# Massless Particles with Infinite Spin

JOSEPH E. MURPHY*
*Physics Department, The City College of the City University of New York, New York, New York 10031*
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Reducible free fields of massless particles with infinite spin, transforming covariantly under the Poincaré group, are constructed. The fields are linear combinations of annihilation and creation operators for massless particles summed over all possible spin values. We give some sufficient conditions for satisfying the causality requirement, and find no necessary connection between spin and statistics. Fermi and Bose fields can be quantized so as to satisfy causality with or without the usual spin-statistics relation. We exhibit covariant propagators for these fields and calculate the scattering amplitude in lowest order for two interaction Lagrangians.

## I. INTRODUCTION

The unitary irreducible representations (UIR) of the Poincaré group have long been known and utilized in physical applications.[1] Even those representations which are thought not to correspond to physical particles have played an important role in both field-theoretical and $S$-matrix descriptions of particle interactions. The spacelike representations have been used in group-theoretical approaches to complex angular momentum and in discussions of tachyons.[2,3] Lightlike representations with "infinite spin" have been used in expansions of the scattering amplitude.[4]

The purpose of this paper is to construct a quantum field theory for massless particles with infinite spin. These correspond to representations of the Poincaré group for which $P^2$ is zero, but $W^2$ (the square of the Pauli-Lubanski spin vector) is not zero. The latter implies that the free particle has an infinite number of helicity states for each allowed value of momentum. The existence of these representations was first pointed out by Wigner,[1] who also developed a classical field theory to describe such particles.[5]

Infinite-component fields transforming according to a UIR of the homogeneous Lorentz group have been used primarily in the description of massive particles.[6,7] With modification of the techniques applied to massive particles, we can describe massless particles with infinite spin. The construction is least cumbersome when we use fields transforming according to a Majorana representation of $SL(2,C)$ and obeying the massless Klein-Gordon equation (but not the first-order Majorana equation). The key idea in constructing these reducible fields is to sum over all faithful representations of the little group for $P^2=0$; i.e., sum over all values of the spin. Only if this is done will the fields have simple covariant transformation properties and possibly be causal.

The transformation laws of states, and of creation and annihilation operators, are reviewed in Sec. II. Some properties of the UIR of the homogeneous Lorentz group, in particular the unitary Majorana representations, are presented in Sec. III. Infinite-dimensional free fields are constructed in Sec. IV, and there we give some sufficient conditions for satisfying the causality condition and find no necessary connection between spin and statistics. We do find that both Fermi and Bose fields can be quantized so as to satisfy the causality condition with or without the usual spin-statistics relation. Finally, in Sec. V we calculate covariant propagators and the scattering amplitude in lowest order for two interaction Lagrangians.

## II. FREE-PARTICLE STATES

We begin with a description of physical one-particle states and their behavior under Poincaré transformations. The physical states are a basis for a UIR of the Poincaré group. The infinitesimal generators are $P^\mu$, $M^{\mu\nu}$ with well-known commutation relations. The basis most commonly used in particle physics is one in which $P^\mu$ and one component of the Pauli-Lubanski vector $W^\mu=\frac{1}{2}\epsilon^{\mu\alpha\beta\gamma}M_{\alpha\beta}P_\gamma$ are diagonal. We can choose the standard momentum vector for a massless particle to be $\tilde{p}=(\epsilon,0,0,1)$; here $\epsilon=\pm1$ is the sign of the energy, which is an invariant as in the massive case. Acting on states with momentum $\tilde{p}$, $W^\mu$ has the form

$$W^\mu=(\epsilon M_{12},\ \epsilon M_{23}+M_{02},\ \epsilon M_{31}-M_{01},\ M_{12}). \quad (2.1)$$

It is convenient to group the $M_{\mu\nu}$ into two Hermitian three-vectors

$$\mathbf{J}=(M_{23},M_{31},M_{12}),\quad \mathbf{F}=(M_{01},M_{02},M_{03}). \quad (2.2)$$

* NDEA Pre-Doctoral Fellow.
[1] E. P. Wigner, Ann. Math. **40**, 149 (1939).
[2] H. Joos, *Lectures in Theoretical Physics VIII A*, edited by W. E. Brittin and A. O. Barut (University of Colorado Press, Boulder, Colo., 1965); M. Toller, Nuovo Cimento **54A**, 295 (1968).
[3] O. M. P. Bilaniuk, V. K. Deshpande, and E. C. G. Sudarshan, Am. J. Phys. **30**, 718 (1962); G. Feinberg, Phys. Rev. **159**, 1089 (1967); M. E. Arons and E. C. G. Sudarshan, *ibid.*, **173**, 1622 (1968).
[4] E. P. Wigner, The Application of Group Theory to The Special Functions of Mathematical Physics, Lecture Notes, Princeton, 1955 (unpublished).
[5] E. P. Wigner, Z. Physik **124**, 665 (1948); V. Bargmann and E. P. Wigner, Proc. Natl. Acad. Sci. (U. S.) **34**, 211 (1948); E. P. Wigner, *Theoretical Physics* (IAEA, Vienna, 1963).
[6] E. Majorana, Nuovo Cimento 9, 335 (1932).
[7] G. Feldman and P. T. Matthews, Ann. Phys. (N. Y.) **40**, 19 (1966); Phys. Rev. **151**, 1176 (1966); **154**, 124 (1967); H. D. I. Abarbanel and Y. Frishman, *ibid.* **174**, 1442 (1968); D. Tz. Stoyanov and I. T. Todorov, J. Math. Phys. 9, 2146 (1968).

