Analytic contributions to the g factor of the electron

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(Received 10 November 1975)

We compute the analytic value of seven more graphs contributing to the g factor of the electron in sixth order. After comparing with other numerical and analytic evaluations, we give an updated "best" theoretical estimate of the g factor, and compare it to the experimental value.

I. INTRODUCTION

Continuing the approach developed in a previous publication,[1] we have computed the analytic values of the contributions of seven more graphs to the g factor of the electron in sixth order. The graphs, grouped into Sets D and C, are shown in Figs. 1 and 2. Our results[2] are displayed in Table I. The analytic results for graphs 22 and 25 (and also for graph 12) were obtained independently by Barbieri, Caffo, and Remiddi.[3]

This paper is organized as follows. In Sec. II we outline the general techniques for graphs of Set D. Some aspects of the computer program and some typical running parameters are given in Sec. III. New aspects encountered in the graphs of Set C are indicated in Sec. IV. A comparison of our results with those of Ref. 3 is discussed in Sec. V. Section VI contains a comparison of our results with various numerical evaluations, and we give the best theoretical values for $(g_e - 2)$, combining our results with numerical evaluations of graphs which are still not known analytically.

II. OUTLINE OF TECHNIQUE

As indicated in I, given a Feynman graph, we write the Feynman integrand in momentum space, multiply by a projection operator, take the trace which extracts the magnetic moment, analytically continue to the Euclidean region, introduce four-dimensional spherical coordinates for each loop momentum, and perform the angular integrations. This is straightforward using the techniques of I. As an example of the approach for graphs of group D, consider the graph in Fig. 3 with the lines labeled as shown. After the angular integrations are done, the remaining integral is (setting the electron mass to 1)

$$\int K^2 dK^2 \int L^2 dL^2 \int M^2 dM^2 \frac{1}{(1+K^2)^2}\frac{1}{(1+M^2)^2}\frac{Z_{KM}}{KM}\frac{Z_{KP}}{KP}\frac{Z_{PL}}{PL}\frac{Z_{KL}}{KL}\frac{1}{L^4} F(K^2, L^2, M^2, Z_{KM}, Z_{KP}, Z_{KL}, Z_{PL}),$$

where

$$P^2 = -1,$$

$$Z_{KL} = \{K^2 + L^2 + 1 - [(K^2 + L^2 + 1)^2 - 4K^2L^2]^{1/2}\}/(2KL),$$

$$Z_{PL} = [L^2 - (L^4 + 4L)^{1/2}]/(2PL).$$

The L^2 integral is from 0 to ∞, while the K^2, M^2 integrations and the values of Z_{KP}, Z_{KM} can be divided into the following regions:

region I: $Z_{PK} = P/K$, $Z_{KM} = K/M$, $0 < K^2 < M^2 < \infty$;

region II: $Z_{PK} = P/K$, $Z_{KM} = M/K$, $0 < M^2 < K^2 < \infty$;

region III: $Z_{KM} = K/M$, $-1 < K^2 < 0$, $K^2 < M^2 < \infty$;

region IV: $Z_{KM} = M/K$, $-1 < K^2 < 0$, $K^2 < M^2 < 0$.

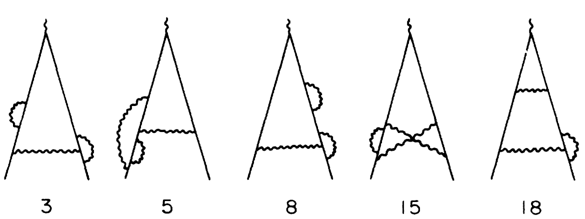

FIG. 1. Graphs of Set D.

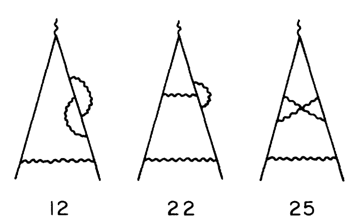

FIG. 2. Graphs of Set C.

TABLE I. Analytic value of the coefficient of $(\alpha/\pi)^3$ in $\frac{1}{2}(g-2)$ from the sum of the graph and its mirror image if they are distinct. Infrared parts proportional to $\log\lambda$ and $\log^2\lambda$ are omitted. They can be found in Ref. 5. (All results are for the Feynman gauge.)

