K_l^I/(1+RK_l^I)=(e^{2i\delta_l^I}-1)/2i\rho, \quad (3.1)$$

where

$$\rho=2q/\sqrt{s}=[1-4\mu^2/s]^{1/2}, \quad (3.2)$$

$R$ is defined below, and $\delta_l$ is the $l$th partial-wave phase shift. This procedure considers elastic unitarity only, and is strictly valid for $4\mu^2\leq s\leq 16\mu^2$.

Equation (3.1) can be written as

$$1/T_l^I=1/K_l^I+R. \quad (3.3)$$

The requirement of unitarity on $R$ is

$$\mathrm{Im}(1/T_l^I)=\mathrm{Im}R=-\rho, \quad s\geq 4\mu^2. \quad (3.4)$$

It is evident that there are many choices for $R$. Lovelace chose an $R$ that vanishes asymptotically to restore crossing symmetry at high energies. We regard the Veneziano model as applicable to low energies and do not impose asymptotic crossing symmetry. On the other hand, we require that the Veneziano representation with satellite terms satisfy $s$- and $p$-wave crossing-symmetry sum rules.

In order to remove the $s^{1/2}$ singularity of $T$ near $s=0$ because of the $s^{-1/2}$ in (3.2), we consider[12]

$$R=-i\rho+(2\rho/\pi)\ln[s^{1/2}(1+\rho)/2\mu]. \quad (3.5)$$

We shall see in Sec. IV that (3.5) is better than $R=-i\rho$ from the point of view of satisfying the crossing-symmetry sum rules.

When (3.5) is substituted into (3.1), we find

$$T_l^I=\bar{K}_l^I/(1-i\rho\bar{K}_l^I), \quad (3.6)$$

where

$$\bar{K}_l^I=K_l^I/[1+\mathrm{Re}(R)K_l^I]. \quad (3.7)$$

An effect of the presence of Re$R$ is to remove the mass degeneracy between the parents and daughters.

The combination of Eqs. (3.1), (3.6), and (3.7) leads to

$$\begin{aligned}\cot\delta_l^I&=1/\rho\bar{K}_l^I=1/\rho K_l^I+\mathrm{Re}(R)/\rho\\&=1/\rho K_l^I+(2/\pi)\ln[s^{1/2}(1+\rho)/2\mu]. \quad (3.8)\end{aligned}$$

In the $s$-wave case, when $K_0^I(s)$ is replaced by its scattering length

$$\mu a_0^I=K_0^I(4\mu^2), \quad (3.9)$$

one obtains the Chew-Mandelstam effective-range formula.[5] In the present model $K_l^I$ is given by Eqs. (2.15)–(2.17).

We note that the position of a pole in $K_l^I$ depends on the linear Regge trajectory. The mass of the corresponding resonance occurring at $\cot\delta_l=0$ is shifted from the trajectory mass. We define the mass $M$ and width $\Gamma$ of a physical resonance by the relations[13]

$$\cot\delta_l^I|_{s=M^2}=0, \quad (3.10)$$

$$d\delta_l^I/ds|_{s=M^2}=1/M\Gamma. \quad (3.11)$$

Let us apply Eqs. (3.10) and (3.11) in detail to the $\rho$ meson. They give the following relations[14] between the trajectory mass $M_1$ defined as $a+bM_1^2=1$ and the physical mass $M_\rho$:

$$M_1^2=(M_\rho^2+4\mu^2eA_{10})/(1+eA_{10}) \quad (3.12)$$

$$=(M_\rho^2-6\mu^2+4\mu^2fA_{10})/(fA_{10}-2\mu^2/M_\rho^2), \quad (3.13)$$

where

$$e=(\rho/3\pi)\ln[M_\rho(1+\rho)/2\mu], \quad (3.14)$$

$$f=\tfrac{1}{6}[(M_\rho/\Gamma_\rho)\rho+1/\pi]\rho^2, \quad (3.15)$$

$$\rho=(1-4\mu^2/M_\rho^2)^{1/2}. \quad (3.16)$$

Equating Eqs. (3.12) and (3.13), and rearranging, we obtain for the width $\Gamma_\rho$

$$\Gamma_\rho=\tfrac{1}{6}M_\rho\rho^3A_{10}\{1+[\tfrac{1}{2}(1+\rho^2)e-\rho^2/6\pi]A_{10}\}^{-1}, \quad (3.17)$$

in contrast to the Veneziano-width formula

$$\Gamma(1,1)=\tfrac{1}{6}M_1\rho_1^3A_{10},$$

where

$$\rho_1=(1-4\mu^2/M_1^2)^{1/2},$$

as can be obtained from Eqs. (2.22), (2.24), and (B1).