The algebra of the Euclidean group in two dimensions is generated by

$$J_3,\quad L_1=\epsilon J_2-F_1,\quad L_2=-\epsilon J_1-F_2, \tag{2.3}$$

which have the commutation relations

$$[J_3,L_1]=iL_2,\quad [J_3,L_2]=-iL_1,\quad [L_1,L_2]=0. \tag{2.4}$$

A set of $E2$ basis vectors can be defined by

$$\begin{aligned} J_3|\rho m\rangle &= m|\rho m\rangle,\\ L_\pm|\rho m\rangle &= (L_1\pm iL_2)|\rho m\rangle=\rho|\rho m\pm 1\rangle,\\ L_+L_-|\rho m\rangle &= (L_1{}^2+L_2{}^2)|\rho m\rangle=\rho^2|\rho m\rangle, \end{aligned} \tag{2.5}$$

with normalization

$$\langle\rho m|\rho' m'\rangle=\delta_{mm'}\delta(\rho-\rho')/\rho. \tag{2.6}$$

The only UIP of $E2$ from which representations of the Poincaré group can be constructed are of two types:

(a) Principal series: $\rho$ real, $0<\rho<\infty$

$$\begin{aligned} m&=0,\ \pm1,\ \pm2,\ \ldots\ \text{(single valued)}\\ &=\pm\tfrac{1}{2},\ \pm\tfrac{3}{2},\ \ldots\ \text{(double valued)}. \end{aligned}$$

(b) Discrete series: $\rho=0$.

Representations (b) are not faithful since $L_1=L_2=0$ in them, and they are all one-dimensional since $J_3$ becomes the Casimir operator in $E2$. Thus $m=0$, $\pm\frac{1}{2}$, $\pm1$, ... labels the one-dimensional representations. From Eq. (2.1), $W^\mu=mP^\mu$ and these representations correspond to the known physical particles of mass zero such as the neutrino, photon, or graviton. Representations (a) have been called "infinite integer spin" and "infinite half-integer spin" by Wigner because the particles have an infinite number of spin states. Numerous arguments can be presented to show that no real particles can exist which would transform according to the infinite-spin representations of the Poincaré group. However, they are not as troublesome as tachyons with spin, which, like the massless infinite-spin particle, would give rise to an infinite heat capacity, but also might be found in a state of negative energy.[8]

If we parametrize the momentum according to

$$p^\mu=(\epsilon e^{\epsilon\alpha},e^{\epsilon\alpha}\sin\theta\cos\varphi,e^{\epsilon\alpha}\sin\theta\sin\varphi,e^{\epsilon\alpha}\cos\theta), \tag{2.7}$$

$$-\infty<\alpha<\infty,\quad 0\leq\theta<\pi,\quad 0\leq\varphi<2\pi, \tag{2.8}$$

then we can define the canonical basis vector $|p,s,\lambda,\epsilon\rangle$ by

$$|p,s,\lambda,\epsilon\rangle=U(L(p,\tilde{p}))|\tilde{p},s,\lambda,\epsilon\rangle. \tag{2.9}$$

$U(L(p,\tilde{p}))$ is the unitary transformation which takes us from momentum $\tilde{p}$ to momentum $p$; it can be written in the form

$$U(L(p,\tilde{p}))=\exp[i(\sin\varphi\,\hat{\imath}-\cos\varphi\,\hat{\jmath})\cdot\mathbf{J}\theta]\times\exp(-i\alpha F_3). \tag{2.10}$$

[8] E. P. Wigner (private communication).

The operators $P^\mu$, $W^0$, $P_\mu P^\mu$, and $W_\mu W^\mu$ then act simply in this basis:

$$P^0|ps\lambda\epsilon\rangle=\epsilon|\mathbf{p}||ps\lambda\epsilon\rangle, \tag{2.11}$$

$$\mathbf{P}|ps\lambda\epsilon\rangle=\mathbf{p}|ps\lambda\epsilon\rangle, \tag{2.12}$$

$$W^0|ps\lambda\epsilon\rangle=\mathbf{P}\cdot\mathbf{J}|ps\lambda\epsilon\rangle=\lambda|\mathbf{p}||ps\lambda\epsilon\rangle, \tag{2.13}$$

$$P_\mu P^\mu|ps\lambda\epsilon\rangle=0, \tag{2.14}$$

$$-W_\mu W^\mu|ps\lambda\epsilon\rangle=s^2|ps\lambda\epsilon\rangle. \tag{2.15}$$

The behavior of these basis vectors under an arbitrary Poincaré transformation $(a,\Lambda)$ is easily found to be

$$U(a,\Lambda)|p,s,\lambda,\epsilon\rangle=e^{ia\cdot\Lambda p}\sum_{\lambda'}D_{\lambda'\lambda}{}^{(s)}(R_W)|\Lambda p,s,\lambda',\epsilon\rangle. \tag{2.16}$$

Here $D_{\lambda'\lambda}{}^{(s)}$ are the matrix elements of finite $E2$ transformations, and $R_W=L^{-1}(\Lambda p,\tilde{p})\Lambda L(p,\tilde{p})$ is an element of $E2$. Now we consider the physical case $\epsilon=+1$ and introduce creation and annihilation operators $a^\dagger(ps\lambda)$ and $a(ps\lambda)$ such that acting on the vacuum $|0\rangle$,

$$|ps\lambda\rangle=a^\dagger(ps\lambda)|0\rangle, \tag{2.17}$$

$$[a(ps\lambda),a^\dagger(p's'\lambda')]_\pm=2p^0\delta(\mathbf{p}-\mathbf{p}')F^2(s)\delta(s-s')/s. \tag{2.18}$$

