simply wrong, but in any case I would suggest that both the hypothesis of Nambu[4] and the mass formulas of Gell-Mann and Okubo[1] are exact in integers, that a strong theoretical support can be given to them in terms of algebraic number theory, and that the methods of algebraic number theory should be used as a test of, or even an approach to, elementary-particle mass formulas.

PHYSICAL REVIEW D VOLUME 2, NUMBER 6 15 SEPTEMBER 1970

Unitary Nonplanar Closed Loops. II*

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(Received 7 May 1970)

Using the methods developed in a previous paper, we generalize the calculation of the dual amplitude for a nonplanar diagram with a single closed loop to the case with an arbitrary number of "twisted vertices." Just as with the four-point function discussed in the previous paper, we find that, if we write the amplitude $M=\int_\Sigma d^4\Pi\, \mathfrak{M}(\Pi)$, $\mathfrak{M}(\Pi)$ is periodic in Π, the period being given by the sum of the four-momenta of the "twisted vertices." We then show in the general case that the prescription of choosing Σ to range over just one period yields the imaginary part required by perturbative unitarity. We verify that M so defined is dual in three different ways. We show explicitly that our result is equivalent to the Kikkawa-Klein-Sakita-Virasoro (KKSV) prescription; we also prove duality directly from the Bardakci-Ruegg–like form without reference to the KKSV structure; and finally we show that duality is manifest within the operator formalism *before* the trace is performed.

I. INTRODUCTION

IN a recent paper,[1] hereafter referred to as I, we have developed techniques for calculating the dual amplitude corresponding to a nonplanar diagram with one closed loop. In I, we worked out explicitly the amplitude for the four-point diagram with one "twisted vertex." The result was found to be almost precisely the form predicted by the duality arguments of Kikkawa, Klein, Sakita, and Virasoro (KKSV).[2] When we wrote the single loop amplitude as $M=\int_\Sigma \mathfrak{M}(\Pi)d^4\Pi$, we found that, for the four-point function, $\mathfrak{M}(\Pi)$ was periodic in Π with the period given by the four-momentum of the twisted vertex, or equivalently, by the sum of the four-momenta of the "untwisted vertices." We then found that by choosing the region of integration, Σ, to range over precisely one period we obtain the correct result for the imaginary part.

In this paper we present the explicit calculation of the arbitrary N-point nonplanar amplitude with m untwisted vertices and $N-m$ twisted vertices. Again we find that, except for a factor

$$\prod_{r=1}\left(\frac{1}{1-x^r}\right)^4,$$

our result is in agreement with the KKSV prescription if one includes *all* lines in the dual diagram. In the arbitrary case we find that $\mathfrak{M}(\Pi)$ is periodic with period given by the sum of the four-momenta of the untwisted vertices.

In Sec. II we derive our result for $\mathfrak{M}(\Pi)$ and write it in Kikkawa-like form. In Sec. III we discuss the singularity structure of $\mathfrak{M}(\Pi)$ and show that by choosing the region Σ to range over just one period, we obtain precisely the result required by perturbative unitarity. In Sec. IV we discuss the duality of the final result; in particular, we discuss duality within the framework of the operator formalism. Most of the details of the calculations are relegated to the appendices.

II. CALCULATION OF UNITARY CLOSED LOOP

Writing the amplitude for Fig. 1 as $M=\int_\Sigma d^4\Pi\mathfrak{M}(\Pi)$, with the region Σ to be specified in accordance with

* Research supported by the U. S. Air Force Office of Scientific Research, Office of Aerospace Research, under Grant No. AF-AFOSR-68-1471A.
† NSF Graduate Fellow.
[1] M. Kaku and C. B. Thorn, Phys. Rev. D 1, 2860 (1970). We refer to formulas from this reference by prefixing a Roman numeral I, e.g., (I,2.3).
[2] K. Kikkawa, S. A. Klein, B. Sakita, and M. A. Virasoro, Phys. Rev. D 1, 3258 (1970); K. Kikkawa, B. Sakita, and M. A. Virasoro, Phys. Rev. 184, 1701 (1969); K. Kikkawa, *ibid.* 187, 2249 (1969).

unitarity, we have[1]

$$\mathfrak{M}(\Pi)=\int_0^1 \prod_{i=1}^N du_i \prod_{i=1}^N u_i^{-\alpha(P_i^2)-1}(1-u_i)^{-c}(1-x)\mathrm{Tr}\{\Omega(\Pi)V^{k_N}u_N{}^R\cdots V^{k_{m+1}}u_{m+1}{}^R\Omega(\Pi-P)V^{k_m}u_m{}^R\cdots V^{k_1}u_1{}^R\}\,, \quad (2.1)$$

where

$$P=\sum_{i=m+1}^N k_i\,,\quad P_i=\Pi+\sum_{i=1}^{i-1}k_i\ (2\le i\le N)\,,\quad P_1=\Pi \text{ and } x=\prod_{i=1}^N u_i\,.$$

Through the use of (I,2.2) and (I,2.3),[1] we reduce the trace in (2.1) to

$$\mathrm{Tr}\left\{e^{\tilde{A}^\dagger(-\Pi)}V^{k_N}\left[\frac{u_N(1-y_m/y_{N-1})}{1-y_m/y_N}\right]^R V^{k_{N-1}}\left[\frac{u_{N-1}(1-y_m/y_{N-2})}{1-y_m/y_{N-1}}\right]^R\cdots V^{k_{m+1}}\left(\frac{-u_{m+1}}{1-y_m/y_{m+1}}\right)^R V^{k_m}u_m{}^R\cdots\right.$$
$$\left.V^{k_2}u_2{}^R V^{k_1}\left[-u_1\left(1-\frac{y_m}{y_N}\right)\right]^R\right\}\prod_{i=m+1}^N\left(1-\frac{y_m}{y_i}\right)^{-P_i\cdot k_i}. \quad (2.2)$$

The y_i's are defined by $y_i=u_{i+1}\cdots u_N$ for $i=1,\ldots,N-1$ and $y_N=1$. Using (I,3.12) to calculate the trace, (2.2) becomes

$$\prod_{i=m+1}^N\left(1-\frac{y_m}{y_i}\right)^{-P_i\cdot k_i}\prod_{r=1}^\infty\left(\frac{1}{1-x^r}\right)^4\prod_{j=1}^N\left(1+\frac{\bar{z}_1\cdots\bar{z}_j}{1-x}\right)^{-\Pi\cdot k_j}\prod_{i<j}(1-\bar{z}_{i+1}\cdots\bar{z}_j)^{-k_i\cdot k_j}$$
$$\times\prod_{ij}\prod_{r=1}^\infty\left[1-x^r\left(\frac{x+(1-x)\bar{z}_{j+1}\cdots\bar{z}_N}{x+(1-x)\bar{z}_{i+1}\cdots\bar{z}_N}\right)\right]^{-k_i\cdot k_j}, \quad (2.3)$$

where we have abbreviated the trace in (2.2) to $\mathrm{Tr}\{e^{A^\dagger(-\Pi)}V^{k_N}\bar{z}^R{}_N\cdots V^{k_1}\bar{z}^R{}_1\}$. Finally, we define variables z_i by

$$z_i=-x-(1-x)\bar{z}_{i+1}\cdots\bar{z}_N\,,\quad z_N=-1\,.$$

It is then a straightforward task to verify that the z_i's satisfy the inequalities

$$-1=z_N\le z_{N-1}\le\cdots\le z_{m+1}\le -x\le 0\le xz_m\le z_1\cdots\le z_m<\infty\,. \quad (2.4)$$

We now change variables in (2.1) from the u_i's to the z_i's and x, obtaining for our final answer

$$\mathfrak{M}(\Pi)=\int_{\mathcal{R}}\frac{dxdz_1\cdots dz_N}{z_1\cdots z_N}(1-x)^2\left(1-x\frac{z_m}{z_1}\right)^{-c}\left(1-\frac{z_1}{z_2}\right)^{-c}\cdots\left(1-\frac{z_{m-1}}{z_m}\right)^{-c}\left(1+\frac{x}{z_{m+1}}\right)^{-c}\left(1-\frac{z_{m+1}}{z_{m+2}}\right)^{-c}\cdots$$
$$(1+z_{N-1})^{-c}x^{-\alpha(\Pi^2)-1}\prod_{r=1}^\infty\frac{1}{(1-x^r)^4}\prod_{j=1}^N|z_j|^{-(\Pi+k_1+\cdots+k_{j-1})k_j-\frac{1}{2}m^2}\prod_{i<j}\left(1-\frac{z_i}{z_j}\right)^{-k_i\cdot k_j}\prod_{ij}\prod_{r=1}^\infty\left(1-x^r\frac{z_i}{z_j}\right)^{-k_i\cdot k_j}. \quad (2.5)$$

$\mathcal{R}$ is the region defined by (2.4).