[10] J. Shapiro and J. Yellin, LRL Report No. UCRL-18500 (unpublished); J. Shapiro, Phys. Rev. **179**, 1345 (1969).

[11] N. G. Antoniou, A. Bartl, and F. Widder, Universität Tübingen report (unpublished).

[12] L. S. Brown and R. L. Goble, Phys. Rev. Letters **20**, 346 (1968); D. G. Ravenhall and R. L. Schult, University of Illinois report (unpublished).

[13] See G. J. Gounaris and J. J. Sakurai, Phys. Rev. Letters **21**, 244 (1968). Equations (3.10) and (3.11) appear reasonable in the present context. However, the definitions of physical mass and width are ambiguous. If other definitions are used, the widths may vary by 20% while the masses are more stable.

[14] In deriving the following equations in this section, we have used the approximation $1/K_l^I(s)\approx(s-M_j^2)/r_l^I(j)$.

More generally, near the Regge pole of trajectory mass $M_j$,

$$K_l^I \approx -r_l^I(j)/(s-M_j^2)\,,\quad r_l^I(j)=H(j,l)\mathcal{g}(l)/2b\,,$$

so that we obtain from (3.8) and (3.10)

$$M_j^2=M^2-2r_l^I(j)(\rho/\pi)\ln[M(1+\rho)/2\mu]\,, \tag{3.18}$$

which expresses the trajectory mass $M_j$ in terms of the physical resonance mass $M$ and $\rho$ is given by (3.16) with $M_\rho$ replaced by $M$. Similarly, the generalization of (3.17) is found from the combination of (3.8) and (3.11):

$$\Gamma=\frac{r_l^I(j)\rho}{M}\times\frac{\rho^2}{\rho^2-2\mu^2/M^2+2\mu^2M_j^2/M^4-r_l^I(j)\rho^2/\pi M^2}\,. \tag{3.19}$$

Equations (3.18) and (3.19) are used to compute the physical masses and widths of the various parents and daughters.

It may appear that when the $T_l$'s are summed to get the scattering amplitude $T$, there is a violation of the Adler self-consistency condition. This is, however, not the case. One finds that when the proper $p_\mu \to 0$ limit is taken only the $s$-wave term survives. Moreover $K_0^0$ and $K_0^2$ vanish at the Adler point as may be seen explicitly by use of Eqs. (2.11)–(2.25). Therefore, $T_0^0$ and $T_0^2$ also vanish.

## IV. CROSSING-SYMMETRY SUM RULES

Although the $K$ matrix is crossing symmetric by construction, this symmetry is broken in the $T$ matrix because of the function $R(s)$. In order to evaluate the degree of violation, we use the $s$- and $p$-wave crossing-symmetry sum rules.[15] There are an infinite number of such sum rules, each of which contains a finite number of partial-wave amplitudes integrated over the region $0 \le s \le 4\mu^2$. We use the first five sum rules; consequently we have only an approximate check on the crossing-symmetry violation. They are recorded here for convenience:

$$S_1=\int_0^{4\mu^2} ds\,(s-4\mu^2)[2A_0^0(s)-5A_0^2(s)]=0\,,$$

$$S_2=\int_0^{4\mu^2} ds\,(s-4\mu^2)(3s-4\mu^2)[A_0^0(s)+2A_0^2(s)]=0\,,$$

$$S_3=\int_0^{4\mu^2} ds\,(s-4\mu^2)[(3s-4\mu^2)A_0^2(s)+(s-4\mu^2)A_1^1(s)]=0\,,$$

$$S_4=\int_0^{4\mu^2} ds\,(s-4\mu^2)\{(10s^2-32s\mu^2+16\mu^4)\times[2A_0^0(s)-5A_0^2(s)]+6(s-4\mu^2)\times(5s-4\mu^2)A_1^1(s)\}=0\,, \tag{4.1}$$

$$S_5=\int_0^{4\mu^2} ds\,(s-4\mu^2)\{(35s^3-180s^2\mu^2+240s\mu^4-64\mu^6)\times[2A_0^0(s)-5A_0^2(s)]-15(s-4\mu^2)\times(21s^2-48s\mu^2+16\mu^4)A_1^1(s)\}=0.$$

In our calculation, the amplitude $A_l^I(s)$ will be either $T_l^I(s)$ or $K_l^I(s)$.