$F(s)$ is a normalization factor, and the other commutators or anticommutators vanish. The transformation properties of these operators follow immediately from Eq. (2.16) and the invariance of the vacuum:

$$U(a)a^\dagger(ps\lambda)U(a)^{-1}=e^{ip\cdot a}a^\dagger(ps\lambda), \tag{2.19}$$

$$U(\Lambda)a^\dagger(ps\lambda)U(\Lambda)^{-1}=\sum_{\lambda'}D_{\lambda'\lambda}{}^{(s)}(R_W)a^\dagger(\Lambda p,s,\lambda'), \tag{2.20}$$

$$U(a)a(ps\lambda)U(a)^{-1}=e^{-ip\cdot a}a(ps\lambda), \tag{2.21}$$

$$U(\Lambda)a(ps\lambda)U(\Lambda)^{-1}=\sum_{\lambda'}D_{\lambda\lambda'}{}^{(s)}(R_W{}^{-1})a(\Lambda p,s,\lambda'). \tag{2.22}$$

When we introduce a complex conjunction matrix $C^{(s)}$ such that

$$D^{(s)}(E)^*=C^{(s)}D^{(s)}(E)C^{(s)-1}, \tag{2.23}$$

$$C^{(s)}C^{(s)*}=(-1)^{2k_0},\quad C^{(s)\dagger}C^{(s)}=I, \tag{2.24}$$

where $k_0=0$ for single-valued representations, $k_0=\frac{1}{2}$ for double-valued ones, the transformation law of the $a^\dagger$ can be made to look like that of the $a$'s. Define $\tilde{a}^\dagger$ by

$$\tilde{a}^\dagger(ps\lambda)=\sum_{\lambda'}(C^{(s)-1})_{\lambda\lambda'}a^\dagger(ps\lambda). \tag{2.25}$$

Then

$$U(a)\tilde{a}^\dagger(ps\lambda)U(a)^{-1}=e^{ia\cdot p}\tilde{a}^\dagger(ps\lambda), \tag{2.26}$$

$$U(\Lambda)\tilde{a}^\dagger(ps\lambda)U(\Lambda)^{-1}=\sum_{\lambda'}D_{\lambda\lambda'}{}^{(s)}(R_W{}^{-1})\tilde{a}^\dagger(\Lambda ps\lambda'). \tag{2.27}$$

Later we shall introduce creation and annihilation operators $b^\dagger$ and $b$ for antiparticles, whose transformation properties are identical to those of $a^\dagger$ and $a$, respectively.

In Sec. IV we want to form the free fields by taking linear combinations of creation and annihilation operators. The fields are assumed to transform according to

$$U(a,\Lambda)\psi(x)U(a,\Lambda)^{-1}=D(\Lambda^{-1})\psi(\Lambda x+a), \quad (2.28)$$

where $D(\Lambda)$ is an infinite-dimensional representation of the Lorentz group. The transformation property under translations forces us to set the field equal to something like a Fourier transform of the creation and annihilation operators. However, the $a(ps\lambda)$ and $b^\dagger(ps\lambda)$ behave under Lorentz transformation in a complicated way on the momentum through the Wigner rotation. The ordinary Fourier transform would not behave like (2.28). In order to construct fields with simple transformation properties we must introduce generalized (infinite-dimensional) spinors as the coefficients of $a$ and $b^\dagger$ in the definition of the field $\psi(x)$.

We should point out some of the differences between the massive and the massless cases. $SU(2)$ is the little group for $p^2>0$; and, in conventional treatments of massive infinite-component fields, the spinors are proportional to matrix elements of a UIR of the homogeneous Lorentz group for the boost $L(p,\tilde{p})$ in an $SU(2)$ basis.[7] We find that our generalized spinors for massless infinite-component fields are proportional to matrix elements of a UIR of the homogeneous Lorentz group for the boost $L(p,\tilde{p})$ in an $E2$ basis.

## III. HOMOGENEOUS LORENTZ GROUP $L$

In this section we briefly review some of the properties of the UIR of the homogeneous Lorentz group. Most of the results can be found in the work of Naimark,[9] and of Gelfand, Minlos, and Shapiro.[10]

The commutation relations of the infinitesimal generators of $L$ are

$$[J_i,J_j]=i\epsilon_{ijk}J_k, \quad (3.1)$$

$$[J_i,F_j]=i\epsilon_{ijk}F_k, \quad (3.2)$$

$$[F_i,F_j]=-i\epsilon_{ijk}J_k. \quad (3.3)$$

All of the irreducible representations are specified by two parameters $(c,k_0)$, where $k_0$ is an integer or half-integer and $c$ is a complex number. If we work in an $E2$ basis, the vectors can be labeled as $|\rho m c k_0\rangle$, and

$$(\mathbf{J}^2-\mathbf{F}^2)|\rho mck_0\rangle=(k_0^2+c^2-1)|\rho mck_0\rangle, \quad (3.4)$$

$$\mathbf{J}\cdot\mathbf{F}|\rho mck_0\rangle=-ik_0c|\rho mck_0\rangle, \quad (3.5)$$

$$J_3|\rho mck_0\rangle=m|\rho mck_0\rangle, \quad (3.6)$$

$$L_\pm|\rho mck_0\rangle=\{(J_2-F_1)\mp i(J_1+F_2)\}|\rho mck_0\rangle =\rho|\rho m\pm 1ck_0\rangle, \quad (3.7)$$

$$0<\rho<\infty, \quad m=k_0,\, k_0\pm1,\, k_0\pm2,\, \ldots. \quad (3.8)$$

[9] M. A. Naimark, *Linear Representations of the Lorentz Group* (Pergamon, New York, 1964).

