Nonperturbative renormalizability for a class of gradient-free models

R. Kotecký
Mathematical Institute of the Academy of Science, Žitná 25, Prague 1, Czechoslovakia

D. Preiss
Department of Mathematics, Charles University, Sokolovská 83, Prague 8, Czechoslovakia
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The augmented field theory proposed by Klauder is shown to be equivalent (in its Euclidean formulation) to the conventional field theory with the term $\log|\phi(x)|$ added to the Lagrangian. Gradient-free models (arising by omitting the kinematical term of the Lagrangian) with the Lagrangian of the form $\mathcal{P}(\phi(x))+\log|\phi(x)|$ (where $\mathcal{P}$ is a polynomial) are studied (in the Euclidean lattice limit formulation). Their generating functionals are explicitly calculated. It is shown that these models (nonrenormalizable in the perturbative approach) are renormalizable in nonperturbative theory. The problem of the limit with "vanishing interaction constant" is considered. Its solution depends heavily on clarifying the notion of the "vanishing interaction constant." When introducing the "physical" mass and interaction constants as parameters of generating functionals, this limit gives the usual free theory even for the models studied by Klauder.

This is a comment concerning a recent proposal by Klauder[1] for an alternative approach to scalar field quantization called the "augmented field theory." Klauder introduces a new interpretation of the formal Lebesgue measure $\mathfrak{D}\phi$ in the functional integral expression

$$S(h)=\mathfrak{N}\int \exp\Big(i\int h(x)\phi(x)d^dx - \int\big(\tfrac{1}{2}\{[\nabla\phi(x)]^2+m_0{}^2\phi^2(x)\} + \lambda_0\phi^4(x)\big)d^dx\Big)\,\mathfrak{D}\phi \qquad (1)$$

for the generating functional of time-ordered Green's functions in the Euclidean formulation.[2] He replaces the usual formal prescription $\mathfrak{D}\phi=\Pi_x d\phi(x)$ by $\mathfrak{D}'\phi$ formally given by

$$\mathfrak{D}'\phi=\prod_x|\phi(x)|^{-1}d\phi(x)\,. \qquad (2)$$

Then he analyzes the gradient-free model [obtained from (1) by omitting the gradient term $(\nabla\phi)^2$] with this new prescription $\mathfrak{D}'\phi$ in mind. In this case he is able to compute nonperturbatively (as a limit of the lattice approximation with the lattice spacing approaching zero and spacetime cutoff going to infinity regularizing $\mathfrak{D}'\phi$ in the lattice approximation in a particular way) the explicit formula for the generating functional. This formula has an interesting property: in the limit of "vanishing interaction constant" λ, it yields the pseudofree model, which differs significantly from the usual free model.

Klauder's "augmented measure" $\mathfrak{D}'\phi$ is formally identical to $\exp[-\int\ln|\phi(x)|\,d^dx]\mathfrak{D}\phi$.

$$\begin{aligned}\mathfrak{D}'\phi&=\prod_x|\phi^{-1}(x)|d\phi(x)\\&=\prod_x\exp[-\ln|\phi(x)|]d\phi(x)\\&=\exp\Big[-\int\ln|\phi(x)|d^dx\Big]\mathfrak{D}\phi\,,\end{aligned}$$

where the last term should be understood as being regularized as $\exp[-b\int\ln|\phi(x)|d^dx]\mathfrak{D}\phi$, which corresponds to the similar regularization of Klauder's $\mathfrak{D}\phi'$ as

$$\prod_x\frac{d\phi(x)}{|\phi(x)|^a}\,.$$

Therefore the augmented field theory is, at least at the formal level, equivalent to the conventional theory with the nonpolynomial term $\ln|\phi(x)|$ added to the Lagrangian, however. Moreover, it is clear that in the lattice approximation all calculations are exactly the same; that shows the rigorous equivalence of both formulations. [We are considering the Euclidean formulation since it fully suffices for the discussion of the problems we are interested in (nonperturbative renormalization, vanishing interaction constants). To recover the Minkowski formulation of augmented field theory one should use a certain analytic continuation of generating functionals (corresponding to continuation in all interaction constants except the one in front of $\ln|\phi|$).]

