is the same as

$$\left(\frac{d\sigma}{d\Omega}\right)_{e^-e^+\to\mu^-\mu^+}.$$

One finds that the s dependence in $A(s)$ completely cancels out, and thus scale invariance is recovered. This will not happen if we are doing an experiment in which the (fractional) longitudinal momentum x of the final nucleon is measured. Then, the x dependence in the cross section is a very complicated function of s; hence, the s dependence no longer cancels out.

Our point here is not that we have proved or disproved scale invariance in e^-e^+ annihilation. Rather, it is that scale invariance can reasonably hold in an annihilation experiment in which all the parameters of the final state are free to vary over their ranges. On the other hand, such scale invariance is unlikely in an experiment in which some of the parameters of the final state are fixed.

The fact that the total hadron cross section from e^+e^- colliding beams of a given (high) energy should be approximately the same as the $\mu^+\mu^-$ cross section was predicted earlier by Bjorken[9] through the study of certain limits of current algebra. In our model, if we consider the jet as a single unit and if the energy of the final hadrons are not detected, the hadron cross section and the $\mu^+\mu^-$ cross section should have the same angular dependence $1+\cos^2\theta$ as well. This is related to the fact that the only charged particles in our theory are Dirac particles. In a theory in which the only charged particles are scalar particles, one would expect a different angular dependence: $\sin^2\theta$.[10] These factors are nothing but d functions. For a theory involving both charged scalar and charged Dirac particles, the angular dependence of the jets is more involved.

Finally, we would like once again to stress the possible fundamental importance of the longitudinal impact parameter τ in experiments of type 2. Although the particular dependence on τ may not be important, the fact that the distribution of final particles is Poisson in the τ space may very well be. This is a feature which is accessible to experimenters in principle, although the problem of separating multiparticle resonant decay from nonresonant production will undoubtedly be difficult.

[9] J. D. Bjorken, Phys. Rev. **148**, 1467 (1966).

[10] For a more complete discussion of these points see, e.g., R. Gatto, in *Springer Tracts in Modern Physics* (Springer-Verlag, Berlin, 1965), Vol. 39.

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Second-Sheet Poles and the Maximum Behavior of Regge Trajectories at Infinity

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We find that Regge trajectories cannot increase faster than linearly as $s\to\infty$ in any direction on the physical sheet of the complex s plane. Further, if $\mathrm{Re}\alpha(s)$ increases more rapidly than $\mathrm{Im}\alpha(s)$ as $s\to+\infty$, the trajectories must be asymptotically linear. These results are valid only to within possible logarithmic factors and are a consequence of the generally assumed absence of resonance poles on the physical sheet. Other assumptions are as follows. (i) $\alpha(s)$ is real analytic in the s plane cut from threshold (s_0) to $+\infty$. (ii) $\mathrm{Im}\alpha(s)\geq 0$ for $s_0\leq s<+\infty$. (iii) $\alpha(s)$ is continuous on the cut and increases no faster than a power of s as $s\to+\infty$, i.e., along the positive real axis. (iv) As $s\to\infty$ in complex directions, $|\alpha|$ increases no faster than a power of exp $(|s|)$.

In this paper we report a previously unnoticed connection between the behavior of Regge trajectories and the fact that resonance poles occur on the second sheet of the s plane. The connection is rather simple, and we find that it places an upper bound on the behavior of trajectories at large s, namely, that as $|s|\to\infty$,

$$\alpha(s)/|s|^{1+\epsilon}\to 0 \tag{1}$$

for any $\epsilon>0$. This result is valid for any trajectory which is real analytic[1] in the s plane cut from threshold (s_0) to $+\infty$ and continuous on the cut, which is bounded for some ρ and some N by $\exp(|s|\rho)$ as $|s|\to\infty$ and by s^N as $s\to+\infty$,[2] and whose imaginary part is positive for $s_0<s<+\infty$. Actually the imaginary part of the trajectory can be negative as long as the trajectory does not encircle the origin in the l plane as s varies from $-\infty$ to $+\infty$ along the real axis.

Our method makes use of the phase representation[3] for $\alpha(s)-N$, where N is a positive integer of the right signature. For this purpose, we define a function $h(s)$

[1] By real analytic, we mean $\alpha^*(s)=\alpha(s^*)$ for s not on the cut.

