# Approximation to Feynman amplitudes valid for small masses*

Richard Talman
*Laboratory of Nuclear Studies, Cornell University, Ithaca, New York 14853*
(Received 7 July 1975; revised manuscript received 13 October 1975)

A simple procedure is given for exploiting the smallness of the electron mass in approximating Feynman amplitudes. Using helicity conservation, the Dirac matrices are factored out leaving amplitudes no more complicated than those for spinless particles. For the case of bremsstrahlung the helicity structure is exhibited and the validity of helicity conservation is studied. To exhibit the simplicity and power of the method an amplitude for trident production is written down by inspection. It is shown to agree with the value obtained by Bjorken and Chen using projection operators and traces.

## I. INTRODUCTION

In analyzing electron processes at high energies the radiation formulas are so complicated that it is difficult to avoid blunders in their use. The purpose of this paper is to develop simpler formulas by exploiting the small value of the electron mass, $m$, relative to the particle energies involved. In this approximation the electron helicity is conserved and as a result there are only two amplitudes, one for each helicity. This simplicity is not explicitly apparent in the conventional method using Dirac matrices, where, at least formally, 16 amplitudes appear. In the method described here, no Dirac matrices appear and Feynman amplitudes are written directly as products of scalar factors, one for each vertex in the diagram.

As well as describing the method we apply it to the process of electron-nucleus bremsstrahlung. For this process it is possible to evaluate the complete cross section, thus obtaining an estimate of the accuracy of the approximation. The formulas are directly applicable as radiative corrections to inelastic electron scattering and were, in fact, derived for that purpose. To minimize confusion this aspect will be discussed only in detail sufficient to investigate the method.[1]

After defining terminology in Sec. II, the rule for obtaining Feynman amplitudes is derived in Sec. III, which contains the main result as formula (17). In the following section the relativistic invariance of the method is discussed. This is somewhat superfluous as the invariance is clear from the derivation. It is given because of its connection with certain phase factors which have different values for different diagrams and hence are a nuisance for the method. In Sec. V examples are considered, including elastic scattering (Mott) and radiative elastic scattering or bremsstrahlung (Bethe-Heitler). In this context the approximation can be viewed as a generalization of the soft-photon emission formula. The Bethe-Heitler formula is derived and its helicity makeup identified. Consideration of the accuracy of the assumption of helicity conservation is more meaningful when applied to cross sections integrated over regions of photon directions containing the three peaks which dominate the cross section. These integrals are evaluated in Sec. VI for both helicity-conserving and helicity-flip cross sections. The error inherent in the approximation is indicated.

The usefulness of helicity conservation was emphasized by Bjorken and Chen[2] in a calculation of trident production. In Sec. VII an amplitude for trident production is written down by inspection. It is shown to agree with the value obtained by Bjorken and Chen, who used conventional projection operators and traces. This illustrates the simplicity and power of the method. In the final section we mention cursorily the relation between our work and previous work on radiative corrections.

## II. TERMINOLOGY

By and large the conventions are those of Berestetskii, Lifshitz, and Pitaevskii.[3] Particle momenta appear in the form $p$ which can mean either the 4-momentum or the magnitude of the spatial part. The latter is normally assumed to be equal to the energy. $\vec{p}$ and $\hat{p}$ are the spatial momentum vector and a unit vector along $\vec{p}$, respectively. Scalar products are indicated by $p\cdot q$ or $p^\mu q_\mu$. The normal spatial components have upper indices. Lowered indices correspond to reversal of the signs of the spatial components. The particles will be assumed to be electrons, though the formulas apply also to positions with the only change being that left-handed electrons couple only to right-handed positrons (in our approximation).

*Unit vectors.* The symbols refer to the Feynman vertex shown in Fig. 1 with $\vec{p}$ and $\vec{p}\,'$ referring

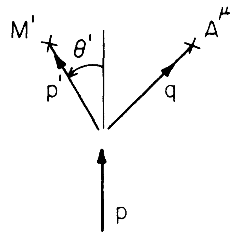


FIG. 1. A Feynman vertex in which an electron changes from 4-momentum $p$ to $p'$ by interacting with a field $A^\mu$.