In light of the above observation it would be interesting to clarify questions concerning the limit of vanishing interaction constants. At first sight the new term $\ln|\phi|$ in the Lagrangian seems to suggest that the peculiar behavior of the gradient-free models, when the interaction constant approaches zero, could be simply due to keeping fixed the constant in front of $\ln|\phi|$. However, we will see, that the generating functional given by Klauder converges to that of the free field when properly interpreting the notion of vanishing interaction constant.

The aim of the present article is to show what may be summarized in the following points:

(i) The augmented field theory is equivalent to the conventional formulation [with the Lebesgue measure $\mathfrak{D}\phi$ in (1)], however, with the nonpolynomial interaction term $\int \ln|\phi(x)| d^d x$ added to the action.

(ii) Generating functionals of gradient-free models given formally by

$$S(h) = \mathfrak{N} \int \exp\Big(i \int h(x)\phi(x) d^d x - \int \Big[\tfrac{1}{2} m_0^2 \phi^2(x) + \mathcal{L}_I(\phi(x)) + b \ln|\phi(x)| \Big] d^d x \Big) \mathfrak{D}\phi$$

with the polynomial term $\mathcal{L}_I(\phi)$ can be explicitly nonperturbatively calculated by the lattice limit. For $\mathcal{L}_I(\phi) = \lambda_0 \phi^4$ Klauder's result is recovered.

(iii) In the nonaugmented case with $b = 0$ nontrivial generating functionals are also obtained.

(iv) The models from the two last points can be considered as renormalizable in nonperturbative theory, although they are nonrenormalizable in the perturbative approach.

(v) When introducing the physical mass and interaction constants as parameters of generating functionals, the usual free theory is obtained in the limit of vanishing interaction constant with the mass kept fixed.

RENORMALIZATION IN THE NONPERTURBATIVE APPROACH

Parameters $m_0, \lambda_0, \ldots$ in formulas of the type (1) are bare parameters, having only symbolic meaning. When solving the model nonperturbatively using the lattice limit, they have to be replaced by cutoff-dependent functions (e.g., functions of lattice spacing). In the case where the model is not superrenormalizable (i.e., it is renormalizable or nonrenormalizable in perturbation theory), it is not known how to transform the information about perturbatively given counterterms into the information about particular lattice dependence of functions $m_0(\Delta), \lambda_0(\Delta), \ldots$. Generally, many particular functions $m_0(\Delta), \lambda_0(\Delta), \ldots$ will be allowed, yielding in the limit the whole set of generating functionals.

Let us explain it in more detail. The task is to give the precise meaning to the formal expression for dynamical measure μ [whose Fourier transform gives the generating functional $S(h) = \int \exp(i\langle\phi, h\rangle d\mu(\phi)$ of Euclidean n-point functions]

$$d\mu(\phi) = \mathfrak{N} \exp\left\{ - \int \Big[\sum_1^q a_k \mathcal{L}_k(\phi(x)) \Big] d^d x \right\} \mathfrak{D}\phi \,. \qquad (3)$$

Here $\mathfrak{N}$ is a normalizing factor [such that $\int d\mu(\phi) = 1$] and $\mathcal{L}_k(\phi(x))$ are the individual terms from which the Langrangian is composed. The formula (3) does not have an *a priori* meaning for the whole infinite-dimensional space, but is well defined on lattice subspaces (with eventual derivatives approximated by differences). The measure μ can be looked for in the form of a limit of these finite-dimensional measures. Here we will understand the lattice subspace as the vector space W_Δ formed by step functions constant on lattice boxes of volumes Δ vanishing outside the around-the-origin-centered space-time box of volume L_Δ. By the lattice limit (denoted $\Delta \to 0$) we then mean the limit in which both the volume Δ vanishes and the volume L_Δ increases to infinity. Approximating measures on a lattice subspace W_Δ will be denoted $d\mu_\Delta(\varphi, a_1(\Delta), \ldots, a_q(\Delta))$. As we generally have no information as to which particular lattice dependence of coefficients $a_k(\Delta)$ yields any reasonable result, we will consider as the outcome of the theory the set of all measures $\mu = \lim_{\Delta \to 0} \mu_\Delta(\varphi, a_k(\Delta))$ corresponding to all choices of $a_k(\Delta)$ for which this limit exists.[3] In the case where there is a parametrization of this set by a finite number of real parameters it is natural to call the model renormalizable (in the nonperturbative sense). Indeed, the unique generating functional can then be chosen just by fixing several constants given from experimental data (e.g., mass, interaction constants).