We observe immediately that $\mathfrak{M}(\Pi)$ is periodic in Π with period P. Indeed, if we change variables via $z_i=xz_i'$ for $1\le i\le m$, we find

$$\prod_{j=1}^N|z_j|^{-(\Pi+k_1+\cdots+k_{j-1})\cdot k_j-\frac{1}{2}m^2}=x^{\Pi\cdot P-\frac{1}{2}P^2}\prod_{j=1}^N|z_j'|^{-(\Pi+k_1+\cdots+k_{j-1})\cdot k_j-\frac{1}{2}m^2}\,,$$

$$\prod_{r=1}^\infty\left(1-x^r\frac{z_i}{z_j}\right)=\left(1-x\frac{z_i'}{z_j}\right)^{-1}\prod_{r=1}^\infty\left(1-x^r\frac{z_i'}{z_j}\right)\quad\text{for}\quad i\in[i,m],\ j\in[m+1,N]$$

$$=\left(1-\frac{z_i}{z_j'}\right)\prod_{r=1}^\infty\left(1-x^r\frac{z_i}{z_j'}\right)\quad\text{for}\quad i\in[m+1,N],\ j\in[i,m]$$

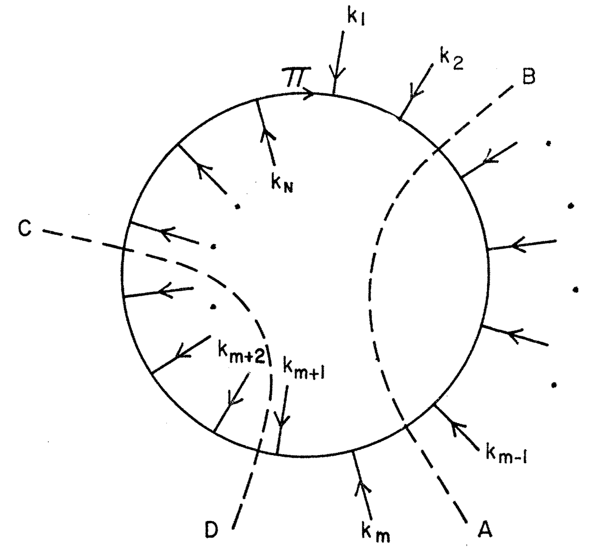

FIG. 1. Diagram representing a nonplanar amplitude with a single closed loop. The dashed lines AB and CD represent two possible channels.

and

$$\left(1-\frac{z_i}{z_j}\right)=\left(1-x\frac{z_i'}{z_j}\right) \text{ for } i\in[1,m],\ j\in[m+1,N]. \blacksquare$$

All of the other factors in the integrand are invariant under this transform. It is now clear that the only change in (2.5) is the substitution $\Pi\rightarrow\Pi-P$. Therefore we have $\mathfrak{M}(\Pi)=\mathfrak{M}(\Pi-P)$ as desired. So, just as in the case of the four-point loop amplitude, if we take Σ = all space, the imaginary part will have an infinite number of equal contributions and will therefore be infinite. We shall show shortly that if we let Σ range over one period, we obtain the correct result for the imaginary part.

But before we turn to this question we again find it convenient to rewrite (2.5) in KKSV form. The KKSV prescription is to associate a variable $\bar{u}_i$ with the ith line in the dual diagram and then write for $\mathfrak{M}$

$$\mathfrak{M}(\Pi)=\int_0^1\prod_j\frac{d\bar{u}_i}{J}\prod_i\bar{u}_i^{-\alpha(P_i^2)-1}, \tag{2.6}$$

where P_i is the four-momentum of the ith line, the product over i ranges over all "distinct" lines, the product over j ranges over N nonintersecting lines, and J is a suitable Jacobian. The dual diagram corresponding to Fig. 1 is illustrated in Fig. 2. One can think of the dual diagram lying on a cylinder, the point $A^{(n)}{}_i$ being identified with the point $A^{(0)}{}_i$ for all $-\infty<n<\infty$. In their original work, Kikkawa, Sakita, and Virasoro did not include lines which went around the cylinder more than one time in (2.6). However, experience with

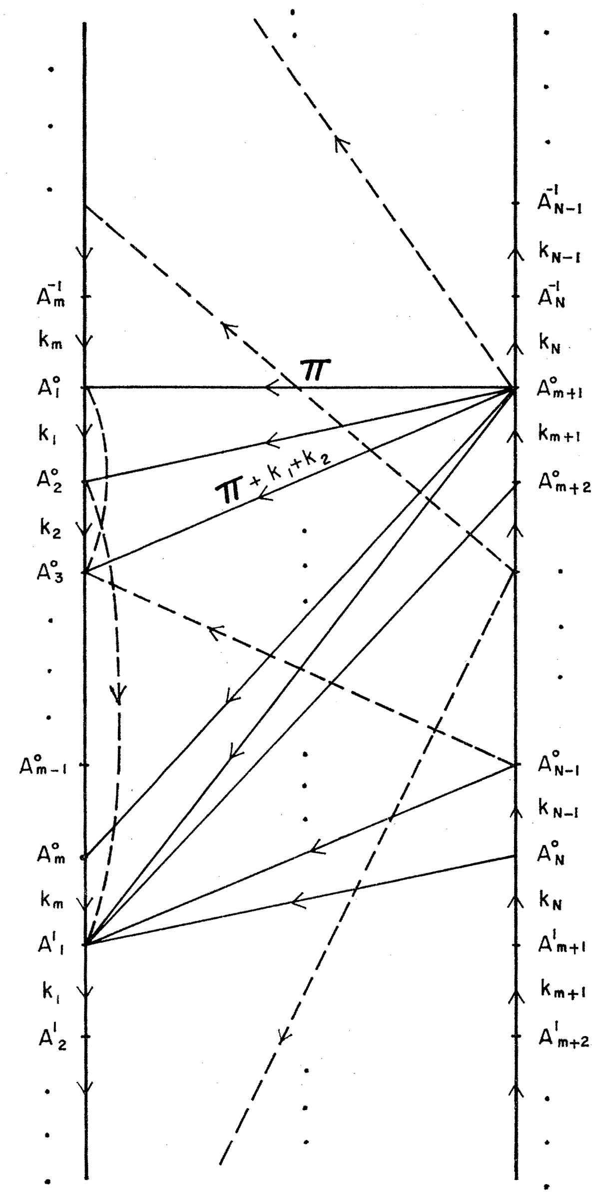

FIG. 2. Dual diagram corresponding to the amplitude represented by Fig. 1. The solid lines correspond to the internal poles as in Fig. 1. There are also an infinite number of other lines such as the dashed lines.

the planar loop[3] has shown us that all such lines really must be included. Nevertheless, for the nonplanar diagram, any two lines which have the same four-momentum are not regarded as distinct. Thus, for example, the line $A^{(n)}{}_iA^{(m)}{}_j$ in Fig. 2 is identified with the line $A^{(n+p)}{}_iA^{(m+p)}{}_j$, with p any integer. On the cylinder two such lines would coincide.