Two criteria are used to determine the magnitude of violation for the $T$ matrix. First is a simple ratio test formed by taking the quotient of the numerical value of the entire sum rule to that of only one term. The second uses the fact that the $K$ matrix is crossing symmetric, so that its use in (4.1) should cause $S_i=0$ identically. However, the numerical analysis with only a finite number of digits introduces round-off errors and the numerical integration introduces further errors. Both of these effects prevent a true zero so that the actual calculation with the $K$ matrix provides a measure of the method's precision. The second criterion is therefore a comparison of the magnitudes of each sum rule using the $T$ matrix to those using the $K$ matrix.

Some preliminary investigations showed clearly that if $R(s)=-i\rho$ is used to calculate the $T$ matrix, the sum rules are not satisfied. The primary cause for this is the square-root singularity of $R(s)$ at $s=0$. On the other hand, $R(s)$ defined by Eq. (3.5) does not have this singularity. It is found that the sum rules are more accurately satisfied and their calculated values are relatively insensitive to changes in the coefficients, $A_{pn}$. This is easily understood when one notes that $RK_l^I$ in the denominator of Eq. (3.1) is now small, making unitarity corrections gentle over the range of integration. The results of the crossing-symmetry sum rules for the parameters chosen in the next section are presented in Sec. VI.

## V. DETERMINATION OF PARAMETERS

The relation between the $\rho$ width and $A_{10}$ corrected for the mass-shift effect as well as unitarity damping is given by Eq. (3.17). We take $\mu=0.138$ GeV, $M_\rho=0.77$ GeV, and $\Gamma_\rho=0.11$ GeV as input and find $A_{10}=1.19$. Equation (3.12) then leads to a trajectory mass for the $\rho$ of $M_1=0.712$ GeV. We find from Eq. (2.24) and

$$1=a+bM_1^2\,, \tag{5.1}$$

that $a=0.4805$ and $b=1.024$. The importance of the mass-shift effect is indicated by the 17% rise over the value $b=0.87$ occurring in the phenomenological $\rho$ trajectory. The next most significant parameters are $A_{20}$ and $A_{11}$, which now determine the $f$ and $\epsilon$ widths, respectively. See Eqs. (B3) and (B2).

[15] J. L. Basdevant, G. Cohen-Tannoudji, and A. Morel, Nuovo Cimento **64A**, 585 (1969); R. Z. Roskies, *ibid.* **65A**, 467 (1970). These sum rules have the same meaning as the crossing relations for partial waves discovered by A. P. Balachandran and J. Nuyts, Phys. Rev. **172**, 1821 (9168).

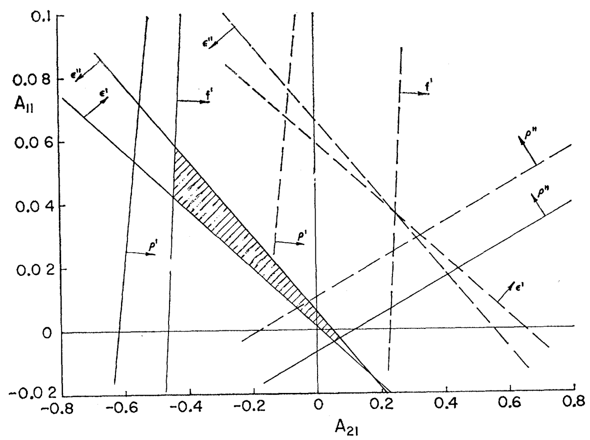


FIG. 1. Domain of positive widths. The value of $A_{10}$ is 1.19. For the solid lines $A_{20}=0$ and for the dashed lines $A_{20}=-0.3121$. $A_{22}$ is determined from the Adler condition as in Eq. (2.25).