[10] I. M. Gelfand, R. A. Minlos, and Z. Ya. Shapiro, *Representations of the Rotation and Lorentz Groupz* (Pergamon, New York, 1963).

We shall use the convenient normalization

$$\langle\rho mck_0|\rho' m' ck_0\rangle=\delta_{mm'}\delta(\rho-\rho')/\rho. \quad (3.9)$$

Our reason for considering the UIR of the Lorentz group in an $E2$ basis should by now be clear. One sees that $\rho$ and $m$ have the same range of values as $s$ and $\lambda$ for a physical state. In fact, it is essential that $|\rho mck_0\rangle$ has the same transformation property for the $E2$ subgroup as the physical state $|\tilde{p},s,\lambda\rangle$.

In the construction of our infinite-component fields we will use two particular representations, $(c,k_0)=(0,\frac{1}{2})$, $(c,k_0)=(\frac{1}{2},0)$, the unitary Majorana representations which support a four-vector $\Gamma_\mu$. The normalization can be chosen so that the action of the $\Gamma_\mu$ on the basis states is given[11]:

$$(\Gamma_0+\Gamma_3)|\rho M\rangle=\rho|\rho m\rangle, \quad (3.10)$$

$$(\Gamma_0-\Gamma_3)|\rho m\rangle=-\left(\rho\frac{\partial^2}{\partial\rho^2}+2\frac{\partial}{\partial\rho}-\frac{4m^2-1}{4\rho}\right)|\rho m\rangle, \quad (3.11)$$

$$(\Gamma_1\pm i\Gamma_2)|\rho m\rangle=i\left(\rho\frac{\partial}{\partial\rho}+\frac{3\mp4m}{2}\right)|\rho m\mp1\rangle. \quad (3.12)$$

In the above equations, and in the following, the labels $(c,k_0)$ are suppressed unless needed. The usual group-theoretic properties of the representation matrices of $L$ in an $E2$ basis take the form

$$\int\rho d\rho\sum_m D_{\rho_1m_1\rho m}(\Lambda_1)D_{\rho m\rho_2m_2}(\Lambda_2) =D_{\rho_1m_1\rho_2m_2}(\Lambda_1\Lambda_2), \quad (3.13)$$

$$D_{\rho m\rho' m'}(\Lambda)=D_{mm'}{}^{(\rho)}(\Lambda)\delta(\rho-\rho')/\rho \quad (3.14)$$

for $\Lambda\in E2$. The $E2$ subgroup of $SL(2,C)$ can be parametrized as

$$E(\varphi,\alpha,\psi)=\begin{pmatrix}e^{-i\varphi/2}&0\\0&e^{i\varphi/2}\end{pmatrix}\begin{pmatrix}1&0\\\alpha&1\end{pmatrix}\begin{pmatrix}e^{-i\psi/2}&0\\0&e^{i\psi/2}\end{pmatrix}, \quad (3.15)$$

$$0\le\varphi+\psi<4\pi, \quad 0\le\varphi-\psi<2\pi, \quad -\infty<\alpha<+\infty. \quad (3.16)$$

The $D_{mm'}{}^{(\rho)}(\varphi,\alpha,\psi)$ are well known[4]:

$$D_{mm'}{}^{(\rho)}(\varphi,\alpha,\psi)=e^{-im\varphi}J_{m-m'}(\rho\alpha)e^{-im'\psi}. \quad (3.17)$$

For some calculations, it is advantageous to use an $E2$ parametrization of the unimodular $2\times2$ matrices $A\in SL(2,C)$, which is of the form

$$A=E_1e^{-a\sigma_3/2}VE_2, \quad (3.18)$$

where $V=e^{i\pi\sigma_2/2}$ and the $E$'s are given by (3.15); $\exp(-a\sigma_3/2)$ gives a boost of velocity $v=\tanh a$ along

[11] S. J. Chang and L. O'Raifertaigh, J. Math. Phys. **10**, 21 (1969).

the $z$ axis. For the Majorana representations,[12]

$$D_{\rho m\rho' m'}(A)=\sum_n D_{mn}{}^{(\rho)}(E_1)r_{\rho\rho' n}(a)D_{-nm'}{}^{(\rho')}(E_2)\,, \quad (3.19)$$

where

$$n=\pm\tfrac{1}{2},\pm\tfrac{3}{2},\ldots \quad \text{for} \quad (c,k_0)=(0,\tfrac{1}{2})\,, \quad (3.20)$$

and

$$n=0,\pm1,\pm2,\ldots \quad \text{for} \quad (c,k_0)=(\tfrac{1}{2},0)\,; \quad (3.21)$$

$r_{\rho\rho' n}(a)$ is given by

$$r_{\rho\rho' n}(a)=[2/(\rho\rho'\Delta)^{1/2}]J_{-2n}(4(\rho\rho'/\Delta)^{1/2})\,, \quad (3.22)$$

$$\Delta=4e^{-a}. \quad (3.23)$$

With this machinery we can construct fields out of particle and antiparticle operators for mass zero and infinite spin such that the fields have simple transformation properties under the Poincaré group.

