Confined solutions of the Thirring model coupled to a Schwinger field*

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In the study of the confined classical solutions of the boson form of the massive Thirring field coupled to a Schwinger field, it is observed that, regardless of their respective magnitudes and signs, the Thirring interaction is dominant over the other one, in determining whether such a solution exists. Confined solutions for the Thirring field are possible if and only if the Thirring coupling is attractive. Solutions are constructed for the cases in which the Thirring model coupling is attractive, repulsive, and equal to zero.

In field-theoretical models with more than one type of interaction, the relative importance of the different interactions may not depend only on the relative magnitudes of the coupling constants. The above statement is not unknown to physicists, but in classifying interactions as weak or strong, one is tempted only to look at the relative magnitudes of the coupling constants. At the back of our minds there is always the possibility of a tiny interaction that may radically change the overall behavior of the model; however, one often thinks that such a situation is highly improbable.

In this note we study the relative importance of the coupling constants in a commonly used model in two-dimensional space-time the Thirring model coupled to a photon field. We study only the classical solutions of the model, and try to see for which combination of the coupling constants we get confined solutions. We show that in order to obtain confined solutions for the Thirring field ψ, λ, the Thirring-model interaction coupling constant should be negative. This makes the effect of the self-interaction attractive. It is of minor importance to the problem what value the Schwinger-model coupling constant e takes.

We construct the classical solutions with $\lambda \neq 0$ first. We find that confined solutions exist for ψ and A^μ if $\lambda < 0$. We also look for the solutions of the model when $\lambda = 0$ and find that in this case only A^μ is confined; ψ is not confined. Quantum-mechanical effects may change this behavior as shown by Coleman[1] and Swieca.[2] For $\lambda > 0$, no normalizable solutions exist for the Thirring model. The interesting fact about our solutions is that the sign of λ also determines whether A^μ is confined or not.

In Sec. I, we obtain the confined solutions for ψ and for A^μ. In Sec. II, we construct the solutions for $\lambda = 0$ and find that ψ is of the traveling-wave type. In constructing our solutions we make use of a method recently employed by Lee *et al.*[3] Our convention for the γ matrices is $\gamma^0 = \sigma_1$, $\gamma^1 = i\sigma_2$, $\gamma^5 = \gamma^0\gamma^1$.

I. SOLUTIONS FOR ψ AND A^μ

We are looking for the classical confined solutions of the Schwinger model[4] coupled to the Thirring model.[5] The equations of motion are as follows[4]:

$$i\partial\!\!\!/\Psi - m\Psi - \tfrac{1}{2}\lambda(\bar{\Psi}\gamma_\mu\Psi)\gamma^\mu\Psi + e\gamma_\mu A^\mu\Psi = 0\,, \tag{1.1}$$

$$\partial_\mu F^{\mu\nu} = e\bar{\Psi}\gamma^\nu\Psi\,. \tag{1.2}$$

We want a stationary form of Ψ:

$$\Psi = e^{-iEt}\psi\,. \tag{1.3}$$

This imposes the condition that A^μ must be a static function or else part of Eq. (1.1) is independent of time whereas the other part is time dependent. So our equations reduce to

$$E\gamma^0\psi - i\gamma^1\partial^1\psi - m\psi - \tfrac{1}{2}\lambda(\bar{\psi}\gamma_\mu\psi)\gamma^\mu\psi - eA^\mu\gamma_\mu\psi = 0\,. \tag{1.4}$$

$$\partial_0 F^{10} = -e\bar{\psi}\gamma^1\psi = 0\,, \tag{1.5}$$

$$(\partial^1)^2 A^0 = -e\bar{\psi}\gamma^0\psi\,. \tag{1.6}$$

Note that in these equations A^1 does not appear with a derivative anywhere. So, we will use A^1 as a degree of freedom employed in obtaining our solutions. We write the energy-momentum tensor for this model and impose its conservation. For our solutions,

$$\partial_0 T^{00} = 0\,, \quad \partial_0 T^{01} = 0$$

trivially, so

$$\frac{d}{dx}T^{10} = 0 \tag{1.7}$$

gives, for our confined solutions,

$$i\bar{\psi}E\gamma^1\psi = 0 \tag{1.8}$$

which is satisfied if we take

$$\psi = R\begin{pmatrix} e^{i\theta_1} \\ e^{i\theta_2} \end{pmatrix}. \tag{1.9}$$

We take T^{11}, which is equal to zero, divide it into two parts and set each part equal to zero. Since we are looking for a solution to the problem, and we have A^1 as a free parameter, we can do this. We set

$$-\partial^1 A^0 \partial^1 A^0 + \partial^1 A^1 \partial^1 A^1 - e\bar{\psi}\gamma^0\psi A^0 = 0 \qquad (1.10)$$

and

$$E\bar{\psi}\gamma^0\psi - m\bar{\psi}\psi - \tfrac{1}{4}\lambda(\bar{\psi}\gamma^0\psi)^2 + \tfrac{1}{2}e\bar{\psi}\gamma^0\psi A^0 = 0\,. \qquad (1.11)$$

We choose A^1 such that Eq. (1.10) is satisfied.