[3] K. Bardakci, M. B. Halpern, and J. A. Shapiro, Phys. Rev. **185**, 1910 (1969); D. Amati, C. Bouchiat, and J. L. Gervais, Nuovo Cimento Letters **2**, 399 (1969); L. Susskind, Yeshiva University report, 1969 (unpublished).

We can divide the set of distinct lines into three groups as follows:

$S_1=\{$distinct lines connecting two points on the left vertical line$\}$,

$S_2=\{$distinct lines connecting two points on the right vertical line$\}$,

$S_3=\{$distinct lines connecting a point on the left to a point on the right$\}$.

Thus S_1 contains the lines $A^0{}_1A^0{}_3, \ldots, A^0{}_1A^0{}_m, A^0{}_1A^1{}_1, A^0{}_1A^1{}_2, A^0{}_1A^1{}_3, \ldots, A^0{}_iA^0{}_{i+2}A^0{}_iA^0{}_{i+3}, \ldots, A^0{}_iA^n{}_j, \ldots,$ i.e., all the lines connecting one of the $A^0{}_i$ $(1\leq i\leq m)$ to a point at least two points below. Similarly, S_2 contains all lines connecting one of the $A^0{}_j$ $(m+1\leq j\leq N)$ to one at least two points below. Finally, S_3 contains all lines connecting one of the $A^0{}_i$ $(m+1\leq i\leq N)$ to one of the points $A^n{}_j$ $(1\leq j\leq m, -\infty<n<\infty)$.

We now show how one can write our expression (2.5) in the form (2.6). We first associate the variable z_i/x^n with the line $A^{(n)}{}_iA^{(n)}{}_{i+1}$; but note that the variable z_m/x^n is associated with the line $A^{(n)}{}_mA^{(n+1)}{}_i$, and $z_N/x^n=-1/x^n$ with $A^{(n)}{}_NA^{(n+1)}{}_{(m+1)}$. Then any of the $\bar{u}_i$'s shall be given by the anharmonic ratio of the variables associated with the four lines emanating from the two connected points. Thus, if the ith line is connected as in Fig. 3, then

$$\bar{u}_i\equiv\frac{(a-b)(c-d)}{(a-d)(c-b)}.$$

For example, the line associated with Π has a u given by

$$u_\Pi=\frac{(z_mx-xz_N)(z_1-z_{m+1})}{(z_mx-z_{m+1})(z_1-xz_N)}.$$

In Appendix A we shall prove that if we define

$$\mathcal{L}_1(z_1,\ldots,z_m,x,k_1,\ldots,k_m)=\prod_{i\in S_1}\bar{u}_i^{-\alpha(P_i)^2-1},$$

$$\mathcal{L}_2(z_{m+1},\ldots,z_N,x,k_{m+1},\ldots,k_N)=\prod_{i\in S_2}\bar{u}_i^{-\alpha(P_i{}^2)-1},$$

$$\mathcal{L}_3(z_i,k_i,x,\Pi)=\prod_{i\in S_3}\bar{u}_i^{-\alpha(P_i{}^2)-1},$$

$$J=(z_m-z_{m-2})\cdots(z_3-z_1)(z_2-xz_m)(z_1-xz_{m-1})(1+z_{N-2})\times(z_{N-1}-z_{N-3})\cdots(z_{m+1}+x)(z_{m-1}-xz_{N-1}),$$

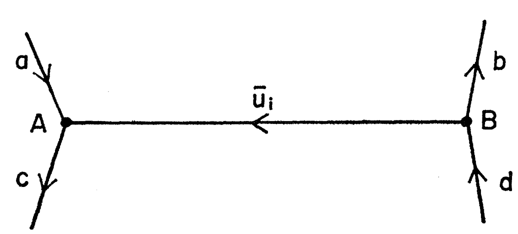

FIG. 3. A typical line AB with associated variable $\bar{u}_i=(a-b)(c-d)/(a-d)(c-b)$.

then (2.5) becomes

$$\mathfrak{M}(\Pi)=\int_{\mathfrak{R}}dx\prod_{i=1}^{N-1}dz_i\frac{1}{J}\prod_{r=1}^{\infty}\left[\frac{1}{(1-x^r)}\right]^4\times\mathcal{L}_1\mathcal{L}_2\mathcal{L}_3(1-x)^2.\quad(2.7)$$

Except for the factor

$$\prod_{r=1}^{\infty}\left[\frac{1}{(1-x^r)}\right]^4(1-x)^2,$$

our result is precisely the KKSV prescription, suitably generalized to take into account all possible lines in the dual diagram.

III. SINGULARITY STRUCTURE OF $\mathfrak{M}(\Pi)$

As we saw in Sec. II, the expression (2.5) for $\mathfrak{M}(\Pi)$ is periodic with period P. Since this fact implies that there are an infinite number of equal contributions to the imaginary part if we take Σ to be all four-space, we have introduced unwanted singularities through our trace calculation. We turn now to a discussion of that problem.

As we know from I, contributions to the imaginary part in a channel with total four-momentum Q arise from double poles in the Π integral of the form

$$R/[\Pi'^2-M_1{}^2][(\Pi'+Q)^2-M_2{}^2].\quad(3.1)$$

From (2.6), we see that there will be a pole in $\mathfrak{M}(\Pi)$ corresponding to any of the $\bar{u}_i$, with $i\in S_3$, going to zero. In terms of the dual diagram (Fig. 2), there is a pole in each of the variables corresponding to each of the lines which cross the diagram. Thus two lines which differ in momentum by the channel momentum represent a contribution to the imaginary part in that channel. We now enumerate the possible contributions to the imaginary part in the various channels.

(1) Channel $k_1, k_2, \ldots, k_m$. Any pair of lines of the form $(A^{(n)}{}_iA^0{}_j, A^{(n+1)}{}_iA^0{}_j)$, where $1\leq i\leq m$, $1\leq j\leq N$, $-\infty<n<\infty$, might contribute to this channel.

(2) Channel $k_i, \ldots, k_j$, $1\leq i<j\leq m$. The lines $(A^{(n)}{}_iA^0{}_k, A^{(n)}{}_{j+1}A^0{}_k)$, with $m+1\leq k\leq N$, $-\infty<n<\infty$, except when $j=m$, in which case the pair is $(A^{(n)}{}_jA^0{}_k, A^{(n+1)}{}_1A^0{}_k)$, represent the possible contributions.

(3) $k_1, \ldots, k_i; k_j, \ldots, k_m$. The possible contributions are $(A^{(n)}{}_jA^0{}_k, A^{(n+1)}{}_{i+1}A^0{}_k)$, $-\infty<n<\infty$, $m+1\leq k\leq N$.

(4) Channel $k_i, \ldots, k_j$, $m+1\leq i<j\leq N$. The contributions are $(A^{(n)}{}_kA^{(0)}{}_i, A^{(n)}{}_kA^{(0)}{}_{j+1})$ except when $j=N$, where the contributions are $(A^{(n)}{}_kA^{(0)}{}_i, A^{(n-1)}{}_kA^0{}_{m+1})$.

(5) Channel $k_{m+1}, \ldots, k_i; k_j, \ldots, k_N$. Here we have $(A^{(n)}{}_kA^0{}_j, A^{(n-1)}{}_kA^0{}_i)$ contributing.