In this model, we consider only the $\rho$, $f$, and $g$ mesons with their daughters $\epsilon$; $\rho'$, $\epsilon'$; and $f'$, $\rho''$, and $\epsilon''$. In order to satisfy the unitarity condition that the widths of all these resonances be non-negative, we require the $H(j,l)$ for $j\leq 3$ to be non-negative. Since the model is applicable in the low-energy region, we need not be concerned about the resonances at higher masses.

To study the non-negativity of widths, we plot $A_{11}$ as a function of $A_{21}$ for the $\rho'$, $\epsilon'$, $f'$, $\rho''$, and $\epsilon''$ with the appropriate $H(j,l)$ set equal to zero. See Eqs. (B4), (B5), and (B7)–(B9). The coefficient $A_{10}$ is fixed at 1.19 so these plots are made for various $A_{20}$'s. The dashed lines in Fig. 1 correspond to the case where $A_{20}=-0.3121$. The lines are labeled by resonance names and by arrows. The arrows indicate the half-plane of positive $H(j,l)$. No lines are shown for the $\rho$, $\epsilon$, $f$, and $g$ mesons, since their widths depend only on $A_{10}$, $A_{20}$, and $A_{11}$, as in Eqs. (B1)–(B3) and (B8). The dashed lines corresponding to $A_{20}=-0.3121$ form a limiting case where there is only a point solution for non-negativity. At the triple intersection of the $f'$, $\epsilon'$, and $\epsilon''$ lines, we have $\Gamma_{f'}=\Gamma_{\epsilon'}=\Gamma_{\epsilon''}=0$. When we increase $A_{20}$ to the value zero, we have the situation represented by the solid lines. The large shaded polygon contains the non-negative solutions. Since the lines move uniformly with $A_{20}$, one can show from Fig. 1 and Eq. (B6) that solutions exist only for $A_{20}$ in the range

$$-0.3121\leq A_{20}\leq 0.595. \tag{5.2}$$

The upper limit comes from requiring $\Gamma_g\geq 0$.

The special case $A_{20}=0$ is of historical interest because the origin then yields the single-term Veneziano result ($A_{11}=A_{20}=A_{21}=A_{22}=0$). In this case the negative $\epsilon'$ width found previously[15] is reproduced as the origin is on the wrong side of the $\epsilon'$ line.

We study two cases in detail. Case I is the limiting case with $A_{20}=-0.3121$, where $\Gamma_{\epsilon'}=\Gamma_{\epsilon''}=\Gamma_{f'}=0$. The second corresponds to $\Gamma_{\epsilon'}=\Gamma_{\epsilon''}=\Gamma_{\rho''}=0$, which is of interest because the deepest daughters have been removed. The nonzero $A_{pn}$ coefficients are listed in Table I for both cases. We do not consider any other cases since the $f$ width decreases as $A_{20}$ rises.

TABLE I. Parameters and results for specific calculations.

| | Case I | | Case II | | Experiment[a] | |
|---|---|---|---|---|---|---|
| $A_{10}$ | 1.19 | | 1.19 | | | |
| $A_{11}$ | 0.0364 | | 0.0161 | | | |
| $A_{20}$ | −0.3121 | | −0.1868 | | | |
| $A_{21}$ | 0.2477 | | 0.2160 | | | |
| $A_{22}$ | −0.1624 | | −0.1872 | | | |
| | Mass (MeV) | Width (MeV) | Mass (MeV) | Width (MeV) | Mass (MeV) | Width (MeV) |
| $\rho$ | 770 | 110 | 770 | 110 | (input) | (input) |
| $\epsilon$ | 992 | 445 | 1013 | 481 | | |
| $f$ | 1302 | 127 | 1295 | 108 | 1264±10 | 151±25 |
| $\rho'$ | 1266 | 68 | 1288 | 99 | | |
| $\epsilon'$ | | 0 | | 0 | | |
| $g$ | 1613 | 59 | 1607 | 50 | 1663±20 | 111±30 |
| $f'$ | | 0 | 1605 | 47 | | |
| $\rho''$ | 1587 | 24 | | 0 | | |
| $\epsilon''$ | | 0 | | 0 | | |
| $\mu a_0{}^0$ | 0.2438 | | 0.2565 | | | |
| $\mu a_0{}^2$ | −0.0629 | | −0.0660 | | −0.051±0.005 | |
| $\mu^3 a_1{}^1$ | 0.0501 | | 0.0521 | | | |

[a] The masses and widths are from Ref. 16 and the scattering length is from J. P. Baton *et al.*, Ref. 18.