GRADIENT-FREE MODELS

We will discuss gradient-free models[4] with a Lagrangian of the type $\sum_{k=1}^{2p} A_k \phi^k(x) + B \ln|\phi(x)|$ in accordance with the above-discussed concept of renormalization in the nonperturbative theory. Thus we consider approximating measures

$$d\mu_\Delta(\varphi, A_1(\Delta), \ldots, A_{2p}(\Delta), B(\Delta)) = \mathfrak{N} \exp\left\{ - \int \Big[\sum_{k=1}^{2p} A_k(\Delta) \varphi^k(x) + B(\Delta) \ln|\varphi(x)| \Big] d^d x \right\} d\varphi \qquad (4)$$

or equivalently

$$d\mu_\Delta(\varphi, A_1(\Delta), \ldots, A_{2p}(\Delta), B(\Delta)) = \mathfrak{N}\exp\left\{-\sum_j\left[\sum_{k=1}^{2p}\Delta A_k(\Delta)(\varphi_j)^k + \Delta B(\Delta)\ln|\varphi_j|\right]\right\}d\varphi\,. \quad (5)$$

It turns out that the set of all possible resulting dynamical measures can be parametrized by $2p+1$ real parameters.

Proposition 1. A measure μ results as a lattice limit of $\mu_\Delta(\varphi, A_1(\Delta), \ldots, A_{2p}(\Delta), B(\Delta))$ if and only if either (a) there are $\alpha_k \in \mathrm{R}$ $(k=0,1,\ldots,2p)$, $0 \leqslant p' \leqslant p$, $\alpha_{2p'} > 0$, and $\alpha_{2p'+1} = \cdots = \alpha_{2p} = 0$ such that the generating functional (Fourier transform of μ) is

$$S(h) = \exp\left[\int_{\mathrm{R}^d}\int_{\mathrm{R}}\{\exp[iuh(x)]-1\} \times \exp\left(-\sum_{k=0}^{2p}\alpha_k u^k\right)\frac{du}{|u|}\,d^dx\right]$$

or (b) there are $\alpha_k, \beta_k \in \mathrm{R}$ $(k=0,\ldots,p-1)$, $\alpha_k \geqslant 0$ such that

$$S(h) = \exp\left[\int\left(i\beta_0 h(x) - \alpha_0 h^2(x) + \sum_{j=1}^{p-1}\alpha_k\{\exp[i\beta_k h(x)]-1\}\right)d^dx\right]$$

or (c) there are α_k, β_k $(k=1,\ldots,p)$, $\alpha_k \geqslant 0$ such that

$$S(h) = \exp\left[\int\left(\sum_{k=1}^{p}\alpha_k\{\exp[i\beta_k h(x)]-1\}\right)d^dx\right].$$

The proof of the proposition will be given in the Appendix; it relies heavily upon the fact we are considering the gradient-free models, which allows the use of the powerful theory of infinitely divisible processes.

The resulting dynamical measures in proposition 1 are given explicitly by their Fourier transforms. This is convenient in illustrating some problems connected with the notion of renormalizability in nonperturbative theory, namely, the direct consequence of the above proposition is that given models are renormalizable—the set of dynamical measures is parametrized by a finite number of real parameters. On the other hand, these models are nonrenormalizable (for $d>0$) in the perturbative approach. Thus we see that these two notions of renormalizability do not coincide.