(6) Channel $k_i, \ldots, k_j; k_r, \ldots, k_l$, with $1\leq i<j\leq m$, $m+1\leq r<l\leq N$. Here the contributions are $(A^{(n)}{}_iA^0{}_r, A^{(n)}{}_{j+1}A^0{}_{l+1})$, $-\infty<n<\infty$, unless $j=m$ or $l=N$, when

$$A^{(n)}{}_{j+1}A^0{}_{l+1}\to A^{(n+1)}{}_1A^0{}_{l+1}$$

or

$$A^{(n)}{}_{j+1}A^0{}_{l+1}\to A^{(n+1)}{}_{j+1}A^0{}_{m+1}.$$

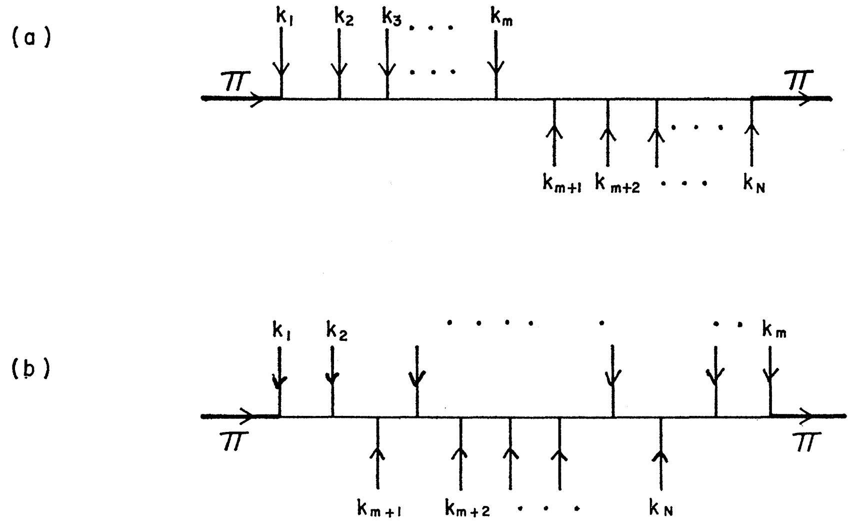

FIG. 4. (a) Diagram representing the tree amplitude to which we applied the tree theorem. (b) An example of a tree diagram representing a factorization scheme different from (a). Duality ensures that both (a) and (b) represent the same amplitude.

(7) Channel $k_1, \ldots, k_i; k_j, \ldots, k_m; k_r, \ldots, k_l$, with $1 \leq i < j \leq m$, $m+1 \leq r < l \leq N$. The contributions are $(A^{(n)}{}_j A^0{}_r,\ A^{(n+1)}{}_{i+1} A^0{}_{l+1})$. If $l=N$, $A^{(n+1)}{}_{i+1} A^0{}_{l+1} \to A^n{}_{i+1} A^0{}_{m+1}$. In all cases there is no contribution unless both incoming and outgoing sets contain at least two particles.

Since the contributions to the imaginary part are due to *double* poles, we must inquire into the conditions for simultaneous poles. In Appendix B we demonstrate that the poles corresponding to any pair of lines can occur simultaneously if and only if the following two conditions hold:

(1) The two lines do not cross one another.
(2) Both lines lie within the same "cycle," i.e., within a parallelogram of side P.

These two conditions can be combined into one if we imagine the dual diagram on a cylinder: The lines corresponding to simultaneous poles do not cross one another, and conversely.

It is easily seen by inspection that each pair of lines quoted in connection with the enumeration of the various channels does indeed satisfy these conditions, and so all correspond to contributions to the imaginary part. On the other hand, for a particular channel, any two such pairs of simultaneous poles are due to the Veneziano variables being in disjoint regions, so that each contribution to the imaginary part occurs additively.

The amplitude to which we applied the tree theorem is represented diagrammatically in Fig. 4(a). It has a set of $N-1$ simultaneous poles corresponding to one of the $u_2, u_3, \ldots, u_N$ being near zero. We applied the tree theorem to get Fig. 1, adding a pole corresponding to u_1 being near zero. In Appendix B we show that the variables corresponding to the set of lines $A^0{}_1 A^0{}_{m+1}, \ldots, A^1{}_1 A^0{}_{m+1}, A^1{}_1 A^0{}_{m+2}, \ldots, A^1{}_1 A^0{}_N$, are zero only if $u_1, u_2, \ldots, u_N$, respectively, goes to zero. Furthermore, any of the other lines in the dual diagram can be zero only if some of the u_i's are near 1. Thus in a channel obtained by cutting any pair of these N lines, e.g., as by the line AB or CD in Fig. 1, we obtain the correct imaginary part by isolating that contribution to the imaginary part corresponding to this pair of lines. Since the contributions are additive, we can achieve this by first cutting off the u_i integrations at $1-\epsilon$, obtaining $\mathfrak{M}_\epsilon(\Pi)$, calculating M_ϵ by choosing Σ to be all four-space, calculating the imaginary part, and finally letting $\epsilon \to 0$ at the end of the calculation. Thus we can write for such a channel

$$\mathrm{Im} M = \int_{\text{all space}} d^4\Pi\, R^0(\Pi)\,, \tag{3.2}$$

where $R^0(\Pi)$ is the contribution corresponding to a pair of lines in the set $A^0{}_1 A^0{}_{m+1}, \ldots, A^1{}_1 A^0{}_{m+1}, \ldots, A^1{}_1 A^0{}_N$.

We make two observations with regard to this procedure. First, since amplitudes represented by diagrams such as that in Fig. 4(b) are equal to the amplitude represented in Fig. 4(a) by duality, application of the tree theorem to each such diagram would yield the same result. Thus there should be contributions to the imaginary part in channels such as $A'B'$ in Fig. 5. Since such a contribution corresponds to some of the u_i's being near 1, the cutoff procedure would yield a zero contribution. But we can include these contributions separately by an analogous cutoff procedure applied to Figs. 4(b) and 5. Notice, however, that a channel such as $C'D'$ in Fig. 5 is the same as CD in Fig. 1, so that there would not be a separate contribution from each diagram.

Our second observation is that the amplitude represented in Fig. 6(a) is not the same as that represented in Fig. 4(a) inasmuch as the cyclic ordering of the particles is different. However, the loop amplitude represented in Fig. 6(b) should be equivalent to that in

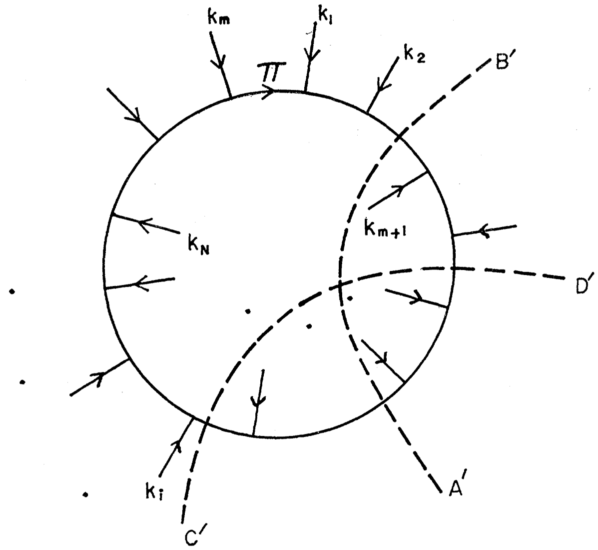

FIG. 5. Applying the tree theorem in the factorization scheme of Fig. 4(b) would yield an amplitude corresponding to this diagram. $A'B'$ and $C'D'$ are two possible channels.

Fig. 1 by duality. Thus, if M is really a dual amplitude we should have in a channel such as $p_1, \ldots, p_m$ a separate contribution for each cyclic ordering $p_i, \ldots, p_m, p_1, \ldots, p_{i-1}$, as well as for each such ordering in the outgoing particles $p_{m+1}, \ldots, p_N$. Since the above cutoff procedure includes only one such contribution, we must take the others into account separately. With these two observations in mind, we can calculate the imaginary part of the dual amplitude corresponding to Fig. 1 in any channel. However, we still have no expression for the real part and so cannot yet define the amplitude M.