## VI. RESULTS AND CONCLUSIONS

We summarize the numerical results first. The mass and width of each resonance for the two specific cases of

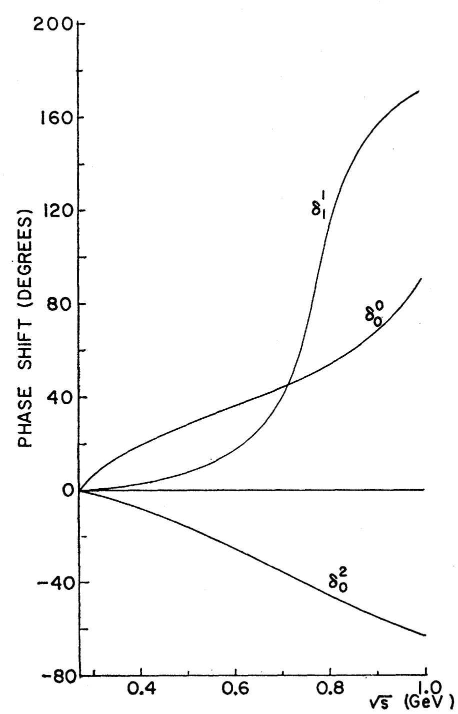


FIG. 2. Phase shifts $\delta_l{}^I$ for case I.

the $A_{pn}$ coefficients are presented in Table I. The large splitting of the $\epsilon$-$\rho$ mass degeneracy is due to the large width of the $\epsilon$. A comparison with the Particle Properties Tables[16] shows that the $f$ width appears acceptable while the $g$ width is a factor of 2–3 too small.

On the basis of the $f$ width, we tend to prefer the solution of case I. The $s$- and $p$-wave phase shifts for case I are presented in Fig. 2. In both cases the phase shift $\delta_0{}^0$ corresponds very well to the experimental solution[17] called set II. On the other hand,[18] the magnitude of $\delta_0{}^2$ grows too large for energies greater than 0.6 GeV. The latter phase shift is slightly worse for case II.

The values of the crossing-symmetry sum rules for the two cases are nearly the same. Table II contains the relevant quantities needed for the two criteria mentioned in Sec. IV using the parameters of case I. Let us consider the second criterion which compares the magnitude of each sum rule using the $T$ matrix to that using the $K$ matrix. Since the magnitudes are always smaller for the $T$ matrix and since the $K$ matrix is crossing symmetric by construction, we conclude that these sum rules show no significant crossing-symmetry violation to the accuracy of our calculation. In support of this conclusion, we note that the $K_l{}^I$ are not positive definite in the region $0\leq s\leq 4\mu^2$ so that smaller sum-rule values are not a trivial consequence of effectively scaling down $T_l{}^I$ compared to $K_l{}^I$. As can be seen in columns 2 and 3 of Table II, the sum rules for the $T$ matrix approximately satisfy the ratio test. A variation in the $A_{pn}$ coefficients could be used to diminish the entries in column 3. However, the accuracy of the calculation is indicated in column 5 and improvement beyond these values is meaningless. The consequences of approximations introduced by numerical techniques are illustrated in the results of sum rule 5, where practically all significance has been lost.

TABLE II. Crossing-symmetry sum-rule results.

| | $T$ matrix | | $K$ matrix | |
|---|---|---|---|---|
| Sum rule | One term | Whole sum rule | One term | Whole sum rule |
| $S_1$ ($10^{-7}$ GeV$^4$) | $-3420$ | $-4.54$ | $-3243$ | 42.08 |
| $S_2$ ($10^{-8}$ GeV$^6$) | $-1005$ | 1.66 | $-951$ | 34.22 |
| $S_3$ ($10^{-8}$ GeV$^6$) | 503.4 | $-1.69$ | 493 | $-23.50$ |
| $S_4$ ($10^{-9}$ GeV$^8$) | 33.72 | $-1.22$ | 21.81 | $-3.19$ |
| $S_5$ ($10^{-11}$ GeV$^{10}$) | $-16.13$ | 1.39 | $-67.20$ | $-46.59$ |