Let us also remark that case (b) of proposition 1 corresponds to the polynomial gradient-free model calculated without any logarithmic term in the Lagrangian.[5] This does not contradict Klauder's proof[1] that $\mathcal{L}(\phi(x)) = A_2\phi^2(x) + A_4\phi^4(x)$ yields only a free model (Gaussian measure). His proof does not work for

$$\mathcal{L}(\phi(x)) = \sum_{k=1}^{4}A_k\phi^k(x)$$

for which we obtain a 4-parameter set of measures with Fourier transforms

$$S(h) = \exp\left(\int(i\beta_0 h(x) - \alpha_0 h^2(x) + \alpha_1\{\exp[i\beta_1 h(x)]-1\})d^dx\right). \quad (6)$$

Klauder's result follows from the fact that the only symmetric (under the transformation $\phi \to -\phi$) measures among these are Gaussian ones. Perhaps this example indicates that it is more reasonable to study $\mathcal{P}(\phi)$ models with $\deg\mathcal{P} \leqslant 2p$ instead of ϕ^{2p} models; of course, the terms of order lower then $2p$ in the former can be considered as originating from counterterms added to the latter.

The explicit formulas of proposition 1 allow us to discuss the behavior of the model with vanishing interaction constant. First, it is necessary to clarify what is meant by vanishing interaction constant. Physical interaction constants are usually specified as values of (amputated) truncated n-point functions for certain impulses. This means that we must first replace the parametrization from proposition 1 by another one given by physical mass and interaction constants. Let us illustrate it on the simpler example of the model with Lagrangian $\mathcal{L}(\phi(x)) = \sum_{k=1}^4 A_k\phi^k(x)$ and with dynamical measures described in (6). The first four truncated n-point functions are

$$S_1(x_1) = \beta_0 + \alpha_1\beta_1\,,$$
$$S_2^T(x_1, x_2) = (2\alpha_0 + \alpha_1\beta_1^{\,2})\delta(x_1 - x_2)\,,$$
$$S_3^T(x_1, x_2, x_3) = \alpha_1\beta_1^{\,3}\delta(x_1 - x_2)\delta(x_2 - x_3)\,,$$
$$S_4^T(x_1, x_2, x_3, x_4) = \alpha_1\beta_1^{\,4}\delta(x_1 - x_2)\delta(x_2 - x_3)\delta(x_3 - x_4)\,.$$

Thus there is a one-to-one correspondence,

$$g_1 = \beta_0 + \alpha_1\beta_1\,,$$
$$\frac{1}{M^2} = 2\alpha_0 + \alpha_1\beta_1^{\,2}\,,$$
$$g_3 = \alpha_1\beta_1^{\,3}\,,$$
$$g_4 = \alpha_1\beta_1^{\,4}$$

between parameters α_0, α_1, β_0, β_1 $(\alpha_0 \geqslant 0,\ \alpha_1 \geqslant 0)$ and "physical" parameters M, g_1, g_3, g_4 $[1/M \geqslant 0,\ g_4 \geqslant 0,\ g_3^{\,2} \leqslant (1/M^2)g_4]$. Direct computation shows that for fixed M and $g_1, g_3, g_4 \to 0$ the corresponding dynamical measure converges to the Gaussian measure (i.e., the free-field model).

In general the "physical" reparametrization will be more difficult to perform,[6] and therefore we will confine ourselves to the following remarks. *A priori* it is not clear which behavior of the parameters from proposition 1 corresponds to vanishing "physical" interaction constants with the mass kept fixed. However, as follows from the last part of the proof of proposition 1 the set of all dynamical measures is connected[7] (and contains Gaussian measures). This means that from every dynamical measure the Gaussian one can be approached by certain limits of parameters from proposition 1 (however, these parameters need not approach 0). The following proposition illustrates this explicitly for the class of dynamical measures given by

$$S(h)=\exp\left(\int_{R^d}\int_{R}\{\exp[iuh(x)]-1\}\times\exp(-\alpha_2u^2-\alpha_4u^4)\frac{du}{|u|}\,d^dx\right),$$

which are the measures studied by Klauder. (He denotes α_2, α_4 by m, λ and considers only the limit m fixed, $\lambda\to 0$.)

Proposition 2. Consider the class of dynamical measures given by generating functionals

$$S(h)=\exp\left(\int_{R^d}\int_{R}\{\cos[uh(x)]-1\}\times\exp(-\alpha_2u^2-\alpha_4u^4)\frac{du}{|u|}\,d^dx\right)$$

with $\alpha_2, \alpha_4\in R$, $\alpha_4>0$. Then there is a reparametrization of its subset by two real parameters M, c such that in the limit M fixed, $c\to 0$ the corresponding models converge to the free ones.