Now we show that we can define the amplitude M by choosing Σ to range over just one period:

$$M=\int_{1\text{ period}} d^4\Pi\, \mathfrak{M}(\Pi)\,. \tag{3.3}$$

In view of the discussion in the first part of this section, we can write the imaginary part in any channel:

$$\operatorname{Im}M=\sum_T \int_{1\text{ period}} R_T(\Pi_T)d^4\Pi\,,$$

where each T corresponds to a distinct contribution to the imaginary part and $\Pi_T=\Pi+P_T$ is the momentum of one of the pairs of contributing lines. We can also write this

$$\operatorname{Im}M=\sum_{n=-\infty}^{\infty}\int_{1\text{ period}}\sum_S R^n{}_S(\Pi_S+nP)\,,$$

where to each distinct pair of lines which contribute to the imaginary part in the given channel but does not encircle the cylinder more than once we assign an index S; $R^0{}_S$ is then the corresponding contribution and $R^n{}_S$ is the contribution of the pair which connects the same points as S but encircles the cylinder $|n|$ times in a direction determined by the sign of n. In Appendix B we show that $R^n{}_S(\Pi+nP)=R^0{}_S(\Pi+nP)$. Thus we can write

$$\operatorname{Im}M=\sum_{n=-\infty}^{\infty}\int_{1\text{ period}}\sum_S R^0{}_S(\Pi+nP)d^4\Pi$$

$$=\int_{\text{all space}} d^4\Pi \sum_S R^0{}_S(\Pi)\,. \tag{3.4}$$

But each S corresponds to one of the contributions to the imaginary part in the given channel which we would have had to include separately using the cutoff procedure. In fact, $\int d^4\Pi\, R^0{}_S(\Pi)$ is precisely the contribution we would have obtained. Thus we can conclude that M defined in (3.3) yields precisely the imaginary part required by unitarity.

IV. DUALITY

We already know that the tree amplitudes are dual. Thus, for example, the amplitudes represented in Figs. 4(a) and 4(b) are equal by duality. Thus applying the tree theorem to Figs. 4(a) and 4(b) yields the same result for the two single-loop diagrams in Figs. 1 and 5. It is not yet obvious, however, that the diagrams in Figs. 4(a) and 6(a) contribute to the same dual amplitude. We do know that M has the required singularity structure to yield a contribution with the same pole structure as in Fig. 6(a) since we have been able to write it in a manifestly dual form. In fact, it is almost obvious that taking the residue of the pole corresponding to the line $A^0{}_iA^0{}_j$ yields precisely the amplitude in in Fig. 6(a) since we can change variables so that the lines $A^{-1}{}_iA^0{}_j$, $A^{-1}{}_{i+1}A^0{}_j$, $\ldots$, $A^{-1}{}_mA^0{}_j$, $A^0{}_1A^0{}_j$, $\ldots$, $A^0{}_iA^0{}_j$, $A^0{}_iA^u{}_{j+1}$, $\ldots$, $A^0{}_iA^0{}_N$, $A^0{}_iA^1{}_{m+1}$, $\ldots$, $A^0{}_iA^1{}_{j-1}$ play the same role as $A^0{}_1A^0{}_{m+1}$, $\ldots$, $A^1{}_1A^0{}_{m+1}$, $A^1{}_1A^0{}_{m+2}$, $\ldots$, $A^1{}_1A^0{}_N$. It is then clear from the structure of the integrand that the only change is that $P_i\to P_i+P'$, and when we integrate over one period the result for M is the same as that due to application of the tree theorem to Fig. 6(a).

But we can also prove this aspect of duality directly from the form (2.5). We shall prove the result for the cases $i=m$, $j=m+1$; $i=1$, $j=N$. It will then obviously hold for the arbitrary case. First consider the region $\mathfrak{R}$:

$$z_N=-1\le z_{N-1}\le\cdots\le z_{m+1}$$
$$\le -x\le 0\le xz_m\le z_1\le\cdots\le z_m<\infty\,.$$

If we define $z_m'=xz_m$, we have the region of integration

$$z_N=-1\le z_{N-1}\le\cdots\le z_{m+1}$$
$$\le -x\le 0\le xz_{m-1}\le z_m'\le z_1\cdots\le z_{m-1}<\infty\,.$$

The volume element is unchanged. In fact the only

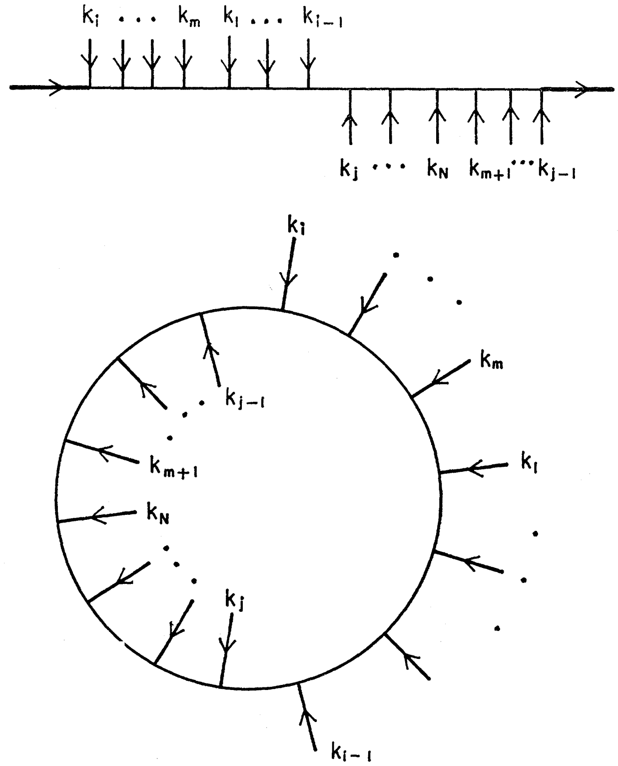

FIG. 6. (a) This tree diagram has a different cyclic ordering than in Fig. 4(a). (b) Applying the tree theorem to Fig. 6(a) would yield a loop amplitude corresponding to this diagram. If the loop amplitude is dual, the diagrams in Figs. 6(b) and 1 represent the *same* amplitude.

changes are the extra factors

$$\left(\frac{1}{x_2}\right)^{-(\Pi+k_1+\cdots+k_{m+1})\cdot k_m-\frac{1}{2}m^2}\left(\frac{1}{x}\right)^{-k_m(k_{m-1}+\cdots+k_N)}$$
$$\times\prod_{i>m}|z_i|^{k_i\cdot k_m}x^{\Pi\cdot k_1-\frac{1}{2}m^2}\prod_{i>m}|z_i|^{k_i\cdot k_m}.$$

The result is the same integral as before but evaluated at $\Pi'=\Pi-k_4$ and, of course, with the change of roles

$$k_1,\, z_1\longrightarrow k_2,\, z_2\longrightarrow\cdots\longrightarrow k_{m-1},\, z_{m-1}\longrightarrow k_m,\, z_m\longrightarrow k_1,\, z_1.$$

Integrating over a period, we have the desired result. We can prove the other case by defining new variables $z_i'=-z_i/z_{N-1}$, $m+1\leq i<N-1$; $z_N'=x/z_{N-1}$. Again one easily finds that the integrand changes only in the roles of the variables: $z_{m+1}\longrightarrow z_{m+2}\longrightarrow\cdots\longrightarrow z_{N-1}\longrightarrow z_N\longrightarrow z_{m+1}$ and the momentum argument $\Pi\longrightarrow\Pi-k_N$. Thus duality can be proved without reference to the specific KKSV structure.