[2] The notation $s\to+\infty$ will be used consistently to indicate that s is approaching infinity along the positive real axis.

[3] See M. Sugawara and A. Tubis [Phys. Rev. **130**, 2127 (1963)] for references on the phase representation.

for s on the first sheet by the integral

$$h(s)\equiv\frac{s}{\pi}\int_{s_0}^{\infty}\frac{\psi_+(s')ds'}{s'(s'-s)}, \tag{2}$$

where $\psi(s)$ is the argument of $\alpha(s)-N$ and $\psi_+(s)\equiv\lim_{\epsilon\to 0}\psi(s+i\epsilon)$. This integral is constructed in such a way that $\exp[h(s)]$ has the same phase (or argument) as $\alpha(s)-N$ for $s_0<s<+\infty$. It follows that the function

$$F(s)\equiv[\alpha(s)-N]e^{-h(s)} \tag{3}$$

is real for real s, that is, $F(s)$ is a single-valued function; it does not have the branch points of $\alpha(s)$. In fact, $F(s)$ has no singularities of any kind in the finite s plane and is therefore an entire function. To see this we must consider the function $h(s)$. First we notice that $\psi_+(s)$ is bounded. This and its continuity guarantee the convergence of (2) as long as the denominator does not vanish, i.e., for all s not on the cut. At all points on the cut except at s_0, the integral exists as a principal-value integral, and we already know that this cut is not present in $F(s)$. However, the point $s=s_0$ must be examined carefully. It is a branch point of $h(s)$ and although F does not have a branch point at s_0, it may have some other kind of singularity at this point if $h(s_0)$ is unbounded. Fortunately, it is very easy to extract the behavior of $h(s)$ near s_0, assuming that $\psi(s)$ satisfies a Hölder condition at s_0,[4]

$$|\psi(s)-\psi(s_0)|\leq K|s-s_0|^u, \tag{4}$$

where $u>0$. Thus,

$$\begin{aligned}h(s)&=\frac{s}{\pi}\int_{s_0}^{s_0+\epsilon}\frac{[\psi_+(s_0)+\psi_+(s')-\psi_+(s_0)]}{s'(s'-s)}ds'+\frac{s}{\pi}\int_{s_0+\epsilon}^{\infty}\frac{\psi_+(s')ds'}{s'(s'-s)}\\&=\psi_+(s_0)\frac{s}{\pi}\int_{s_0}^{s_0+\epsilon}\frac{ds'}{s'(s'-s)}+\frac{s}{\pi}\int_{s_0}^{s_0+\epsilon}\frac{\psi_+(s')-\psi_+(s_0)}{s'(s'-s)}ds'+\frac{s}{\pi}\int_{s_0+\epsilon}^{\infty}\frac{\psi_+(s')ds'}{s'(s'-s)}\\&=-\frac{\psi_+(s_0)}{\pi}\ln\left[\frac{(s_0-s)(s_0+\epsilon)}{s_0(s_0+\epsilon-s)}\right]+\frac{s}{\pi}\int_{s_0}^{s_0+\epsilon}\frac{[\psi_+(s')-\psi_+(s_0)]}{s'(s'-s)}ds'+\frac{s}{\pi}\int_{s_0+\epsilon}^{\infty}\frac{\psi_+(s')ds'}{s'(s'-s)}.\end{aligned} \tag{5}$$

The third integral in (5) is clearly bounded at s_0 and for the second one we have

$$\begin{aligned}\left|\frac{s}{\pi}\int_{s_0}^{s_0+\epsilon}\frac{[\psi_+(s')-\psi_+(s_0)]}{s'(s'-s)}ds'\right|&\leq\frac{Ks}{s_0\pi}\int_{s_0}^{s_0+\epsilon}\frac{|s'-s_0|^u}{|s'-s|}ds'\\&\leq\frac{Ks}{s_0\pi}\int_{s_0}^{s_0+\epsilon}|s'-s_0|^{u-1}ds'\leq\frac{Ks}{\pi s_0}\frac{\epsilon^u}{u}.\end{aligned}$$