## IV. REDUCIBLE FREE FIELDS AND LOCALITY

We can now construct an infinite-component field satisfying

$$\partial_\mu\partial^\mu\psi(x)=0 \quad (4.1)$$

and containing particle and antiparticle operators. In order to construct a local field with a simple covariant transformation law, we must consider a reducible field with all values of $s$. Consider the operator

$$\psi_{\rho m}{}^{(-)}(x)=(2\pi)^{-3/2}\int d^4p\;\delta(p^2)\theta(p^0)\int sds\sum_\lambda D_{\rho ms\lambda}$$
$$\times(L(p,\tilde{p}))e^{-ip\cdot x}a(ps\lambda)\,. \quad (4.2)$$

Equation (4.1) is satisfied by construction. Applying an arbitrary Poincaré transformation $(a,\Lambda)$ and using Eqs. (2.15), (3.12), and (3.13), one finds

$$U(a,\Lambda)\psi_{\rho m}{}^{(-)}(x)U(a,\Lambda)^{-1}$$
$$=\int\rho' d\rho'\sum_{m'}D_{\rho m\rho' m'}(\Lambda^{-1})\psi_{\rho' m'}{}^{(-)}(\Lambda x+a)\,. \quad (4.3)$$

Alternatively, we could have assumed this transformation law for the reducible field, and then determined that

$$\psi^{(-)}(x)=(2\pi)^{-3/2}\int d^4p\;\delta(p^2)\theta(p^0)$$
$$\times\int sds\sum_\lambda u(ps\lambda)e^{-ip\cdot x}a(ps\lambda) \quad (4.4)$$

satisfies (4.3) if the $(\rho m)$ component of the infinite spinor is given by

$$U(ps\lambda)_{\rho m}=D_{\rho ms\lambda}(L(p,\tilde{p}))=\langle\rho m|U(L(p,\tilde{p}))|s\lambda\rangle\,. \quad (4.5)$$

We will assume that there is a distinct antiparticle, and introducing creation and annihilation operators $b^\dagger$ and $b$, construct the field operator,

$$\psi_{\rho m}{}^{(+)}(x)=(2\pi)^{-3/2}\int d^4p\;\delta(p^2)\theta(p^0)\int sds\sum_\lambda e^{ip\cdot x}$$
$$\times D_{\rho ms\lambda}(L(p,\tilde{p}))\tilde{b}^\dagger(ps\lambda)\,, \quad (4.6)$$

$$\tilde{b}^\dagger=\sum_{\lambda'}(C^{(s)-1})_{\lambda\lambda'}{}_\pi{}^\dagger(ps\lambda)\,. \quad (4.7)$$

The $b,b^\dagger$ are taken to satisfy

$$[b(ps\lambda),b^\dagger(p',s',\lambda')]_\pm$$
$$=2p^0\delta(\mathbf{p}-\mathbf{p}')F^2(s)\delta(s-s')/s\,. \quad (4.8)$$

$\psi^{(+)}(x)$ has the same transformation law as $\psi^{(-)}(x)$, so that the combination

$$\psi_{\rho m}(x)=\alpha\psi_{\rho m}{}^{(-)}(x)+\beta\psi_{\rho m}{}^{(+)}(x) \quad (4.9)$$

transforms according to

$$U(a,\Lambda)\psi(x)U(a,\Lambda)^{-1}=D(\Lambda^{-1})\psi(\Lambda x+a) \quad (4.10)$$

and obeys the massless Klein-Gordon equation (4.1).

The locality of the fields can be investigated using the expression for $\psi(x)$ given in (4.9) and its adjoint, along with the assumed commutation (or anticommutation) relation of the $a$'s and $b$'s:

$$[\psi_{\rho m}(x),\psi_{\rho' m'}(y)]_\pm$$
$$=(2\pi)^{-3}\int d^4p\;\delta(p^2)\theta(p^0)\int sds\,F^2(s)$$
$$\times\sum_\lambda D_{\rho ms\lambda}(L(p,\tilde{p}))D_{s\lambda\rho' m'}(L^{-1}(p,\tilde{p}))$$
$$\times\{|\alpha|^2e^{-ip\cdot(x-y)}\pm|\beta|^2e^{ip\cdot(x-y)}\}\,. \quad (4.11)$$

For this expression to vanish for $(x-y)$ spacelike, it must be proportional to the causal function[13]

$$-i\Delta(x-y)=(2\pi)^{-3}\int d^4p\;\delta(p^2)\theta(p^0)$$
$$\times[e^{-ip\cdot(x-y)}-e^{ip\cdot(x-y)}]\,, \quad (4.12)$$

or a finite number of derivatives of it. Now, we wish to determine some particular $\psi(x)$ which cause (4.11) to vanish for $(x-y)$ spacelike (the microcausality condition). We set $\alpha=\beta=1$, and will assume $F^2(s)$ is a monomial (the extension to a polynomial in $s$ is obvious):

$$F^2(s)=s^N. \quad (4.13)$$

Further, let $D(\Lambda)$ be one of the Majorana representa-

[12] G. Mack and G. J. Iverson, J. Math. Phys. 11, 1581 (1970).