We shall close our paper by a discussion of the aspects of duality which can be seen directly within the operator formalism, without actually performing the trace. In Appendix C we prove the following identity which is crucial for our discussion (see Fig. 7):

$$\begin{aligned}\Omega(\Pi)V^{k_1}D(R,(\Pi-k_1)^2)\Omega(\Pi-k_1)V^{k_2}D(R,(\Pi-k_1-k_2)^2)\\ =V^{k_2}D(R,(\Pi-k_2)^2)\Omega(\Pi-k_2)V^{k_1}\\ \times D(R,(\Pi-k_1-k_2)^2)\Omega(\Pi-k_1-k_2).\end{aligned}\tag{4.1}$$

This identity allows us to see, for example, that Figs. 4(a) and 4(b) are in fact equal. We can also use (4.1) to prove that $\mathfrak{M}(\Pi)$ is periodic without carrying out the trace:

$$\begin{aligned}\mathfrak{M}(\Pi)=&\mathrm{Tr}\{\Omega(\Pi)V^{k_N}D(R,P_i^2)\cdots\\ &V^{k_{m+1}}D(R,P_{m+1}^2)\Omega(P_{m+1})[1-\mathcal{P}_S(P_{m+1})]\\ &\times V^{k_m}D(R,P_m^2)\cdots V^{k_1}D(R,\Pi^2)\}\\ =&\mathrm{Tr}\{[1-\mathcal{P}(P_{m+1})]V^{k_m}D(R,P_m^2)\cdots\\ &V^{k_1}D(R,\Pi^2)\Omega(\Pi)V^{k_N}D(R,P_N^2)\\ &\cdots V^{k_{m+1}}D(R,P_{m+1}^2)\Omega(P_{m+1})\}\\ =&\mathrm{Tr}\{[1-\mathcal{P}(P_{m+1})]\Omega(P_{m+1})V^{k_N}D(R,(P_N-P)^2)\\ &\cdots V^{k_{m+1}}D(R,(P_{m+1}-P)^2)\Omega(P_{m+1}-P)\\ &\times V^{k_m}D(R,(P_m-P)^2)\cdots V^{k_1}D(R,(\Pi-P)^2)\}\\ =&\mathfrak{M}(\Pi-P).\end{aligned}$$

Identity (4.1) expresses also the equivalence of the amplitudes represented in Figs. 1 and 6(b). Thus we can know without explicitly carrying out the trace that our amplitude is dual once we recognize that integrating

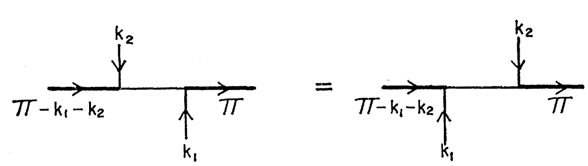

FIG. 7. Diagrammatic representation of the identity (4.1).

over just one period yields the correct value for the imaginary part. We have found no way to see this completely within the operator formalism: It seems necessary to write the amplitude in KKSV form before this is clear.

Note added in manuscript. After this work was completed I was informed that David J. Gross, A. Neveu, J. Scherk, and John H. Schwarz, Phys. Rev. D **2**, 697 (1970), have also obtained many of the results in this paper.

ACKNOWLEDGMENTS

I should like to thank Stanley Mandelstam for many fruitful discussions. I would also like to thank him and Joe Weis for reading the manuscript and making helpful suggestions.

APPENDIX A

We now prove that (2.7) is the same as (2.5). We first consider the factor $\mathfrak{L}_3$. The set of factors corresponding to the lines emanating from $A^0{}_{m+1}$ are

$$\prod_{r=-\infty}^{\infty}\left\{\left[\frac{(x^{r+1}z_m-xz_N)(x^rz_1-z_{m+1})}{(x^{r+1}z_m-z_{m+1})(x^rz_1-xz_N)}\right]^{-\alpha(\Pi+rP)-1}\left[\frac{(x^rz_1-xz_N)(x^rz_2-z_{m+1})}{(x^rz_1-z_{m+1})(x^rz_2-xz_N)}\right]^{-\alpha(\Pi+rP+k_1)-1}\cdots\left[\frac{(x^rz_{m-1}-xz_N)(x^rz_m-z_{m+1})}{(x^rz_{m-1}-z_{m+1})(x^rz_m-xz_N)}\right]^{-\alpha(\Pi+rP+K_m)-1}\right\}$$

$$=x^{-\alpha(\Pi)-1}\left(\frac{z_N}{z_{m+1}}\right)^{-\alpha(\Pi)-1}\prod_{r=0}^{\infty}\prod_{i=1}^{m}\left(\frac{1-x^{r+1}z_N/z_i}{1-x^rz_{m+1}/z_i}\right)^{-(\Pi-rP+K_i)\cdot k_i-\frac{1}{2}m^2}\prod_{r=1}^{\infty}\prod_{i=1}^{m}\left(\frac{1-x^{r-1}z_i/z_N}{1-x^rz_i/z_{m+1}}\right)^{-(\Pi+rP+K_i)\cdot k_i-\frac{1}{2}m^2}.$$

To derive this result, we first wrote the infinite product as a product over positive r and a product over negative r, then rewrote each cross-ratio in the form $(1-x^ru_1)(1-x^ru_2)/(1-x^ru_3)(1-x^ru_s)$, with $r\geq 0$, and finally expanded the exponents and performed considerable cancellation. We have abbreviated $K_i=k_1+\cdots+k_{i-1}$, $K_1=0$.

We must also include all the lines emanating from $A^0{}_{m+2}, \ldots, A^0{}_N$. They are precisely the same with the substitutions $(xz_N, z_{m+1})\rightarrow(z_{i+m}, z_{i+m+1})$ and $\Pi\rightarrow\Pi+k_{m+1}+\cdots+k_{m+i}$. When all of these factors are multiplied together, a further drastic cancellation occurs, so that $\mathfrak{L}_3$ boils down to the relatively simple form

$$\mathfrak{L}_3=x^{-\alpha(\Pi)-1}\prod_{i=m+1}^{N}(-z_i)^{-(\Pi+k_{m+1}+\cdots+k_{i-1})\cdot k_i-\frac{1}{2}m^2}\prod_{i=1}^{m}z_i^{-(\Pi+K_i)\cdot k_i-\frac{1}{2}m^2}$$
$$\prod_{i,s\text{ either }i\in[1,m]\text{ or }j\in[1,m]}\ \prod_{r=1}^{\infty}\left(1-\frac{x^rz_i}{z_j}\right)^{-k_i\cdot k_j}\prod_{i\in[1,m];\,j\in[m+1,N]}(z_i-z_j)^{-k_i\cdot k_j}.$$

We next consider the lines involving the variables $1, \ldots, m$, i.e., those comprising the factor $\mathfrak{L}_1$. When we combine all such lines emanating from $A^0{}_1$ and cancel everything in sight, we obtain

$$\left[\frac{(z_3-xz_m)(z_2-z_1)}{(z_3-z_1)(z_2-xz_m)}\right]^{-\alpha(k_1+k_2)-1}\cdots\left[\frac{(z_1/x-xz_m)(z_m-z_1)}{(z_1/x-z_1)(z_m-xz_m)}\right]^{-\alpha(P)-1}\left(\frac{1-x}{1-x^2z_m/z_1}\right)^{-\alpha(-P+k_1)-1}$$
$$\times\prod_{r=1}^{\infty}\left[\left(\frac{1-x^rz_1/z_2}{1-x^{r+1}z_m/z_2}\right)^{-(-rP+k_1)-\frac{1}{2}m^2}\cdots\left(\frac{1-x^rz_1/z_m}{1-x^{r+1}}\right)^{-(-rP+k_1+\cdots+k_{m-1})\cdot k_m-\frac{1}{2}m^2}\left(\frac{1-x^{r+1}}{1-x^{r+2}z_m/z_1}\right)^{-(-(r+1)P)\cdot k_1-\frac{1}{2}m^2}\right].$$