We then use Eq. (1.4) to write Eq. (1.11) in the form

$$E\bar{\psi}\gamma^0\psi - m\bar{\psi}\psi + i\bar{\psi}\gamma^1 \frac{d}{dx}\psi = 0\,. \qquad (1.12)$$

This gives us an equation for $\theta_1 - \theta_2$ in Eq. (1.9), which is totally independent of the interaction,

$$2E - 2m\cos(\theta_1 - \theta_2) + \frac{d}{dx}(\theta_1 - \theta_2) = 0\,. \qquad (1.13)$$

We define

$$\theta = \theta_1 - \theta_2\,.$$

Equation (1.13) gives

$$\tfrac{1}{2}\theta(x) = \tan^{-1}(\alpha\tanh\beta x)\,, \qquad (1.14)$$

where

$$\alpha^2 = \frac{m-E}{m+E}\,, \qquad (1.15)$$

$$\beta^2 = m^2 - E^2\,. \qquad (1.16)$$

we have assumed that $m > E$.

We get R also from Eq. (1.11). It gives

$$R^2 = \frac{2}{\lambda}\left[\frac{E-m}{\cosh^2\beta x(1+\alpha^2\tanh^2\beta x)}\right] + \frac{e}{\lambda}A^0 \qquad (1.17)$$

which provides us with a differential equation for A^0,

$$\left(\partial_1{}^2 + \frac{2e^2}{\lambda}\right)A^0 = \frac{4e(m-E)}{\lambda(1+\alpha^2\tanh^2\beta x)\cosh^2\beta x}\,. \qquad (1.18)$$

The solution of this equation is

$$A^0 = \frac{4e(m-E)}{\lambda i a'}\theta(\pm x)e^{\pm i a' x} \times\left[\pm\int_{\mp\infty}^{x}\frac{dx'\,e^{\mp i a' x'}}{\cosh^2\beta x'(1+\alpha^2\tanh^2\beta x')}\right], \qquad (1.19)$$

where

$$a' = (e^2/2\lambda)^{1/2}\,, \quad x \neq 0\,.$$

We see that if and only if $\lambda < 0$ will this be a confined solution for A^0, and for ψ, as given in Eq. (1.17). Actually this statement is true for $|\lambda| \neq 0$. The case with $\lambda = 0$ will be treated separately.

Note that the sign of e has no effect on the confinement properties of the solution. We need $\lambda < 0$ irrespective of whether $e \lessgtr 0$ or $e/|\lambda| \lessgtr 1$. The relative magnitude of e with respect to λ does not make any difference. The magnitude of the Thirring-interaction coupling constant may be much smaller than the electromagnetic one, imitating weak interactions, or much larger, imitating the strong interactions. For all these cases λ has to be negative to get a bound state.

The integral in Eq. (1.19) can be evaluated in a closed form if $a/2\beta$ is an integer. Here

$$a = a'/i = (e^2/2|\lambda|)^{1/2}\,. \qquad (1.20)$$

This requirement sets a relation between m and E, which should not be taken too seriously:

$$\frac{a}{2\beta} = n \qquad (1.21)$$

gives

$$E^2 = m^2 - \frac{e^2}{2n^2|\lambda|}\,. \qquad (1.22)$$

To get an idea of the solution, we choose $a/2\beta = 1$ and obtain for R^2, one-half of the probability density of the Thirring field,

$$R^2 = \frac{2}{|\lambda|}\frac{(m-E)}{\cosh^2\beta x(1+\alpha^2\tanh^2\beta x)} - \frac{e\beta}{\sqrt{2}m|\lambda|^{3/2}}\theta(\pm x)e^{\mp ax}\left\{\ln\left(e^{\mp 4\beta x} + \frac{2E}{m}e^{\pm 2\beta x} + 1\right) - 2\frac{E}{\beta}\left[\tan^{-1}\left(e^{\pm 2\beta x}\frac{m}{\beta} + \frac{E}{\beta}\right) - \tan^{-1}\left(\frac{E}{\beta}\right)\right]\right\}. \qquad (1.23)$$