We conclude that only the first term in (5) is unbounded at s_0 and that $h(s)$ has a logarithic singularity at s_0 unless $\psi_+(s_0)$ vanishes. Using (3) and (5), we see that for s near s_0

$$F(s)\to(s_0-s)^{\psi_+(s_0)/\pi}[\alpha(s_0)-N]. \tag{6}$$

Since $\mathrm{Im}\alpha(s_0)=0$,[5] $\psi_+(s_0)$ must be 0 or π, i.e., $\alpha(s_0)$ is on the real axis. An exception to this occurs if $\alpha(s_0)$ is unbounded, as is the case with Coulomb trajectories. This will be discussed later. Here we assume that $\alpha(s_0)$ is finite and choose $N>\alpha(s_0)$, in which case $\psi_+(s_0)=\pi$ and $F(s)$ is seen to be analytic at s_0; in fact, it has a zero there. Therefore, $F(s)$ is an entire function, and it is important to know something about its behavior at infinity. Frye and Warnock[6] have shown that as $|s|\to\infty$,

$$e^{-h(s)}\to s^{\psi_+(\infty)/\pi}e^{\lambda(s)}, \tag{7}$$

where $|\lambda(s)|<\epsilon\ln s$ for large s and for any $\epsilon>0$. Since $|\alpha|<\exp(|s|^\rho)$ by assumption, it follows from (3) and (7) that for some ρ and for $|s|$ large enough,

$$|F(s)|<e^{|s|^\rho}.$$

The greatest lower bound of the numbers ρ for which the above is valid is called the order of F.

This establishes that $F(s)$ is an entire function of finite order. It is at this point that we use the fact that the resonance poles are on the second sheet in the s plane. Recall that the scattering amplitude can have a pole corresponding to a state of a given mass, spin, and width (say, m, J, and Γ, respectively) if and only if

$$\alpha(m^2-im\Gamma)-J=0. \tag{8}$$

This follows from the usual Regge expression for the scattering amplitude continued into the complex s plane (including the second Riemann sheet),

$$A(s,t)=-\pi\sum_n\frac{\beta_n(s)(2\alpha_n+1)}{2\sin\pi\alpha_n}\times[P_{\alpha_n}(-z)\pm P_{\alpha_n}(z)]+B(s,t), \tag{9}$$

[4] Even if ψ does not satisfy a Hölder condition at s_0, the second term in Eq. (5) will be smaller than the first term, and it is therefore not important for our considerations.

[5] J. B. Bronzan, Phys. Rev. **187**, 2253 (1969).

[6] G. Frye and R. L. Warnock, Phys. Rev. **130**, 478 (1963).

where $B(s,t)$ is the background integral. The continuation of $B(s,t)$ into the complex s plane can be achieved by deforming the integration contour—the integral in its usual form does not converge for complex s. Since the analytic continuation of $B(s,t)$ has no poles in s, it is clear from (9) that $A(s,t)$ can have dynamical poles if and only if $\alpha(s)$ is a positive integer of the right signature. Of course, $\beta(s)$ could have poles, but they would not be associated with a bound state or resonance since they would be present in every angular momentum state. In other words, if we choose N to be a positive integer of the right signature such that $N>\alpha(s_0)$, then $\alpha(s)-N$ cannot vanish on the first sheet of the s plane. The importance of this for our considerations lies in the fact that $F(s)$ is an entire function, and in using Eq. (3) we can restrict our attention to the first sheet of the s plane if we wish. In other words, there is only one point in the s plane at which $F(s)$ is zero, namely, s_0. Recalling that $F(s)$ is of finite order, we can use Picard's little theorem[7] to conclude that

$$F(s)=(s-s_0)e^{P(s)}, \tag{10}$$

where $P(s)$ is a polynomial. Thus

$$\alpha(s)=(s-s_0)e^{P(s)}e^{h(s)}+N. \tag{11}$$

However, (11) and (7) imply that for large s

$$\alpha(s)\rightarrow s^{1-\psi_+(\infty)/\pi}e^{cs^n}, \tag{12}$$

where $P(s)\rightarrow cs^n$, n being the degree of the polynomial. This shows that $\alpha(s)$ will increase exponentially either as $s\rightarrow-\infty$ or as $s\rightarrow+\infty$, depending on the sign of c and the value of the integer n. The former is unphysical and violates the bounds of Greenberg and Low,[8] since in the physical region of the t channel the scattering amplitude has the form $t^{\alpha(s)}$ for large t and negative s. Therefore, if we consider only those trajectories which increase no faster than a power of s as $s\rightarrow+\infty$, we must choose $n=0$. This leaves us with a relatively simple expression for α, namely,