[13] W. Pauli, Phys. Rev. 58, 716 (1940).

tions. The relevant term in (4.11) is

$$A_{\rho m\rho' m'}(p)$$

$$=\int sds\sum_{\lambda} s^N D_{\rho m s\lambda}(L(p,\tilde{p}))D_{s\lambda\rho' m'}(L^{-1}(p,\tilde{p}))$$

$$=\int sds\sum_{\lambda}\langle\rho m|U(L(p,\tilde{p}))|s\lambda\rangle \times s^N\langle s\lambda|U(L^{-1}(p,\tilde{p}))|\rho' m'\rangle. \quad (4.14)$$

Now using

$$(\Gamma_0+\Gamma_3)|s\lambda\rangle=s|s\lambda\rangle, \quad (4.15)$$

$$U(L(p,\tilde{p}))(\Gamma_0+\Gamma_3)U(L^{-1}(p,\tilde{p}))=p^\mu\Gamma_\mu, \quad (4.16)$$

we can write $A_{\rho m\rho' m'}(p)$ as

$$A_{\rho m\rho' m'}(p)=p^{\mu 1}\cdots p^{\mu N}\langle\rho m|\Gamma_{\mu 1}\cdots\Gamma_{\mu N}|\rho' m'\rangle, \quad (4.17)$$

and the (anti) commutator is

$$[\psi_{\rho m}(x),\psi_{\rho' m'}{}^\dagger(y)]_\pm=i^N\langle\rho m|\Gamma_{\mu 1}\cdots\Gamma_{\mu N}|\rho' m'\rangle \times\frac{\partial}{\partial(x-y)_{\mu 1}}\cdots\frac{\partial}{\partial(x-y)_{\mu N}}(2\pi)^{-3}\int d^4p\,\delta(p^2)\theta(p^0) \times[e^{-ip\cdot(x-y)}\pm(-1)^N e^{ip\cdot(x-y)}]. \quad (4.18)$$

If we quantize with the commutator, the field is causal for even $N$, for both infinite integer and infinite half-integer fields. With the anticommutator the field is causal for odd $N$. To keep the usual spin-statistics relations, we can quantize infinite-integer spin fields with the commutator and even $N$, and the infinite half-integer spin fields with the anticommutator and odd $N$.

Obviously, one can exhibit fields for massless particles with infinite spin which are causal and violate the usual connection between spin and statistics. This has been found to be the case also for infinite-component fields describing massive particles.[7] We can achieve the same results by putting the factor $F(s)$ in the definition of the field, e.g.,

$$\psi_{\rho m}{}^{(-)}(x)=(2\pi)^{-3/2}\int d^4p\,\delta(p^2)\theta(p^0)e^{-ip\cdot x} \times\int sds\sum_{\lambda} D_{\rho m s\lambda}(L(p,\tilde{p}))F(s)a(p,s,\lambda), \quad (4.19)$$

and imposing the relation

$$[a(ps\lambda),a^\dagger(p's'\lambda')]_\pm=2p^0\delta(\mathbf{p}-\mathbf{p}')\delta_{\lambda\lambda'}\delta(s-s')/s. \quad (4.20)$$

In the following, we set $F^2(s)=s^{2(k_0+N)}=s^M$, where $k_0$ indicates the Majorana representation appropriate for infinite integer or infinite half-integer spin, and $N=0, 1, 2, \ldots$. Then we quantize with the usual spin-statistics relation. The infinite spinor commutator or anticommutator becomes

$$[\psi(x),\psi^\dagger(y)]_\pm=-i(\Gamma_{\mu 1}\cdots\Gamma_{\mu M})i^M \times\frac{\partial}{\partial(x-y)_{\mu 1}}\cdots\frac{\partial}{\partial(x-y)_{\mu M}}\Delta(x-y). \quad (4.21)$$

## V. PROPAGATORS AND INTERACTIONS

In theories of finite-component fields, calculation of the propagator begins with

$$\langle 0|T\{\varphi_\sigma(x)\varphi_{\sigma'}{}^\dagger(y)\}|0\rangle =\theta(x-y)\langle 0|\varphi_\sigma(x)\varphi_{\sigma'}{}^\dagger(y)|0\rangle +(-1)^{2j}\theta(y-x)\langle 0|\varphi_{\sigma'}{}^\dagger(y)\varphi_\sigma(x)|0\rangle, \quad (5.1)$$

where $j$ is the spin of the field. In momentum space, the effective propagator takes the form

$$S_{\sigma\sigma'}(q)=t_{\sigma\sigma'}(q)/(q^2-\mu^2), \quad (5.2)$$

with $\mu$ equal to the particle mass and $t_{\sigma\sigma'}(q)$ a polynomial of degree $2j$ in $q$. We can attempt the analogous calculation for the infinite-spin fields. Consider

$$\langle 0|T\{\psi_{\rho m}(x)\psi_{\rho' m'}{}^\dagger(y)\}|0\rangle =\theta(x-y)\langle 0|\psi_{\rho m}(x)\psi_{\rho' m'}{}^\dagger(y)|0\rangle +(-1)^{2k_0}\theta(y-x)\langle 0|\psi_{\rho' m'}{}^\dagger(y)\psi_{\rho m}(x)|0\rangle. \quad (5.3)$$

Using the (anti) commutation relations of the creation and annihilation operators, this becomes

$$i^M\langle\rho m|\Gamma_{\mu 1}\cdots\Gamma_{\mu M}|\rho' m'\rangle \times\left(\theta(x-y)\frac{\partial}{\partial(x-y)_{\mu 1}}\cdots\frac{\partial}{\partial(x-y)_{\mu M}}i\Delta_+(x-y) +\theta(y-x)\frac{\partial}{\partial(x-y)_{\mu 1}}\cdots\frac{\partial}{\partial(x-y)_{\mu M}}i\Delta_+(y-x)\right), \quad (5.4)$$

$$i\Delta_+(x)=\frac{1}{(2\pi)^3}\int\frac{d^3p}{2p^0}e^{-ip\cdot x} =\frac{1}{(2\pi)^2}\left[\frac{1}{x^2}-i\pi\delta(x^2)\epsilon(x)\right]. \quad (5.5)$$

Here we encounter a familiar difficulty. If the $\theta$ function could be commuted past the derivates in (5.4), the propagator would be covariant, which would guarantee a Lorentz-invariant $s$ matrix. As it stands, (5.4) has in general noncovariant terms. The standard procedure is to add noncovariant terms to the interaction density so that the effective propagator used in calculations is what one gets by placing the derivatives in (5.4) to the left of the $\theta$ functions. For finite-dimensional fields, the noncovariant terms are generated automatically in the transition from $\mathcal{L}(x)$ to $\mathcal{H}(x)$.[14] We assume the same is

[14] See, e.g., H. Umezawa, *Quantum Field Theory* (North-Holland, Amsterdam, 1956), Chap. X.