Graph No.	Analytic Value
3	$-\frac{349}{288}-\frac{239}{144}\pi^2+\frac{127}{1080}\pi^4-\frac{33}{8}\zeta(3)-\frac{2}{9}\ln^4 2-\frac{16}{3}a_4+\frac{38}{9}\pi^2\ln 2-\frac{16}{9}\pi^2\ln^2 2$
5	$-\frac{47}{48}+\frac{3}{8}\pi^2-\frac{11}{432}\pi^4-\frac{23}{12}\zeta(3)-\frac{1}{18}\ln^4 2-\frac{4}{3}a_4+\frac{5}{6}\pi^2\ln 2-\frac{4}{9}\pi^2\ln^2 2$
8	$-\frac{763}{288}+\frac{19}{48}\pi^2-\frac{1}{90}\pi^4+\frac{5}{12}\zeta(3)-\frac{1}{9}\pi^2\ln 2$
15	$-\frac{415}{144}+\frac{74}{27}\pi^2-\frac{223}{2160}\pi^4+\frac{89}{12}\zeta(3)+\frac{5}{18}\ln^4 2+\frac{20}{3}a_4-6\pi^2\ln 2+\frac{20}{9}\pi^2\ln^2 2$
18	$-4+\frac{463}{216}\pi^2-\frac{11}{80}\pi^4+\frac{25}{8}\zeta(3)+\frac{1}{6}\ln^4 2+4a_4-\frac{53}{18}\pi^2\ln 2+\frac{4}{3}\pi^2\ln^2 2$
3+5+8+15+18	$-\frac{211}{18}+\frac{863}{216}\pi^2-\frac{23}{144}\pi^4+\frac{59}{12}\zeta(3)+\frac{1}{6}\ln^4 2+4a_4-4\pi^2\ln 2+\frac{4}{3}\pi^2\ln^2 2$
22 [a]	$\frac{235}{288}-\frac{1429}{864}\pi^2-\frac{5}{216}\pi^4+\frac{11}{2}\zeta(3)+\frac{5}{9}\ln^4 2+\frac{40}{3}a_4+\frac{4}{3}\pi^2\ln 2-\frac{19}{18}\pi^2\ln^2 2$
25 [a]	$-\frac{173}{288}+\frac{287}{96}\pi^2-\frac{37}{864}\pi^4-\frac{17}{3}\zeta(3)-\frac{7}{12}\ln^4 2-14a_4-3\pi^2\ln 2+\frac{11}{6}\pi^2\ln^2 2$

$$a_4=\sum\frac{1}{2^n n^4}=0.517\,479\ldots$$

$$\zeta(3)=\sum\frac{1}{n^3}=1.202\,0569\ldots$$

[a] Reference 2.

In each of regions III and IV, the integrands are evaluated with $Z_{PK}=P/K$, then with $Z_{PK}=K/P$, and one takes the difference between the two values.

F is a polynomial in its variables, except that some terms also have a factor $\ln(1-Z_{KL}Z_{PK}Z_{PL})$ arising from the angular integrals. The M^2 integral is essentially trivial, since the integrand is a rational function of M^2. Performing this integration leaves a two-dimension integral, whose integrand is a polynomial in K^2, L^2, Z_{KP}, Z_{KL}, Z_{PL} divided by powers of $(1+K^2)$, K^2, L^2, some terms of which also have a factor $\ln(1-Z_{KL}Z_{PK}Z_{PL})$ and some also have a factor $\ln(1+K^2)$, from performing the M^2 integration. The regions of integration are now

region A: $Z_{PK}=K/P$, $-1<K^2<0$, $0<L^2<\infty$;

region B: $Z_{PK}=P/K$, $-1<K^2<\infty$, $0<L^2<\infty$.

At this point, we looked for a variable change which would eliminate the square roots in the Z's, leaving a rational function as integrand, apart from the two logarithmic factors. The variables chosen were

$$w=-Z_{PK}Z_{KL}Z_{PL}\,,$$

$$u=\tfrac{1}{2}[-L^2+(L^4+4L)^{1/2}].$$

Since Z_{PK} is a different function of K and P in regions A and B, the expressions for K^2 and L^2 in terms of u,w will differ in these regions.