We have also recorded the $s$- and $p$-wave scattering lengths in Table I. By a polynomial expansion of Eq. (3.9) in $b\mu^2$, we find the ratio of the $s$-wave scattering lengths is given approximately by[19]

$$a_0{}^0/a_0{}^2=-(3\times 2^{b\mu^2}+\tfrac{1}{2}). \tag{6.1}$$

Since the Weinberg $-\frac{7}{2}$ ratio is not obeyed,[20] we calculate the parameter $\xi$ introduced by Olsson and Turner,[21]

$$a_0{}^0/a_0{}^2=-\tfrac{7}{2}(1-5\xi/14)/(1+\tfrac{1}{2}\xi). \tag{6.2}$$

For both cases we obtain $\xi=-0.12$, which can be compared with $\xi=1$ of Schwinger[22] and $\xi=0$ of Weinberg and Ref. 11. We note that unlike the work of Lipinski,[4] our unitarization procedure does not change $a_0{}^0$ and $a_0{}^2$ although $a_1{}^1$ is changed. Finally, we compare the dispersion-theoretic sum rule derived by Olsson.[23] Using experimental data to evaluate the dispersion integrals, he has found

$$2\mu a_0{}^0-5\mu a_0{}^2-18\mu^3 a_1{}^1=-0.07\pm 0.02. \tag{6.3}$$

For this sum rule we obtain $-0.0997$ and $-0.0948$ in cases I and II, respectively.

We have considered a model for $\pi$-$\pi$ scattering which satisfies the requirements of elastic unitarity, approximate crossing symmetry, and the Adler self-consistency condition. The direct comparison of our model with experiment is favorable with the exception of $\delta_0{}^2$ when $s^{1/2}>0.6$ GeV and the $g$ width. The difficulty with $\delta_0{}^2$ would have to be rectified by changing either $R(s)$ or the number of satellite terms in view of the fact that the inelastic channels are not significant up to about 1 GeV.

The importance of additional terms in the Veneziano series has not been investigated. On the other hand, it is impossible to maintain the non-negativity conditions of Sec. V and the Adler condition without including one of the $p=2$ terms.[24] Therefore, the present model is the simplest generalization of the one-term Veneziano model.

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[19] This approximation fails for large satellite coefficients. On the other hand, the expression remains valid for a one-term Veneziano model.

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[22] J. Schwinger, Phys. Letters **24B**, 473 (1967).

[23] M. G. Olsson, University of Wisconsin report (unpublished).

[24] We ignore the non-negativity solution for the one-term model suggested in Ref. 10; namely, $a\gtrsim 0.496$ because it leads to $b\lesssim 0.2$ GeV$^{-2}$ by use of $\frac{1}{2}=a+b\mu^2$.

## APPENDIX A: COMPLEX TRAJECTORIES AND NON-REGGE BEHAVIOR

In this study we relate the complex parent trajectory to the $n=0$ satellite coefficients $A_{p0}$ using elastic unitarity. It is well known that the imaginary part of a Regge trajectory is related to the total width $\Gamma_j$ of the resonance when $\mathrm{Re}\alpha=j$,

$$\mathrm{Im}\alpha(M_j^2)=bM_j\Gamma_j. \tag{A1}$$

Elastic unitarity sets the partial width equal to the total width so that the latter can be expressed in terms of the $A_{p0}$ coefficients,

$$\Gamma_j=\Gamma(j,j)=[g(j)/2bM_j](1-4\mu^2/M_j^2)^{1/2}H(j,j), \tag{A2}$$

from Eq. (2.22). The factor $H(j,j)$ is obtained from Eq. (2.19):

$$H(j,j)=[\tfrac{1}{2}(j-a-4b\mu^2)]^j \times\frac{\Gamma(1+\frac{1}{2}j)\Gamma(\frac{1}{2}+\frac{1}{2}j)}{\Gamma(j+\frac{3}{2})}\sum_{p=1}^{j}\frac{(-)^pA_{p0}}{\Gamma(j-p+1)}. \tag{A3}$$

Inverting Eq. (A3), the $A_{p0}$'s are found in terms of the $\mathrm{Im}\alpha(M_j^2)$'s:

$$A_{j0}=\sum_{\nu=0}^{j}\frac{(-)^{\nu+1}\,\mathrm{Im}\alpha(M_\nu^2)}{g(\nu)(1-4\mu^2/M_\nu^2)^{1/2}(\nu-a-4b\mu^2)^\nu} \times\frac{\Gamma(2\nu+2)}{\Gamma^2(\nu+1)\Gamma(j-\nu+1)}. \tag{A4}$$

Following Roskies,[25] we require that the trajectory be a Herglotz function with only a right-hand cut and satisfy the condition $\lim_{s\to\infty}\mathrm{Im}\alpha(s)/s^{1-\delta}=\infty$ for all $\delta>0$. These conditions ensure Regge behavior for a one-term Veneziano model.