to electron momenta before and after the vertex:

$$\hat{n}_1 = \frac{\hat{p}-\hat{p}'}{|\hat{p}-\hat{p}'|} = \frac{1}{2\sin\frac{1}{2}\theta'}(\hat{p}-\hat{p}'), \tag{1}$$

$$\hat{n}_2 = \frac{\hat{p}+\hat{p}'}{|\hat{p}+\hat{p}'|} = \frac{1}{2\cos\frac{1}{2}\theta'}(\hat{p}+\hat{p}'), \tag{2}$$

$$\hat{n}_\perp = \frac{\hat{p}'\times\hat{p}}{|\hat{p}'\times\hat{p}|} = \frac{1}{\sin\theta'}(\hat{p}'\times\hat{p}). \tag{3}$$

Components of a vector $\vec{A}$ are defined by

$$\vec{A} = A^1\hat{n}_1 + A^2\hat{n}_2 + A^\perp\hat{n}_\perp. \tag{4}$$

*Dirac matrices.* We use the following representation:

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}, \tag{5}$$

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6}$$

*Free spinors.* These spinors are

$$u_{\vec{p}} = \begin{pmatrix} (E+m)^{1/2} w \\ (E-m)^{1/2}\hat{p}\cdot\vec{\sigma}w \end{pmatrix} \simeq \sqrt{p}\left(1+\gamma^0\frac{m}{2p}\right)\begin{pmatrix} w \\ \hat{p}\cdot\vec{\sigma}w \end{pmatrix}, \tag{7}$$

where $E^2 = p^2 + m^2$. Also,

$$\bar{u}_{\vec{p}'} = ((E'+m)^{1/2}w'^*, \; -(E'-m)^{1/2}w'^*\hat{p}'\cdot\vec{\sigma})$$
$$\simeq \sqrt{p'}\,(w'^*, \; -w'^*\hat{p}'\cdot\vec{\sigma})\left(1+\frac{m}{2p'}\gamma^0\right). \tag{8}$$

*Rotation operators.* $R$ is a rotation operator which transforms two component spinors appropriately to that rotation around $\hat{n}_\perp$ which rotates $\hat{p}$ into $\hat{p}'$:

$$R = \cos\tfrac{1}{2}\theta' + i\sin\tfrac{1}{2}\theta'\,\hat{n}_\perp\cdot\vec{\sigma} \tag{9}$$

$$= \frac{1}{2\cos\frac{1}{2}\theta'}(1+\hat{p}'\cdot\vec{\sigma}\,\hat{p}\cdot\vec{\sigma}). \tag{10}$$

*Helicity states.* The electron helicity $\lambda$ can have the values $\pm\frac{1}{2}$. $\lambda$ always appears multiplied by 2. Where a factor $2\lambda$ appears it can be replaced by $\pm$ to make the formula look simpler. Two-component spinors belonging to states of definite helicity are denoted by $w_\lambda$ which satisfy

$$\hat{p}\cdot\vec{\sigma}w_\lambda = 2\lambda w_\lambda, \quad \lambda = \pm\tfrac{1}{2}$$
$$\hat{p}'\cdot\vec{\sigma}w'_\lambda = 2\lambda w'_\lambda, \tag{11}$$

$$Rw_\lambda = w'_\lambda e^{i\phi}, \tag{12}$$

where $\phi$ is a phase angle which corresponds to the ambiguity of rotations around $\hat{p}'$ in the definition of $w'_\lambda$. This phase factor will be discussed below.

## III. SIMPLIFIED RULES FOR EVALUATION OF FEYNMAN AMPLITUDES

In following an electron line through a Feynman diagram one picks up vertex factors and propagator factors alternately. We will group together a vertex and the propagator following it. (This seems to make the last vertex special, which, however, turns out not to be the case.) The successive factors are to be regarded as operators which operate consecutively on the initial spinor $\bar{u}_{p_0}$. The complete amplitude is obtained as the scalar product of the resulting spinor with a specific final-state spinor $\bar{u}_{\vec{p}_1}$.

Figure 1 shows a Feynman vertex in which the electron momentum is changed from $p$ to $p'$ under the influence of the field $A^\mu$. The Feynman rules yield for the helicity-nonflip part, $M_N$, of the amplitude containing this vertex

$$M_N u_{p_0}(\lambda) = M'\frac{p'\cdot\gamma}{p'^2-m^2}\,e\sqrt{4\pi}\,A\cdot\gamma\sqrt{p}\begin{pmatrix} w_\lambda \\ \hat{p}\cdot\vec{\sigma}w_\lambda \end{pmatrix}m'. \tag{13}$$

In this formula $m'$ represents numerical factors from previous operators while $M'$ represents operators still to come. We have neglected $m$ in the numerator but not in the denominator, and we do not differentiate between the energy and momenta of virtual particles. The notation has already incorporated helicity conservation. That is, the intermediate electrons have the same helicity as the incident electron. To justify this it is sufficient to show that the operator in (13) is proportional to the rotation operator $R$.