With this choice of variables it is easily verified

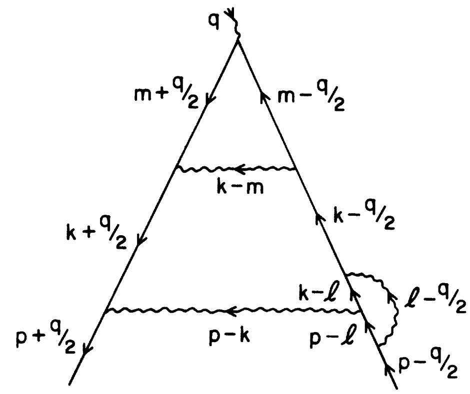

FIG. 3. A typical graph of Set D with momenta labeled.

that regions A and B are mapped into the same u, w region $0<w<1-u$, $0<u<1$, that all denominators can be factored into linear functions of w, and that the arguments of the logarithms are also linear functions of w. Consequently, the w integrals can be evaluated with the observation that

$$\int dw(c+dw)^{-1}\ln(a+bw)$$

involves, at worst, Spence functions (dilogarithms) while

$$\int dw(e+fw)^{-1}\ln(a+bw)\ln(c+dw)$$

involves, at worst, trilogarithms.[4]

After evaluating the w integrals, we are left with a one-dimensional u integral, whose integrand contains rational functions, logarithms, Spence functions, and trilogarithms. The only denominator factors which appear are u, $(1-u)$, $(2-u)$, $(1-u+u^2)$. The final integration is then performed, and the results are shown in Table I.

The discussion so far has avoided the problem of ultraviolet or infrared divergences. The ultraviolet divergences are canceled by the appropriate renormalization counterterms. To assure pointwise cancellation of these divergences the change of variable from K, L space to u, w space must also be done on the counterterm. It is only upon such a pointwise convergent function that one can legitimately perform that required change of variables.

As usual, the infrared divergences were trickier. We found it easiest to introduce a cutoff in the K^2 integral, making the lower limit $-1+\epsilon$ instead of -1. We then followed the technique of Levine and Wright[5] in choosing counterterms which rendered the graph infrared finite. Then we let ϵ tend to 0. The extra infrared pieces present in the Levine and Wright counterterms were computed analytically by hand.

III. THE COMPUTING PROGRAM

We did all our calculations by computer, using the symbolic algebra program[6] ASHMEDAI. Performing the traces was essentially trivial, since ASHMEDAI has built-in trace algorithms. The angular and M^2 integrations were also quite simple. They were done by explicit substitution involving about a dozen basic forms, easily derivable by hand. In order to proceed with the integration over the variable w, we had to perform a partial fraction decomposition of the rational functions of w. Although straightforward to program, this was the most time-consuming aspect of the calculation.

The integrand was then fed into an analytic integrator, also written in ASHMEDAI, whose basic technique is integration by parts. Because the integrand was in partial-fraction form, the dependence of each term entering the integrator was of the form RAT(w)*TRANS(w) where

$$\mathrm{RAT}(w)=\{w^p, p\geq 0\} \quad\text{or}\quad \{(aw+b)^{-n}, n\geq 1\},$$

$$\mathrm{TRANS}(w)=\{1\} \quad\text{or}\quad \{\log(cw+d)\} \quad\text{or}$$

$$\{\log(cw+d)\log(ew+f)\}.$$

First, the power of the rational factor RAT(w) was examined. If the power was not -1, an integration by parts was done and the result was again "partial fractioned." This reduced the number of transcendentals in TRANS by 1, and the process was repeated. If the power of the rational was -1, the integral was found in a table. Thus the only integrals we had to put in by hand were ones with linear denominators. We required 32 such integrals. They were either trivial or could be found in Lewin's book.[4,7]

After this integration, we had an integrand which was a function of u, to be integrated from 0 to 1. Again the same procedure was used, except that the possible set of transcendentals was larger, including logarithms, Spence functions, and trilogarithms. (If we assign degree 1 to logarithms, 2 to Spence functions, and 3 to trilogarithms, the degree of the worst transcendentals was 3.) In this case we had to put in 184 integrals by hand. Most were taken from the tables of Barbieri, Mignaco, and Remiddi,[8] and some we derived ourselves.

As a typical indication of running characteristics, we consider graph 18. The trace, angular integrals, and M^2 integrals required 2 min, the partial fractions 18 min, and the two integration passes 15 min, for a total of 35 min on a Univac 1108. The maximum number of terms at any stage is about 2400, and the number entering the u integrator is 437.