There is also a similar set of factors corresponding to lines emanating from $A^0{}_2, \ldots, A^0{}_m$. When one includes them all, he finds that everything collapses beautifully to

$$\mathfrak{L}_1=(1-z_{m-1}/z_m)^{-c}\cdots(1-z_1/z_2)^{-c}(1-z_{m-2}/z_m)\cdots(1-z_1/z_3)(1-xz_m/z_2)(1-xz_{m-1}/z_1)$$
$$\times\prod_{\substack{i<j,\\ i,j\in[1,m]}}\left(1-\frac{z_i}{z_j}\right)^{-k_i\cdot k_j}\prod_{r=1}^{\infty}\prod_{i,j\in[1,m]}\left(1-\frac{x^rz_i}{z_j}\right)^{-k_i\cdot k_j}.$$

Obviously $\mathfrak{L}_2$ is the analogous expression:

$$\mathfrak{L}_2=\left(1-\frac{z_{N-1}}{z_N}\right)^{-c}\cdots\left(1-\frac{z_{m+1}}{z_{m+2}}\right)^{-c}\left(1-\frac{xz_N}{z_{m+1}}\right)^{-c}\left(1-\frac{z_{N-2}}{z_N}\right)\cdots\left(1-\frac{z_{m+1}}{z_{m+3}}\right)\left(1-\frac{xz_N}{z_{m+2}}\right)\left(1-\frac{xz_{N-1}}{z_{m+1}}\right)$$
$$\times\prod_{\substack{i<j,\\ i,j\in[m+1,N]}}\left(1-\frac{z_i}{z_j}\right)^{-k_i\cdot k_j}\prod_{r=1}^{\infty}\prod_{i,j\in[m+1,N]}\left(1-x^r\frac{z_i}{z_j}\right)^{-k_i\cdot k_j}.$$

Thus we have

$$\mathfrak{L}_1\mathfrak{L}_2\mathfrak{L}_3=\prod_{i=1}^{N}|z_i|^{-(\Pi+k_1+\cdots+k_{i-1})\cdot k_i-\frac{1}{2}m^2}x^{-\alpha(\Pi^2)-1}\prod_{i<j}\left(1-\frac{z_i}{z_j}\right)^{-k_i\cdot k_j}\prod_{ij}\prod_{r=1}^{\infty}\left(1-x^r\frac{z_i}{z_j}\right)^{-k_i\cdot k_j}\left(1-\frac{z_{m-1}}{z_m}\right)^{-c}\cdots$$

$$\left(1-\frac{xz_m}{z_1}\right)^{-c}(1+z_{N-1})^{-c}\cdots\left(1+\frac{x}{z_{m+1}}\right)^{-c}\left(\prod_{i=1}^{N}\frac{1}{z_i}\right)J.$$

Thus (2.7) becomes

$$\mathfrak{M}(\Pi)=\int_{\mathfrak{R}}dx\prod_{i=1}^{N-1}\frac{dz_i}{z_i}\prod_{r=1}^{\infty}\left[\frac{1}{(1-x^r)}\right]^4(1-x)^2\prod_{i=1}^{N-1}|z_i|^{-(\Pi+k_1+\cdots+k_{i-1})\cdot k_i-\frac{1}{2}m^2}x^{-\alpha(\Pi)^2-1}$$

$$\times\prod_{i<j}\left(1-\frac{z_i}{z_j}\right)^{-k_i\cdot k_j}\prod_{ij}\prod_{r=1}^{\infty}\left(1-x^r\frac{z_i}{z_j}\right)^{-k_i\cdot k_j}\left(1-\frac{z_{m-1}}{z_m}\right)^{-c}\cdots\left(1-\frac{xz_m}{z_1}\right)^{-c}(1+z_{N-1})^{-c}\cdots\left(1+\frac{x}{z_{m+1}}\right)^{-c}.$$

But this is precisely (2.5).

APPENDIX B

We now turn to a discussion of the conditions for simultaneous singularities in the function $\mathfrak{M}(\Pi)$. That is, we would like to know which of the factors in the KKSV form can give rise to singularities simultaneously. It is well known that in the KKSV description two factors which correspond to two intersecting lines of the dual diagram cannot yield simultaneous singularities. This is precisely the condition we have quoted in the text. In fact one can prove that these conditions are correct by detailed consideration of the individual factors occurring in (2.7). We shall not present this proof here except to say that the constraints on the variables z_i are precisely those required to assure that the KKSV prediction is true.

It is another matter to verify that the poles we introduced via the tree theorem, i.e., those arising when any of the u_i, $1\leq i\leq N$, are near zero, precisely correspond to the pairs of lines $A^0{}_{m+1}A^0{}_1,\ \ldots,\ A^0{}_{m+1}A^0{}_m$, $A^0{}_{m+1}A_1{}^1$, $A^0{}_{m+2}A^1{}_1,\ \ldots,\ A^0{}_NA^1{}_1$. We first express the u_i's in terms of the z_i's:

$$\begin{aligned}u_1&=x(1+z_m)/(x+z_1)\,,\\ u_i&=(x+z_{i-1})/(x+z_i)\,,\quad 2\leq i\leq m\\ u_{m+1}&=(x+z_m)/(z_m-z_{m+1})\,,\\ u_j&=(z_m-z_{i-1})/(z_m-z_j)\,,\quad m+2\leq i\leq N.\end{aligned}$$

Now consider the variable for the line $A^0{}_iA^0{}_{m+1}$, $2\leq i\leq m$:

$$\bar{u}_i\equiv\bar{u}_{i,m+1}=\frac{(z_{i-1}+x)(z_i-z_{m+1})}{(z_{i-1}-z_{m+1})(z_i+x)}=u_i\left(\frac{z_i-z_{m+1}}{z_{i-1}-z_{m+1}}\right).$$

For the lines $A^0{}_1A^0{}_{m+1}$ and $A^1{}_1A^0{}_{m+1}$ we have

$$\bar{u}_1\equiv\bar{u}_{1,m+1}=\frac{(xz_m+x)(z_1+1)}{(xz_m+1)(z_1+x)}=u_1\left(\frac{z_1-z_{m+1}}{xz_m-z_{m+1}}\right),$$

$$\bar{u}_{m+1}\equiv\bar{u}_{1,m+1}{}'=\frac{(z_m+x)(z_1/x-z_{m+1})}{(z_m-z_{m+1})(z_1/x+x)}=u_{m+1}\left(\frac{z_1-xz_{m+1}}{z_1+x^2}\right).$$

On the other hand, for the lines $A^1{}_1A^0{}_j$, $m+2\leq j\leq N$, we have

$$\bar{u}_j\equiv\bar{u}_{1,j}{}'=\frac{(z_m-z_{j-1})(z_1/x-z_j)}{(z_m-z_j)(z_1/x-z_{j-1})}=u_j\left(\frac{z_1-xz_j}{z_1-xz_{j-1}}\right).$$

The factors multiplying u_i in the formula for $\bar{u}_i$ are always greater than 1 as can be easily seen. Thus $\bar{u}_i\rightarrow 0$ only if $u_i\rightarrow 0$. Furthermore, if all of the u_i's satisfy $u_i\leq 1-\epsilon$, none of the factors multiplying u_i can be infinite. So when we make the cutoff, we have $\bar{u}_i\rightarrow 0$ if and only if $u_i\rightarrow 0$. It is not hard to see that for any of the other lines going to zero, some of the u_i's are near 1. For example, consider the line $A^r{}_{i+1}A^0{}_{m+1}$:

$$\frac{(z_ix^r+x)(z_{i+1}x^r+|z_{m+1}|)}{(z_ix^r+|z_{m+1}|)(z_{i+1}x^r+x)}\rightarrow 0$$

if

$$z_ix^r,\ x,\ \frac{z_i}{z_{i+1}},\ \frac{x}{z_{i+1}x^r},\ \frac{z_ix^r}{|z_{m+1}|},\ \frac{x}{|z_{m+1}|}\rightarrow 0.$$

If $r\geq 1$, $z_{i+1}\rightarrow\infty$, which implies $z_m\rightarrow\infty$, which implies $u_{m+1},\ \ldots,\ u_N\rightarrow 1$. If $r\leq -1$, $z_i/x|z_{m+1}|\rightarrow 0$, which implies $z_i/x\rightarrow 0$, which implies $u_1,\ \ldots,\ u_m\rightarrow 1$.