The unit vectors $\hat{n}_1$, $\hat{n}_2$, and $\hat{n}_\perp$ have been chosen to make this proportionality simple. For example, consider the case

$$\vec{A} = A^2\hat{n}_2\,,$$

$$\begin{aligned}
-p'\cdot\gamma\vec{n}_2\cdot\vec{\gamma}\sqrt{p}\begin{pmatrix} w_\lambda \\ \hat{p}\cdot\vec{\sigma}w_\lambda\end{pmatrix} &= -p'\sqrt{p}\begin{pmatrix} \hat{p}'\cdot\vec{\sigma}\hat{n}_2\cdot\vec{\sigma} & \hat{n}_2\cdot\vec{\sigma} \\ \hat{n}_2\cdot\vec{\sigma} & \hat{p}'\cdot\vec{\sigma}\hat{n}_2\cdot\vec{\sigma}\end{pmatrix}\begin{pmatrix} w_\lambda \\ \hat{p}\cdot\vec{\sigma}w_\lambda\end{pmatrix} \\
&= -p'\sqrt{p}\,(\hat{p}'\cdot\vec{\sigma}\hat{n}_2\cdot\vec{\sigma} + \hat{n}_2\cdot\vec{\sigma}\,\hat{p}\cdot\vec{\sigma})\begin{pmatrix} w_\lambda \\ 2\lambda w_\lambda\end{pmatrix} \\
&= \frac{-p'\sqrt{p}}{\cos\frac{1}{2}\theta'}\,(1+\hat{p}'\cdot\vec{\sigma}\,\hat{p}\cdot\vec{\sigma})\begin{pmatrix} w_\lambda \\ 2\lambda w_\lambda\end{pmatrix} \\
&= -p'\sqrt{p}\;2R\begin{pmatrix} w_\lambda \\ 2\lambda w_\lambda\end{pmatrix} \\
&= -2(pp')^{1/2}\sqrt{p'}\begin{pmatrix} w'_\lambda \\ \hat{p}'\cdot\vec{\sigma}w'_\lambda\end{pmatrix}.
\end{aligned}$$

Similar manipulations can be performed on all components of $\vec{A}$. As a result (13) becomes

$$M_N u_{\vec{p}_0}(\lambda) = M'm'\frac{e(4\pi)^{1/2}}{p'^2 - m^2}\,2(pp')^{1/2}(A^0\cos\tfrac{1}{2}\theta' - A^2 + A^\perp 2\lambda i\sin\tfrac{1}{2}\theta')\,(p')^{1/2}\begin{pmatrix} w'_\lambda \\ \hat{p}'\cdot\sigma w'_\lambda\end{pmatrix}. \tag{14}$$

Notice that the term with $A^1$ vanishes identically and that only the term with $A^\perp$ depends on the electron helicity. The factor $(p')^{1/2}$ has been included with the secondary electron to facilitate combining factors from successive vertices.

In order to avoid writing common factors we define

$$\overline{M} = 2(pp')^{1/2}(A^0\cos\tfrac{1}{2}\theta' - A^2 + A^\perp 2\lambda i\sin\tfrac{1}{2}\theta') \tag{15}$$

$$= (pp')^{1/2}\left[A^0 2\cos\tfrac{1}{2}\theta' - \vec{A}\cdot\left(\frac{\hat{p}+\hat{p}'}{\cos\frac{1}{2}\theta'}\right) + 2\lambda i\,\frac{\vec{A}\cdot(\hat{p}'\times\hat{p})}{\cos\frac{1}{2}\theta'}\right]. \tag{16}$$

For the final vertex there is no propagator factor, but there is a factor of $\gamma^0$ in the definition of $\bar{u}_{\vec{p}_1}$ and again the spinor is rotated to be along the final electron direction. The same factor, given by (15) or (16), appears. Combining the factors from all the vertices we obtain the amplitude as a product of simple kinematic factors with no Dirac matrices:

$$\bar{u}_{\vec{p}_1} M_N u_{\vec{p}_0}(\lambda) = \prod_{\text{Internal lines}}\frac{1}{p'^2 - m^2}\prod_{\text{Vertices}} e\sqrt{4\pi}\,(pp')^{1/2}\left[A^0 2\cos\tfrac{1}{2}\theta' - \frac{\vec{A}\cdot(\hat{p}+\hat{p}')}{\cos\frac{1}{2}\theta'} + 2\lambda i\,\frac{\vec{A}\cdot(\hat{p}'\times\hat{p})}{\cos\frac{1}{2}\theta'}\right]e^{i\phi}, \tag{17}$$