Proof. First introduce the intermediate reparametrization given by

$$-\alpha_2u^2-\alpha_4u^4=-\frac{b}{c^4}[(u^2-c^2)^2-c^4]\,.$$

The formula for "physical" mass[8]

$$\frac{1}{M^2}=\int_R|u|\exp\{-(b/c^4)[(u^2-c^2)^2-c^4]\}du=\frac{e^bc^2}{\sqrt{b}}\int_{-\sqrt{b}}^{\infty}e^{-s^2}ds \tag{7}$$

is an implicit equation $b=b(M,c)$. Let us solve it asymptotically for the case where b is large:

$$\frac{e^b}{\sqrt{b}}\sim\frac{1}{\sqrt{\pi}M^2c^2},$$

$$b-\tfrac{1}{2}\ln b\sim-\ln(\sqrt{\pi}M^2c^2)\,.$$

From the formula for asymptotic expansion of inverse functions[9] we have

$$b\sim-\ln(\sqrt{\pi}M^2c^2)+\tfrac{1}{2}\ln\ln\frac{1}{\sqrt{\pi}M^2c^2}\,.$$

Thus let us take the parametrization given by

$$\alpha_4=\frac{-\ln(\sqrt{\pi}M^2c^2)+\frac{1}{2}\ln\ln(\pi^{-1/2}M^{-2}c^{-2})}{c^4},\qquad \alpha_2=\frac{+2\ln(\sqrt{\pi}M^2c^2)-\ln\ln(\pi^{-1/2}M^{-2}c^{-2})}{c^2} \tag{8}$$

which is well defined for every positive M for small c. Now introducing (8) into (7) one readily verifies that

$$\int_R|u|\exp(-\alpha_2u^2-\alpha_4u^4)du\to\frac{1}{M^2}\text{ for }c\to 0\,.$$

Moreover, for each $\epsilon>0$ the following holds:

$$\int_{R\setminus(-\epsilon,\epsilon)}u^2\exp(-\alpha_2u^2-\alpha_4u^4)du\to 0\text{ for }c\to 0\,.$$

This implies the desired result.

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APPENDIX: PROOF OF PROPOSITION 1

Let us consider any gradient-free model with the measure formally described in (3). Suppose that for some coefficients $a_k(\Delta)$ the approximating measures $\mu_\Delta(\varphi, a_k(\Delta))$ converge[3] to the measure μ. Consider functions h of the type $h=\sum_{m=1}^{M}h_m\chi_{V_m}$ with $h_m\in R$ and χ_{V_m} characteristic functions of disjoint lattice boxes of volume V (each of them containing V/Δ boxes of lattice Δ). Then [since $\mathcal{L}_k(\phi(x))$ contain no derivatives]

$$S_\Delta(h)=\mathfrak{N}_\Delta\int_{W_\Delta}\exp(i\langle\varphi,h\rangle)\times\exp\left(-\sum a_k(\Delta)\int\mathcal{L}_k(\varphi(x))d^dx\right)d\varphi=\prod_m\left(\int\exp[ish_m-Q_\Delta(s)]ds\right)^{V/\Delta}, \tag{9}$$

where

$$Q_\Delta(s)=\Delta\sum a_k(\Delta)\mathcal{L}_k(s/\Delta)+\alpha(\Delta)\,, \tag{10}$$

with $\alpha(\Delta)$ chosen such that

$$\int_{-\infty}^{\infty}\exp[-Q_\Delta(s)]ds=1\,.$$

Let us denote $d\sigma_\Delta(s)=\exp[-Q_\Delta(s)]ds$. Using (9) particularly for $h=h_1\chi_1$ with χ_1 a characteristic function of the unit box $[0,1]^d$, we obtain $\sigma_\Delta*\sigma_\Delta*$