That $R^n{}_S(\Pi+nP)=R^0{}_S(\Pi+nP)$ for each A follows from a simple change of variables. Thus, for example, suppose S is the pair of lines $A^{-1}{}_iA^0{}_{m+1}$, $A^0{}_jA^0{}_{m+1}$; the corresponding variables are

$$\frac{(xz_{i-1}+x)(xz_i+|z_{m+1}|)}{(xz_{i-1}+|z_{m+1}|)(xz_i+x)}\quad\text{and}\quad\frac{(z_{j-1}+x)(z_j+|z_{m+1}|)}{(z_{j-1}+|z_{m+1}|)(z_j+x)}.$$

This pair contributes to

$$R^0{}_S(\Pi-k_i-\cdots-k_m,\ \Pi+k_1+\cdots+k_{i-1}).$$

The pair which contributes to

$$R^n{}_S(\Pi-k_i-\cdots-k_m+nP,\ \Pi+k_1+\cdots+k_{j-1}+nP)$$

is therefore

$$\frac{(x^{n+1}z_{i-1}+x)(x^{n+1}z_i+|z_{m+1}|)}{(x^{n+1}z_{i-1}+|z_{m+1}|)(x^{n+1}z_i+x)}$$

and

$$\frac{(x^n z_{j-1}+x)(x^n z_j+|z_{m+1}|)}{(x^n z_{j-1}+|z_{m+1}|)(x^n z_j+x)}.$$

Now if we change variables to z_s', where $z_s'=x^n z_s$ for $1\leq s\leq m$ and $z_s'=z_s$ otherwise, then the only change in the integrand of (2.7) is in $\mathfrak{L}_3$ and that change is merely the shift of roles so that a factor involving $x^m z_s$ now plays the role of the factor which originally involved $x^{m-n}z_s$. It follows, therefore, that when we calculate $R^n{}_S(\Pi-k_i-\cdots-k_m+nP,\ \Pi+k_i+\cdots+k_{j-1}+nP)$, we obtain precisely $R^0{}_S(\Pi-k_i-\cdots-k_m+nP,\ \Pi+k_1+\cdots+k_{j-1}+nP)$. It is obvious that this is true for any pair S.

APPENDIX C

We now turn to the proof of the identity (4.1). Writing

$$\Omega(\Pi)^{k_1}D(R,(\Pi-k_1)^2)\Omega(\Pi-k_1)V^{k_2}D(R,(\Pi-k_1-k_2)^2)$$

$$=\int_0^1 dz_1dz_2\, z_2{}^{-\alpha((\Pi-k_1)^2)-1}z_1{}^{-\alpha((\Pi-k_1-k_2)^2)}(1-z_1)^{-c}$$

$$\times(1-z_2)^{-c}\Omega(\Pi)V^{k_1}z_2{}^R\Omega(\Pi-k_1)z_1{}^R,$$

we find by repeated use of (I,2.2) and (I,2.3) that

$$\Omega(\Pi)V^{k_1}z_2{}^R\Omega(\Pi-k_1)$$

$$=(1-z_2)^R\Omega(\Pi)V^{k_1}\left(\frac{-z_2}{1-z_2}\right)^R(1-z_2)^{-\Pi\cdot k_1+m^2}$$

$$=z_2{}^{A^\dagger(-\Pi)}\left(\frac{1-z_2}{-z_2}\right)^R V^{k_1}\left(\frac{-z_2}{1-z_2}\right)^R(1-z_2)^{-\Pi\cdot k_1+m^2}z_2{}^{\frac{1}{2}\Pi^2}.$$

Combining this result with the identity

$$V^{k_1}y^RV^{k_2}=y^R(-y)^{-k_1\cdot k_2}V^{k_2}y^{-R}V^{k_1}y^R,$$

we have

$$\Omega V^{k_1}z_2{}^R\Omega V^{k_2}z_1{}^R=z_2{}^{A^\dagger(-\Pi)}V^{k_2}\left(\frac{1-z_2{}^R}{-z_2}\right)V^{k_1}\left(\frac{-z_1z_2}{1-z_2}\right)^R z_2{}^{\frac{1}{2}\Pi^2-k_1\cdot k_2}(1-z_2)^{-(\Pi-k_2)\cdot k_1+m^2}$$

$$=V^{k_2}(1-z_2)^R\Omega(\Pi-k_2)V^{k_1}\left(\frac{-z_1z_2}{1-z_2}\right)^R z_2{}^{\Pi\cdot k_2-m^2-k_1\cdot k_2}(1-z_2)^{-(\Pi-k_2)\cdot k_1+m^2}$$

$$=V^{k_2}[1-z_2(1-z_1)]^R\Omega(\Pi-k_2)V^{k_1}\left(\frac{z_1z_2}{1-z_2(1-z_1)}\right)^R\Omega(G-k_1-k_2)$$

$$\times z_2{}^{\Pi\cdot k_2-m^2-k_1\cdot k_2}(1-z_2)^{-(\Pi-k_2)\cdot k_1+m^2}\left(\frac{1-z_2}{1-z_2(1-z_1)}\right)^{(\Pi-k_2)\cdot k_1-m^2}$$

$$=V^{k_2}[1-z_2(1-z_1)]^R\Omega(\Pi-k_2)V^{k_1}\left(\frac{z_1z_2}{1-z_2(1-z_1)}\right)^R\Omega(\Pi-k_1-k_2)$$

$$\times z_2{}^{\Pi\cdot k_2-k_1\cdot k_2-m^2}(1-z_2(1-z_1))^{-(\Pi-k_2)\cdot k_1+m^2}.$$

Defining $\bar{z}_2=1-z_2(1-z_1)$, $\bar{z}_1=z_1z_2/[1-z_2(1-z_1)]$, and inversely, $z_2=1-\bar{z}_2(1-\bar{z}_1)$, $z_2=\bar{z}_1\bar{z}_2/[1-\bar{z}_2(1-\bar{z}_1)]$, we find

$$\times V^{k_1}D\Omega V^{k_2}D=\int_0^1 d\bar{z}_1d\bar{z}_2\left(\frac{\bar{z}_2}{z_2}\right)z_2{}^{-\alpha((\Pi-k_1)^2)-1}\bar{z}_1{}^{-\alpha((\Pi-k_1-k_2)^2)-1}\left(\frac{\bar{z}_2}{z_2}\right)^{-\alpha((\Pi-k_1-k2)^2)-1}\left(\frac{\bar{z}_2}{z_2}\right)^{-\frac{1}{2}m^2-1}(1-\bar{z}_1)^{-c}(1-\bar{z}_2)^{-c}$$

$$\times V^{k_2}\bar{z}_2{}^R\Omega V^{k_1}\bar{z}_1{}^R\Omega z_2{}^{\Pi\cdot k_2-k_1\cdot k_2-m^2}z^{-(\Pi-k_2)\cdot k_1+m^2}$$

$$=\int_0^1 d\bar{z}_1d\bar{z}_2\,\bar{z}_2{}^{-\alpha((\Pi-k_2)^2)-1}\bar{z}_1{}^{-\alpha((\Pi-k_1-k_2)^2)-1}(1-\bar{z}_1)^{-c}(1-\bar{z}_2)^{-c}V^{k_2}z_2{}^R\Omega V^{k_1}z_1{}^R\Omega$$

$$=V^{k_2}D\Omega V^{k_1}D\Omega,$$

as desired.