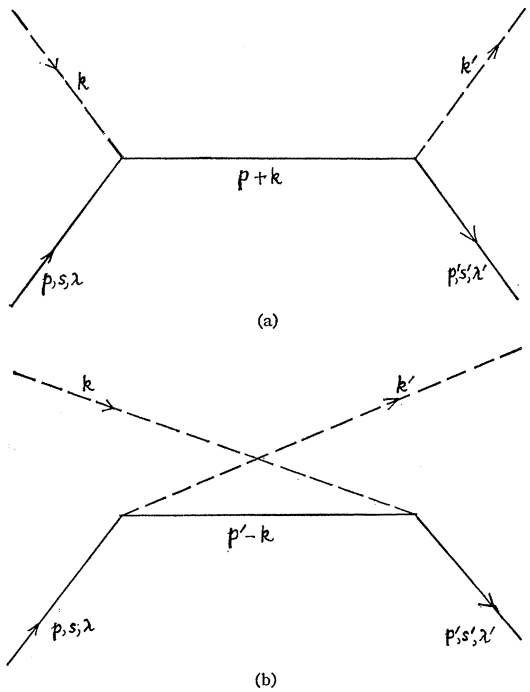


FIG. 1. Lowest-order Feynman diagrams for $\psi(p')+\varphi(k)\rightarrow\psi(p')+\varphi(k')$.

true for these infinite-component fields, so that the effective covariant propagator becomes

$$S_{pmp'm'}(x-y)$$

$$=i^M\langle pm|\Gamma_{\mu1}\cdots\Gamma_{\mu M}|p'm'\rangle\frac{\partial}{\partial(x-y)_{\mu1}}\cdots\frac{\partial}{\partial(x-y)_{\mu M}}$$

$$\times\{\theta(x-y)i\Delta_+(x-y)+\theta(y-x)i\Delta_+(y-x)\}. \quad (5.6)$$

The term in angular brackets is just the usual mass-zero–spin-zero propagator $-i\Delta_F(x-y)$. Writing (5.6) in matrix form then

$$S(x-y)=-i(i)^M\Gamma_{\mu1}\cdots\Gamma_{\mu M}$$

$$\times\frac{\partial}{\partial(x-y)_{\mu1}}\cdots\frac{\partial}{\partial(x-y)_{\mu M}}\Delta_F(x-y). \quad (5.7)$$

In momentum space, (5.7) is given by

$$iS(q)=\Gamma_{\mu1}\cdots\Gamma_{\mu M}q^{\mu1}\cdots q^{\mu M}/q^2. \quad (5.8)$$

Now we wish to illustrate the calculation of the scattering amplitude to lowest order for two interaction Lagrangians. Creation and annihilation operators can be projected out of $\psi(x)$ and $\psi^\dagger(x)$ by using

$$a(ps\lambda)=(2\pi)^{-3/2}\int d^3x\,u^\dagger(ps\lambda)e^{ip\cdot x}i\overleftrightarrow{\partial}_0\psi(x), \quad (5.9)$$

$$a^\dagger(ps\lambda)=(2\pi)^{-3/2}\int d^3x\,\psi^\dagger(x)i\overleftrightarrow{\partial}_0\,u(ps\lambda)e^{-ip\cdot x}. \quad (5.10)$$

Recall that the infinite spinors $u_{\rho m}(ps\lambda)$ have components

$$u_{\rho m}(ps\lambda)=\langle\rho m|U(L(p,\tilde{p}))|s\lambda\rangle. \quad (5.11)$$

Consider the interaction

$$\mathcal{L}_I(x)=g\varphi(x)\psi^\dagger(x)\psi(x), \quad (5.12)$$

where $\varphi(x)$ is the field for a massive scalar particle, and

$$\psi^\dagger(x)\psi(x)\equiv\int\rho d\rho\sum_m\psi_{\rho m}{}^\dagger(x)\psi_{\rho m}(x). \quad (5.13)$$

The lowest-order scattering amplitude is given by (see Fig. 1)

$$A=-i(g/2\pi)^2\delta(p+k-p'-k')(T_a+T_b), \quad (5.14)$$

where

$$T_a=\frac{\langle s'\lambda'|U(L(p',\tilde{p}))^{-1}\Gamma_{\mu1}\cdots\Gamma_{\mu M}U(L(p,\tilde{p}))|s\lambda\rangle(p+k)^{\mu1}\cdots(p+k)^{\mu M}}{(p+k)^2+i\epsilon}, \quad (5.15)$$

$$T_b=\frac{\langle s'\lambda'|U(L(p',\tilde{p}))^{-1}\Gamma_{\mu1}\cdots\Gamma_{\mu M}U(L(p,\tilde{p}))|s\lambda\rangle(p-k')^{\mu1}\cdots(p-k')^{\mu M}}{(p-k')^2+i\epsilon}. \quad (5.16)$$