IV. GRAPHS OF SET C

If we label the momenta of graph 25 as shown in Fig. 4, we can easily perform the traces and the angular integrations. The remaining integral is then

$$\int K^2dK^2\int L^2dL^2\int M^2dM^2\frac{1}{(K^2+1)^2}\frac{1}{(L^2+1)^2}\frac{Z_{LM}{}^2}{L^2M^2(1-Z_{LM}{}^2)}$$

$$\times\frac{Z_{KM}{}^2}{K^2M^2(1-Z_{KM}{}^2)}\frac{Z_{KL}}{KL}\frac{Z_{KP}}{PK}F(K^2,L^2,M^2,Z_{KM},Z_{LM},Z_{KL},Z_{KP}),$$

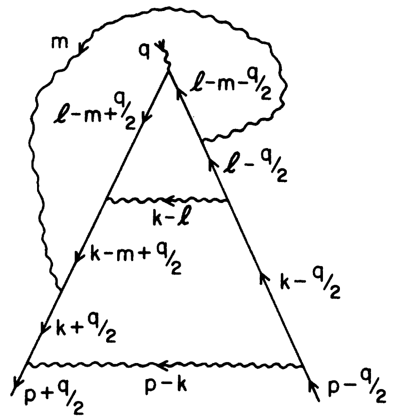

FIG. 4. A typical graph of Set C with momenta labeled.

where

$$Z_{LM}=\{L^2+M^2+1-[(L^2+M^2+1)^2-4L^2M^2]^{1/2}\}/(2LM),$$

$$Z_{KM}=\{K^2+M^2+1-[(K^2+M^2+1)^2-4K^2M^2]^{1/2}\}/(2KM).$$

The M^2 integration is from 0 to ∞, while the K^2, L^2 integrations and the values of Z_{KL}, Z_{KP} can be divided into four regions:

region 1: $Z_{KL}=K/L,\quad Z_{KP}=P/K,\quad 0<K^2<L^2<\infty$;

region 2: $Z_{KL}=L/K,\quad Z_{KP}=P/K,\quad 0<L^2<K^2<\infty$;

region 3: $Z_{KL}=K/L,\quad -1<K^2<0, K^2<L^2<\infty$;

region 4: $Z_{KL}=L/K,\quad -1<K^2<0, K^2<L^2<0$.

In each of regions 3 and 4, the integrands are evaluated at $Z_{KP}=P/K$, then at $Z_{KP}=K/P$, and the two values are subtracted. F is a polynomial in its variables, except that some terms have a factor $\ln(1-Z_{KL}Z_{LM}Z_{KM})$ which arises from doing the angular integrations

The main difference between this integrand and those of Set D are as follows:

(1) There is no trivial integral in the sense that the M^2 integral was trivial in Set D.

(2) Both square roots involve two integration variables. In Set D, one of the square roots depended only on L^2.

We found that it was still possible to eliminate all square roots from the integrand by defining the variables

$$u=LZ_{LM}/M,$$
$$v=KZ_{KM}/M,$$
$$w=-Z_{KL}Z_{KM}Z_{LM}.$$

The integrand now consisted of rational functions of u, v, w with some terms also multiplied by $\ln(1+w)$.

For definiteness, we concentrate on region 1, although the results for other regions are similar. In this region, it is easily verified that all denominators in the integrand can be written as a product of linear functions of u, without introducing any square roots. Thus, after performing partial fractions in u, the u integration is trivial.

The problem was that the denominators in the remaining v, w integrals involved quadratic functions of v and w, which could not be factored without introducing square roots. In general, for a given term, we found that we could either evaluate the v integral or the w integral straightforwardly or else we found another variable change involving both v and w which enabled us to evaluate one integral easily. Thus we reduced the integrand to a one-dimensional form, with limits 0 and 1, and put it through our one-dimensional integrator. The set of integrals is larger than those of graphs D, but we were able to evaluate all of them except for one combination which will be discussed further in the next section.

It also turns out that the extraction of the infrared divergences was much harder than in Set D. Here, letting the lower limit be $K^2=-1+\epsilon$ affected all three integrals in the variables u, v, w rather than only one integral as in Set D. This greatly complicated the work of regions 3 and 4.