We have not been able to make a general study of the convergence property of the asymptotic limit of the Veneziano amplitude (2.2),

$$F(\alpha(s),\alpha(t))\sim-\sum_{p=1}^{\infty}A_{p0}\Gamma(p-\alpha(t))[-\alpha(s)]^{\alpha(t)}. \tag{A5}$$

However, we have chosen several trajectories that satisfy Roskies's conditions and calculated the $A_{j0}$'s numerically. A simple class of such trajectories is given by functions of the form[26]

$$\alpha=x+\frac{cx}{[A/(x-x_0)^p+\ln(x_0-x)]^m}, \tag{A6}$$

where $x=a+bs$ and $x_0=a+4b\mu^2$, for appropriate choice of the parameters $A$, $C$, $p$, and $m$. The parameters are determined to be consistent with $\Gamma_\rho=0.11$ GeV and $\Gamma_f=0.15$ GeV.

Using a large number of possible values of $A$, $C$, $p$, and $m$ in (A6), we have calculated the coefficients $A_{j0}$ numerically from (A4) using the $\mathrm{Im}\alpha$ derived from (A6). We find in every case that the series in (A5) diverges, leading to non-Regge asymptotic behavior.

## APPENDIX B: DECAY WIDTHS

In the absence of unitarity and mass-shift corrections, decay rates may be calculated from Eq. (2.22). The factor $H(j,l)$ is given in the general case by Eq. (2.19). A list of $H(j,l)$'s for $j\leq 3$ is given below. Following each formula is a symbol in parentheses giving the name of the resonance.

$$H(1,1)=(a-\tfrac{1}{3})A_{10}\quad(\rho), \tag{B1}$$

$$H(1,0)=(1-a)A_{10}-2A_{11}\quad(\epsilon), \tag{B2}$$

$$H(2,2)=\tfrac{3}{5}a^2(A_{10}-A_{20})\quad(f), \tag{B3}$$

$$H(2,1)=a[(1-a)A_{10}+(1+a)A_{20}+A_{21}-A_{11}]\quad(\rho'), \tag{B4}$$

$$H(2,0)=a(2a-1)A_{10}+(4-a-2a^2)A_{20} +(2-a)A_{21}+(16+a)A_{11}\quad(\epsilon'), \tag{B5}$$

$$H(3,3)=(4/35)a_3{}^3(\tfrac{1}{2}A_{10}-A_{20})\quad(g), \tag{B6}$$

$$H(3,2)=(1/15)a_3{}^3[3(1-a)A_{10} +6(1+a)A_{20}-2A_{11}+4A_{21}]\quad(f'), \tag{B7}$$

$$H(3,1)=a_3\{[\tfrac{1}{5}a_3{}^2+\tfrac{1}{4}(a^2-2a-\tfrac{1}{3})]A_{10} +[\tfrac{1}{2}(a+3)(1-a)-\tfrac{2}{5}a_3{}^2]A_{20} -\tfrac{2}{3}aA_{21}+\tfrac{1}{3}(a+16)A_{11}\}\quad(\rho''), \tag{B8}$$

$$H(3,0)=\tfrac{1}{2}(1-a)[a_3{}^2+\tfrac{1}{4}(1+a)(a-3)]A_{10} +[(1+a)a_3{}^2+\tfrac{1}{4}(a+3)^2(a-3)]A_{20} -(\tfrac{1}{3}a_3{}^2+\tfrac{1}{4}a^2+8a+95/4)A_{11} +[\tfrac{2}{3}a_3{}^2+\tfrac{1}{2}(a+3)(a-3)]A_{21}\quad(\epsilon''), \tag{B9}$$

where we have used $a_3=\frac{1}{2}(1+3a)$. In writing these results, we have systematically eliminated $A_{22}$ wherever it occurs by use of the Adler condition (2.25).

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[26] A detailed study of the use of such trajectories is under investigation by the authors.