$\cdots * \sigma_\Delta \to \sigma$ (the convolution being taken $1/\Delta$ times). The theory of infinitely divisible random variables implies that the Fourier transform of the measure σ is given by

$$\hat{\sigma}(v) = \exp\left[i\beta v + \int_{-\infty}^{\infty} \psi(v,u) d\gamma(u)\right], \tag{11}$$

where $\beta \in \mathbb{R}$, $\psi(v,u)$ $(v, u \in \mathbb{R})$ is the continuous extension of

$$\left(e^{ivu} - 1 - \frac{ivu}{1+u^2}\right)\frac{1+u^2}{u^2}$$

and γ is a finite measure on $\mathbb{R}$.[10] Thus from (9) and (11)

$$S(h) = \exp\left[\int_{\mathbb{R}^d} \left(i\beta h(x) + \int_{-\infty}^{\infty} \psi(h(x),u)\, d\gamma(u)\right) d^d x\right] \tag{12}$$

(for h of the type considered and from continuity for all h). Moreover,

$$\beta = \lim_{\Delta\to 0} \frac{1}{\Delta} \int \frac{u}{1+u^2} d\sigma_\Delta(u) \tag{13}$$

and

$$d\gamma = \lim_{\Delta\to 0} \frac{1}{\Delta} \frac{u^2}{1+u^2} d\sigma_\Delta(u), \quad \text{weak limit (Ref. 11).} \tag{14}$$

This means that for gradient-free models we reduced the task of determining all generating functionals $S(h)$ to the following one-dimensional problem: Find all $\beta \in \mathbb{R}$ and measures γ on $\mathbb{R}$ such that for some coefficients $a_k(\Delta)$ the formulas (13) and (14) hold.

Turning back to the particular model from the proposition let us introduce some notation:

$$R_\Delta(u) = \sum_{k=1}^{2p} A_k(\Delta)\Delta^{1-k} u^k ,$$

$$\alpha'(\Delta) = \alpha(\Delta) - \Delta B(\Delta) \ln\Delta ,$$

$$P_\Delta(u) = Q_\Delta(u) + \ln\Delta$$
$$= R_\Delta(u) + \Delta B(\Delta)\ln|u| + \alpha'(\Delta) + \ln\Delta .$$

Similarly to Ref. 5 one can prove the following alternative: Either there exists a subsequence of $\{P_\Delta\}$ which is locally uniformly convergent (l.u.c.) on $\mathbb{R}\backslash\{0\}$ or there exists a finite subset $F \subset \mathbb{R}$ and a subsequence of P_Δ l.u.c. on $\mathbb{R}\backslash F$ to a function with values $\pm\infty$.

Suppose the first alternative holds. Then $\alpha'(\Delta) + \ln\Delta \to \alpha \in \mathbb{R}$ and the sequence $R_\Delta(u)$ is l.u.c. to $R(u)$ on $\mathbb{R}$. Since γ is finite,

$$\int_{-\epsilon}^{\epsilon} \exp\{-[R_\Delta(u) + \Delta B(\Delta)\ln|u| + \alpha'(\Delta)]\} du \to 1$$

for every $\epsilon > 0$. Thus

$$\int_{-\epsilon}^{\epsilon} \exp\{-[R_\Delta(u) + \alpha'(\Delta) + \ln\Delta]\} \frac{\Delta}{|u|^{\Delta B(\Delta)}} du \to 1 .$$

Because for $\epsilon \to 0$, $\Delta \to 0$ the exponential part of the integrand approaches $e^{-\alpha}$ we have

$$2e^{-\alpha} \int_0^{\epsilon} \frac{\Delta}{|u|^{\Delta B(\Delta)}} du \to 1 .$$

This means that $\Delta B(\Delta) = 1 - 2e^{-\alpha}\Delta + o(\Delta)$. The formula (14) then yields

$$d\gamma(u) = \frac{|u|}{1+u^2} \exp\{-[R(u) + \alpha]\} du .$$

The assertion (a) from the proposition now follows directly from (12) and (13).

When the second alternative holds, the only possible resulting measures γ are sums of finitely many δ functions. Their number and positions correspond to number and positions of local minima of P_Δ for small Δ. We will discuss it now in more detail. Without loss of generality we can suppose that $\Delta B(\Delta) \to B \in [-\infty, 1]$. Let us distinguish the following cases.