If we want to see how e affects these solutions, we look for the value of the integral

$$I = \int dx\,\psi^\dagger\psi\,,$$

which will give the extent of the solution in space, in one sense. We cannot set this value to 1 and get conditions on the energy as done in Ref. 3, since we have already assumed a value for F in choosing $a/2\beta$ equal to 1. We get

$$I = \frac{8}{|\lambda|}\tan^{-1}(\alpha) - \frac{e\sqrt{2}}{m|\lambda|^{3/2}}\left\{\ln\left(2+\frac{2E}{m}\right)\left(1+\frac{E}{m}-\frac{E}{\beta}\right) + \pi\left(1-\frac{E^2}{m^2}+\frac{E}{\beta}\right) + \frac{4E}{a}\tan^{-1}\left(\frac{E}{\beta}\right) + 2\tan^{-1}(\alpha)\left[-\left(1-\frac{E^2}{m^2}\right) - \frac{E}{\beta}\left(1+\frac{E}{m}\right)\right]\right\}. \tag{1.24}$$

Here the curly brackets give a positive contribution. So when λ increases, the extent of the wave function decreases, so it is more bound. Also, when e increases, I gets smaller. Note that in Eq. (1.1) we have defined e such that e means attraction. To be repulsive e must be negative. When e is negative, I increases. If we are bold enough to draw conclusions from the above expression obtained for a special value for $a/2\beta$, we see that there is a limit to the attractive values e can take without making I negative. e can take enough repulsive values though. As e gets to be more repulsive, there is more extent to the bound system. It becomes looser, but never quite reaches the extent of an unrenormalizable state.

II. SOLUTIONS FOR $\lambda = 0$

In this section we study the same model with $\lambda=0$, i.e., in the absence of the Thirring self-interaction:

$$T^{01} = 0 = i\bar{\psi}E\gamma^1\psi \tag{2.1}$$

gives

$$\psi = R\begin{pmatrix} e^{i\theta_1} \\ e^{i\theta_2}\end{pmatrix}. \tag{2.2}$$

Since this model is gauge invariant, and as seen above, only the difference $\theta_1 - \theta_2$ has a differential equation set for it, we will take

$$\psi = R\begin{pmatrix} e^{i\theta} \\ 1\end{pmatrix} \tag{2.3}$$

for simplicity.

Treating A^1 as done above, imposing the confined solution condition, we get

$$E\bar{\psi}\gamma^0\psi - m\bar{\psi}\psi + \tfrac{1}{2}e\bar{\psi}\gamma^0\psi A^0 = 0. \tag{2.4}$$

The field equation for ψ gives the same equation for θ with the solution

$$\tfrac{1}{2}\theta(x) = \tan^{-1}(\alpha\tanh\beta x). \tag{2.5}$$

Equation (2.4) also gives

$$2R^2E - 2m\cos\theta R^2 + eR^2 = 0 \tag{2.6}$$

which for $R \neq 0$ gives

$$A^0 = \frac{2}{e}\left(\frac{m-E}{\cosh^2\beta x(1+\alpha^2\tanh^2\beta x)}\right), \tag{2.7}$$

a confined solution.

In order to get an expression for R, we study the field equation for ψ,

$$i\partial\!\!\!/\psi - m\psi + eA^0\gamma^0\psi = 0. \tag{2.8}$$

For ψ given as in Eq. (2.3), we get

$$-mRe^{i\theta} + ER + i\frac{dR}{dx} + eA^0R = 0. \tag{2.9}$$

The solution for this equation is of the form

$$R = e^{-i(-Ex+I_1+I_2)}, \tag{2.10}$$

where

$$I_1 = \int -eA^0dx = \frac{-2(m-E)}{\alpha\beta}\tan^{-1}(\alpha\tanh\beta x), \tag{2.11}$$

$$I_2 = \int me^{i\theta}dx = \frac{m}{2i}\frac{\alpha}{\beta}\left\{\left[\frac{2}{(1+\alpha^2)}\ln\left(z^2 - \frac{2E}{m}z + 1\right)\right] - \frac{4(\alpha^2-1)}{(1+\alpha^2)^2}\frac{m}{\beta}\tan^{-1}\left(\frac{zm-E}{\beta}\right)\right\}, \tag{2.12}$$

and

$$z = \exp[i2\tan^{-1}(\alpha\tanh\beta x)]. \tag{2.13}$$

As $x \to \pm\infty$, $z \to \exp[2i\tan^{-1}(\pm\alpha)]$, so both I_2 and I_1 go to a constant factor. R, as given by Eq. (2.10), is clearly not a confined solution. So for $\lambda=0$, we could not find confined solutions for ψ.

As a final remark we want to note that for $\lambda > 0$ the Thirring model, without the photon interaction, does not have confined solutions. A calculation similar to those in Ref. 3 gives, for $e=0$ and $E > m$,

$$R^2 = \frac{2(E-m)}{\lambda\cos^2\beta' x(1+\alpha'^2\tanh^2\beta' x)}, \tag{2.14}$$

where

$$\alpha'^2 = (E-m)/(E+m), \tag{2.15}$$

$$\beta'^2 = E^2 - m^2. \tag{2.16}$$

This solution is not normalizable.

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