Note that we are considering in general an inelastic collision, $s'\neq s$. As a second example we consider the interaction of $\psi(x)$ with a vector field $B^\mu(x)$ given by

$$\mathcal{L}_I(x)=gB^\mu(x)\psi^\dagger(x)\Gamma_\mu\psi(x), \quad (5.17)$$

where $\Gamma_\mu$ is the Majorana vector. The lowest-order Feynman graphs are identical to Fig. 1 except that the spin state of the initial and final vector particles must be given. The amplitude is

$$A=-i(g/2\pi)^2\delta(p+k-p'-k')e^\alpha(k',\sigma')e^\beta(k,\sigma)T_{\alpha\beta}, \quad (5.18)$$

$$T_{\alpha\beta}=\langle s'\lambda'|U(L(p',\tilde{p}))^{-1}\Gamma_\alpha\Gamma_{\mu1}\cdots\Gamma_{\mu M}\Gamma_\beta U(L(p,\tilde{p}))|s\mu\rangle\left(\frac{(p+k)^{\mu1}\cdots(p+k)^{\mu M}}{(p+k)^2+i\epsilon}+\frac{(p-k')^{\mu1}\cdots(p-k')^{\mu M}}{(p-k')^2+i\epsilon}\right). \quad (5.19)$$

Here $e^\alpha(k,\sigma)$ is the polarization four-vector for the vector boson with helicity $\sigma$ and momentum $k$.

These interactions were chosen only to illustrate the form that amplitudes involving massless infinite-spin particles would take. The amplitudes are very similar in form to those that one would obtain for massive infinite-component fields interacting with a scalar or vector field. However, the two interactions shown could not be used to construct a theory of interacting particles because, even in lowest-order perturbation theory, the amplitudes have a singularity in the physical region.

## VI. DISCUSSION AND CONCLUDING REMARKS

No unusual problems arise in constructing fields to describe massless particles with infinite spin. A simple covariant transformation law is realized considering a reducible field, one which contains all spins, and transforms according to a unitary representation of the homogeneous Lorentz group. Only by utilizing a reducible field are we able to satisfy causality. The causality condition does not impose a necessary connection between spin and statistics for these fields. This is also the case for infinite-component fields of massive particles,[7] and seems to be a characteristic of infinite-component fields in general.

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# Nonexistence of Schwinger Terms in Spinor Quantum Electrodynamics*

DAVID G. BOULWARE AND JACK HERBERT
*Physics Department, University of Washington, Seattle, Washington 98105*
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The equal-time commutator between the space and time components of the electromagnetic current in spinor electrodynamics is calculated, to order $\alpha$, using the technique of Johnson, Low, and Bjorken; it is found to be a $c$ number. An argument is presented which indicates that the commutator is a $c$ number in all orders of perturbation theory.

DURING the past few years there has been some controversy concerning the nature of the Schwinger terms in fermion quantum electrodynamics. Straightforward application of the canonical commutation relations leads to the conclusion that the equal-time commutator of the time and space components of the electromagnetic current,[1]

$$\begin{gathered} j^\mu=\bar{\psi}\gamma^\mu\psi\,, \\ \{\psi(\mathbf{r},t),\bar{\psi}(\mathbf{r}',t)\}=\gamma^0\delta(\mathbf{r}-\mathbf{r}')\,, \end{gathered} \tag{1}$$

is identically zero. On the other hand, the basic postulates of quantum field theory imply that the vacuum expectation value of the commutator cannot vanish unless the current vanishes identically.[2] There are, however, no general arguments which indicate the nature of other matrix elements; boson currents yield operator Schwinger terms, but those of free currents are $c$ numbers. Several different answers have appeared in the literature for the case of quantum electrodynamics, but no convincing calculations of the commutators have been given. On the one hand, arguments have been given that the commutator is zero, based on the general structure of the photon scattering matrix element,[3] and, on the other, Brandt[4] has argued on the basis of a point-separation limiting procedure that the commutator is nonvanishing and, in particular, proportional to

$$(\mathbf{A}\cdot\mathbf{A}\delta^{kl}+2A^kA^l)\nabla_l\delta(\mathbf{r})\,.$$

In this paper, we take the position that the commutator is a directly calculable quantity, given the relevant matrix elements of the operators whose commutators we wish to calculate. In particular, given the matrix elements

$$\langle\alpha|j^\mu(x)j^\nu(x')|\beta\rangle=K^{(+)\mu\nu}(x,x') \tag{2}$$

or

$$i\langle\alpha|T^*(j^\mu(x)j^\nu(x'))|\beta\rangle=K^{\mu\nu}(x,x')\,, \tag{3}$$

where the $T^*$ product is the *covariant* current-correlation function,[5] including any seagulls necessary for covariance, the equal-time commutator, if it exists, can

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[1] The metric is $(-1,1,1,1)$ and the Dirac matrices are defined by $\{\gamma^\mu,\gamma^\nu\}=-2g^{\mu\nu}$.

[2] J. Schwinger, Phys. Rev. Letters **3**, 296 (1959); S. Okubo Nuovo Cimento **44A**, 1015 (1966). See also D. G. Boulware and S. Deser, Phys. Rev. **151**, 1278 (1966).

[3] D. G. Boulware, Phys. Rev. **151**, 1024 (1966); T. Nagylaki, *ibid.* **158**, 1534 (1967).

[4] R. Brandt, Phys. Rev. **166**, 1795 (1968).

[5] L. S. Brown, Phys. Rev. **150**, 1338 (1966); R. P. Feynman (unpublished).