1. $B \in (-\infty, 1)$. Then the functions $R_\Delta(u) + \alpha'(\Delta) + \ln\Delta$ converge l.u. to $+\infty$ on $\mathbb{R}\backslash F$ and since they are polynomials of degree $\leq 2p$, they converge l.u. to $+\infty$ even on $\mathbb{R}\backslash\{\beta_1, \ldots, \beta_p\}$. Moreover, one of the β_k's equals 0 since the contrary would immediately imply that $\gamma(\mathbb{R}) = +\infty$. This means that the assertion (b) from the proposition holds.

2. $B = -\infty$. Denote $P_\Delta(u) = |\Delta B(\Delta)|[\tilde{R}_\Delta(u) - \ln|u|]$. If $\tilde{R}_\Delta$ converges l.u. on $\mathbb{R}$ a contradiction follows [$P_\Delta \to +\infty$ l.u. on $(-\epsilon, \epsilon)$ which gives $\gamma(\mathbb{R}) = +\infty$]. If $\tilde{R}_\Delta \to +\infty$ l.u. on $\mathbb{R}\backslash\{\beta_1, \ldots, \beta_p\}$, then, similarly as in case 1, one of the β_k's equals 0, which yields again the assertion (b).

3. $B = 1$. As in case 1 the functions $R_\Delta(u) + \alpha'(\Delta) + \ln\Delta$ converge l.u. to $+\infty$ on $\mathbb{R}\backslash\{\beta_1, \ldots, \beta_p\}$. When one of the β_k's equals 0 the assertion (b) holds. If not then direct computation shows that γ does not contain a δ function in 0; this together with the formulas (12) and (13) yields the assertion (c) of the proposition.

It remains to prove that every measure from the proposition can be obtained as a limit of some sequence $\mu_\Delta(\varphi, A_1(\Delta), \ldots, A_{2p}(\Delta), B(\Delta))$. The preceding analysis shows that this holds for measures from the assertion (a) of the theorem [it suffices to consider $A_k(\Delta) = \alpha_k$ for $k = 1, \ldots, 2p$ and $B(\Delta) = \Delta^{-1} - 2e^{-\alpha_0}\Delta^{-2}$]. The rest of the proof follows from the fact that every measure from (b) can be approximated by measures from (c) and these in turn can be approximated by those from (a).

[1]J. R. Klauder, Phys. Rev. D 14, 1952 (1976).

[2]All results given in the following text hold for every space-time dimension d.

[3]The convergence $\mu_\Delta \to \mu$ is understood in the following sense: μ is a cylindrical measure on $L_{1,\mathrm{loc}}(R^d)$ and

$$\int e^{i\langle\varphi,h\rangle} d\mu_\Delta(\varphi) \to \int e^{i\langle\phi,h\rangle} d\mu(\phi)$$

for every bounded measurable function h with compact support.

[4]The polynomial gradient-free models were studied (from the point of view of dynamical measures) by W. Kainz, Lett. Nuovo Cimento 12, 217 (1975), and H. G. Dosch, Nucl. Phys. B96, 525 (1975), under the name ultralocal static models. However, they do not carry out any renormalization and the lattice limit. It was also studied in the series of papers by J. R. Klauder, Acta Phys. Austriaca 41, 237 (1975) and the lecture in *Proceedings of the XIV Schladming Conference on Nuclear Physics*, edited by P. Urban (Springer, Berlin, 1975) [Acta Phys. Austriaca Suppl. 14 (1975)], p. 581.

[5]R. Kotecký and D. Preiss, Lett. Math. Phys. 2, 21 (1977).

[6]This could be very difficult even in a fixed lattice. See R. Schrader, Commun. Math. Phys. 49, 131 (1976); 50, 97 (1976).

[7]The set of measures from (a) is obviously connected, and the measures from (b) and (c) are in its closure (in weak topology).

[8]
$$\frac{1}{M^2}\delta(x_1 - x_2) = S_2^T(x_1, x_2) = \left.\frac{\delta^2(\ln S(h))}{\delta h(x_1)\delta h(x_2)}\right|_{h=0}.$$

[9]See N. Bourbaki, *Fonctions d'une variable rêele (thêorie êlementaire)* (Hermann, Paris, 1951), Chap. 5, Appendix, proposition 5.

[10]L. Breiman, *Probability* (Addison-Wesley, Reading, Mass., 1968), p. 194.

[11]Breiman, *Probability* (see Ref. 10), p. 198.