These complications greatly increased the algebraic complexity of the program. Instead of a typical running time of 35 min for a graph of Set D, graphs of Set C required about 150 min each. The maximum number of terms at any stage increased from about 2400 to 24000, while the final integration had about 1400 terms as input.

In this way, we computed the values of graphs (22) and (25).

V. COMPARISON WITH REF. 3

We brought the calculation of graphs 22 and 25 to the stage where all integrals have been evaluated analytically, except for the combination

$$\int_0^1 dx\,\ln(1-x+x^2)\left(\frac{S(-x)}{x}-\frac{2}{3}\frac{S(x)}{x}+\frac{2S(-x)}{1-x}+\frac{4}{3}\frac{\ln x\ln(1-x)}{x}+\frac{2\ln(1-x)\ln(1+x)}{x}\right) \qquad (5.1)$$

TABLE II. Comparison of the exact values of graphs with different numerical evaluations. The numbers are quoted for the sum of the graph and its mirror image if they are distinct.

Graph	Analytic Value	Levine and Wright (Ref. 5)	Cvitanović and Kinoshita (Ref. 10)	Carroll and Yao (Ref. 11)
3	6.546 895	6.541(13)	6.5413(104)	
5	0.828 277	0.832(13)	0.8280(61)	
8	−0.084 174	−0.083(6)	−0.0805(36)	
15	−3.968 285	−3.951(40)	−3.9609(96)	
18	−4.193 950	−4.201(13)	−4.1857(87)	
3+5+8+15+18	−0.871 237	−0.862(46)	−0.8578(181)	−0.875(23)
12	−0.122 981 [a]	−0.123(20)		
22	−0.007 146	−0.005(20)	−0.009(5) [b]	
25	−1.286 977	−1.286(13)	−1.286(6) [b]	
12+22+25	−1.417 104	−1.414(31)		−1.421(20)

[a] Reference 3. [b] Reference 13.

which occurs in both graphs. $S(x)$ is defined by

$$S(x) = -\int_0^x \frac{\ln(1-t)}{t}\,dt. \quad (5.2)$$

Equation (5.1) was easily evaluated numerically to 15 decimal places.

At this point, we learned of the results of Barbieri *et al.*[3] They computed analytically the sum of graphs

$$(25)+(22)+\tfrac{1}{2}(12)$$

and separately evaluated graph (12), so that they obtained a value for the contribution of graphs of Set C to g_e:

$$\tfrac{1}{2}(g_e-2)_c = (25)+(22)+(12).$$

We could use their values to compute (25)+(22), and compare this with ours. The numerical results agreed completely, and assuming this was not fortuitous we were able to give an analytic value to our undone integral Eq. (5.1).[9] Later, Barbieri *et al.*[3] computed the analytic values of (22) and (25) separately, and our results agreed completely. In view of the history of errors in higher-order quantum electrodynamics (QED) calculations, this agreement was very reassuring. In view of this independent confirmation, we did not feel it was worthwhile to compute (12) separately, although we could have done this fairly straightforwardly.

VI. COMPARISON WITH NUMERICAL CALCULATIONS AND EXPERIMENT

A comparison of our results with the previous numerical evaluations is shown in Table II.

Let us define a_6 by

$$\tfrac{1}{2}(g_e-2) = \frac{\alpha}{2\pi} - 0.328\,478\ldots\left(\frac{\alpha}{\pi}\right)^2 + a_6\left(\frac{\alpha}{\pi}\right)^3.$$

Then, combining the now-known analytic values[12] with the numerical values for graphs not yet known analytically, the best estimate for the contribution of the sum of all 28 graphs without fermion loops to a_6 is

$S_{28} = 0.933 \pm 0.048$, Levine and Wright (Ref. 5),

$S_{28} = 0.915 \pm 0.015$, Cvitanović and Kinoshita (Ref.13),

$S_{28} = 0.797 \pm 0.038$, Carroll and Yao (Ref. 11).

Combining these with the known analytic value for the graphs with vacuum polarization insertions[14]

$$S_{VP} = -0.09474$$

and the numerical values for the graphs with light-by-light scattering[15]

$$S_{LL} = 0.368 \pm 0.008,$$

we find that the theoretical estimates for a_6 are

$a_6 = 1.206 \pm 0.049$, Levine and Wright,

$a_6 = 1.188 \pm 0.017$, Cvitanović and Kinoshita,

$a_6 = 1.070 \pm 0.039$, Carroll and Yao.