$$\alpha(s)=a(s-s_0)e^{h(s)}+N, \tag{13}$$

where $a\equiv[N-\alpha(s_0)]/s_0$. The primary advantage of this representation is that it allows us to determine the asymptotic behavior of α in terms of the phase of $\alpha-N$ at $s=+\infty$. Using (7) and ignoring $e^{\lambda(s)}$, which can only contribute logarithmic factors, we find that as $|s|\rightarrow\infty$ in any direction,

$$\alpha(s)-N\rightarrow s^{1-\psi_+(\infty)/\pi}. \tag{14}$$

Of course, $\psi_+(\infty)$ is not known in general, but the important point about (14) is that with a single stroke we can restrict $\psi_+(\infty)$ to the range

$$0<\psi_+(\infty)<2\pi \tag{15}$$

without excluding any cases presently thought to be of much physical interest—merely exclude winding trajectories, i.e., those that wind about the origin ($l=0$) as s is increased from s_0 to $+\infty$. Needless to say, the stronger condition that $\mathrm{Im}\alpha\geq0$ for $s_0\leq s<+\infty$ could be used in place of the exclusion of these winding trajectories and (15) would then be replaced by $0<\psi_+(\infty)<\pi$. In either case it follows that (14) prevents trajectories from increasing faster than linearly as $|s|\rightarrow\infty$.

Let us restrict our attention for a moment to infinitely rising trajectories for which

$$\mathrm{Re}\alpha(s)\rightarrow+\infty$$

and

$$\mathrm{Im}\alpha(s)\rightarrow+\infty$$

as $s\rightarrow+\infty$. In this case we obtain a lower bound on $\alpha(s)$ similar to that found earlier by Khuri,[9]

$$\alpha(s)\gtrsim s^{1/2+} \tag{16}$$

as $|s|\rightarrow\infty$. Of course, the trajectories are asymptotically linear only if $\mathrm{Re}\alpha(s)$ increases more rapidly than $\mathrm{Im}\alpha(s)$. Notice that (14) and the conclusions stemming from it are not restricted to large values of s on the positive real axis. This has a bearing on the question of whether infinitely rising trajectories must also be infinitely falling (as $s\rightarrow-\infty$). That this in fact is the case can be seen from (14) by recalling that $h(s)$ is real for $s<s_0$. Therefore e^h is positive and $\alpha(s)\rightarrow-\infty$ as $s\rightarrow-\infty$.

It should be remarked that we could have assumed that the zeros of $\alpha(s)-N$ on the first sheet were of order one instead of assuming that the resonance poles were located on the second sheet of the s plane.

As mentioned earlier, our derivation of the asymptotic form of α is not correct for trajectories which are unbounded at threshold, e.g., Coulomb trajectories. However, by modifying the proof slightly we can show that Eqs. (13) and (14) are valid for such trajectories if we assume that $\alpha-N$ has a zero on the first sheet of order one. Recall that in the case of Coulomb scattering, $\alpha(s)\rightarrow$ const as $|s|\rightarrow\infty$ and $\psi_+(\infty)=\pi$, which is consistent with Eq. (14).

A simple example of a trajectory for which the integral $h(s)$ can be explicitly evaluated is

$$\alpha(s)=as+b-\lambda(s_0-s)^{1/2}. \tag{17}$$

By considering the contour integral

$$\int_c\frac{\ln[\alpha(s')-N]}{s'-s}ds', \tag{18}$$

where c is a circle at infinity plus a curve surrounding the cut from s_0 to $+\infty$, one finds that

$$h(s)=\ln\left(\frac{\alpha(s)-N}{a(s-s_0)}\right),$$

which is equivalent to Eq. (13).

[7] A. I. Markushevich, *Theory of Functions of a Complex Variable* (Prentice Hall, Englewood Cliffs, N. J., 1965), Vol. II, pp. 226–228.

[8] O. W. Greenberg and F. E. Low, Phys. Rev. **124**, 2047 (1961).

[9] N. N. Khuri, Phys. Rev. Letters **18**, 1094 (1967).