The best experimental value of a_6, obtained by using Refs. 16 and 17,

$$\tfrac{1}{2}(g_e-2)_{\text{expt}} = (1159\,656.7 \pm 3.5)\times 10^{-9},$$

$$\alpha_{\text{expt}}^{-1} = 137.035\,987 \pm 0.000\,028$$

gives

$$(a_6)_{\text{expt}} = 1.53 \pm 0.33.$$

Of the estimated error, 0.28 is due to the uncertainty in g_e, 0.05 to the error in α. Higher-order corrections, hadronic effects, muon loops, or weak-interaction contributions are not expected to change a_6 by more than 0.01. A reduction in the experimental error in g_e by a factor of 10 would give an error in a_6 of 0.08 and provide a very stringent test of quantum electrodynamics, or else an alternative precise determination of the fine-structure constant α.

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*Work supported in part by the U. S. Atomic Energy Research and Development Administration under Contract No. E(11-1)-3066.

†Deceased.

‡Alfred P. Sloan Fellow. Work supported in part by the National Science Foundation under Contract No. MPS 75-14073.

[1]M. J. Levine and R. Roskies, Phys. Rev. D 9, 421 (1974), hereinafter referred to as I.

[2]Our results for graphs 22 and 25 depend on an integral identity which we did not prove. See the discussion in Sec. V.

[3]R. Barbieri, M. Caffo, and E. Remiddi, Phys. Lett. 57B, 460 (1975).

[4]For properties of dilogarithms or trilogarithms, see L. Lewin, *Dilogarithms and Associated Functions* (MacDonald, London, 1958); K. S. Kölbig, J. A. Mignaco, and E. Remiddi, BIT 10, 38 (1970); R. C. Perisho, U.S. AEC Report No. COO-3066-45, 1975 (unpublished).

[5]M. J. Levine and J. Wright, Phys. Rev. D 8, 3171 (1973).

[6]M. J. Levine, U.S. AEC Report No. CAR-882-25, 1971 (unpublished). See also M. J. Levine, in Proceedings of the Third International Colloquium in Advanced Computing Methods in Theoretical Physics, edited by A. Visconti, Marseille, 1973 (unpublished), and R. C. Perisho, ASHMEDAI User's Guide, U.S. AEC Report No. COO-3066-44, 1975 (unpublished).

[7]More details on the integrator can be found in M. J. Levine and R. C. Perisho, U.S. AEC Report No. COO-3066-37, 1974 (unpublished). As usual, all basic integrals are verified numerically before inclusion in our program. Some of the forms given in Lewin were put into more suitable form using the algebraic program REDUCE. [See, e.g., A. C. Hearn, in Proceedings of the Third International Colloquium on Advanced Computing Methods in Theoretical Physics, Marseille, 1973, edited by A. Visconti (unpublished).]

[8]R. Barbieri, J. A. Mignaco, and E. Remiddi, Nuovo Cimento 11, 824 (1972), Appendix B.

[9]Our conjecture is that the analytic expression for (5.1) is

$$-\frac{35}{18}\zeta(3)\ln 2+\frac{5}{54}\pi^2\ln^2 2-\frac{20}{9}a_4-\frac{5}{54}\ln^4 2+\frac{391}{11664}\pi^4.$$

[10]P. Cvitanović and T. Kinoshita, Phys. Rev. D 10, 4007 (1974). Our results are taken from their Table V, sometimes by combining the analytic value of known graphs with their estimate for a sum of graphs.

[11]R. Carroll and Y. P. Yao, Phys. Lett. 48B, 125 (1974); R. Carroll, Phys. Rev. D 12, 2344 (1975).

[12]We use the results of Ref. 1, Ref. 3, and this paper.

[13]P. Cvitanović (private communication).

[14]J. A. Mignaco and E. Remiddi, Nuovo Cimento 60A, 519 (1969); R. Barbieri and E. Remiddi, Phys. Lett. 49B, 468 (1974).

[15]J. Calmet and A. Peterman, Phys. Lett. 47B, 369 (1973); C. Chang and M. J. Levine (unpublished).

[16]J. C. Wesley and A. Rich, Phys. Rev. A 4, 1341 (1971).

[17]P. T. Olsen and E. R. Williams, presented at the AMCO-5 Conference, Paris, June, 1975 (unpublished).