where the final phase factor is still to be discussed. This phase factor comes from the phase definition of helicity states mentioned with formula (12). In different diagrams the successive rotations must rotate $\hat{p}_0$ into $\hat{p}_1$, but since they are performed in different orders the complete rotations will differ by rotations around $\hat{p}_1$. In adding amplitudes of different diagrams compensation for these phase factors must be made, and that is the role of the final factor in (17). These phase factors can be further elucidated along with a discussion of the relativistic invariance of formula (17).

## IV. PHASE FACTORS AND RELATIVISTIC INVARIANCE

Formula (17) can be written in a form which makes it look more like a relativistic invariant. Define a quantity $\bar{n}^\lambda$ which, in any frame, is a unit vector normal to the plane containing $\hat{p}$ and $\hat{p}'$, i.e.,

$$\bar{n}^\lambda = (\hat{n}_\perp)^\lambda\,.$$

In the coordinate system indicated in Fig. 2 the $t, x, y, z$ coordinates of $\bar{n}^\lambda$ are

$$\bar{n}^\lambda = (0, 0, 1, 0)\,. \tag{18}$$

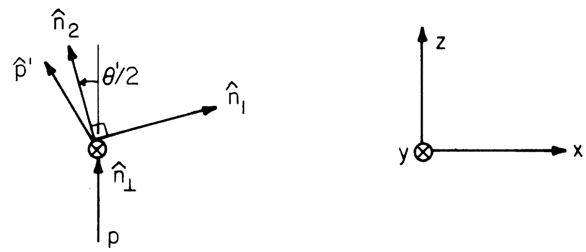


FIG. 2. Unit vectors and coordinate frame.

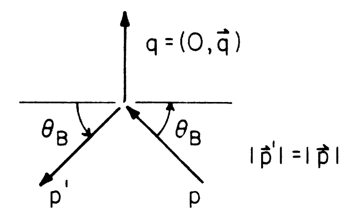


FIG. 3. Breit frame in which the virtual photon has no energy.

$\bar{n}^\lambda$ transforms as a 4-vector for Lorentz transformations in the plane of $\hat{p}$ and $\hat{p}'$. Using $\bar{n}^\lambda$ we can define another quantity $\bar{d}_\lambda$ with similar transformation properties,

$$\bar{d}^\lambda = \epsilon^{\lambda\mu\nu\rho} p_\mu p'_\nu \bar{n}_\rho \,.$$

With a makeshift but supposedly transparent notation, $\bar{d}^\lambda$ can be written as a determinant:

$$\bar{d} = \begin{vmatrix} \hat{t} & \hat{n}_1 & \hat{n}_\perp & \hat{n}_2 \\ p & -p\sin\frac{1}{2}\theta' & 0 & -p\cos\frac{1}{2}\theta' \\ p' & p'\sin\frac{1}{2}\theta' & 0 & -p'\cos\frac{1}{2}\theta' \\ 0 & 0 & -1 & 0 \end{vmatrix} . \tag{19}$$

The scalar product with vector $A$ can be written similarly:

$$A\cdot\bar{d} = \begin{vmatrix} A^0 & -\vec{A} \\ p & -\vec{p} \\ p' & -\vec{p}' \\ 0 & -\hat{n}_\perp \end{vmatrix} \tag{20}$$

$$= (pp')^{1/2}\sin\tfrac{1}{2}\theta'(pp')^{1/2}\left[A^0 2\cos\tfrac{1}{2}\theta' - \frac{\vec{A}\cdot(\hat{p}+\hat{p}')}{\cos\frac{1}{2}\theta'}\right] . \tag{21}$$

Defining the invariant

$$Q = 2(pp')^{1/2}\sin\tfrac{1}{2}\theta' , \tag{22}$$

we can condense the expression for $\bar{M}$:

$$\bar{M} = \frac{2A\cdot\bar{d}}{Q} + 2\lambda i\, Q A\cdot\bar{n} . \tag{23}$$

Although $\bar{M}$ is not a relativistic invariant, it can be shown that $|\bar{M}|$ is. It is sufficient to show for Lorentz transformations along $\hat{n}_\perp$ with infinitesimal speed $\beta$ that the change in $|\bar{M}|$ cancels to first order in $\beta$. To simplify the algebra this can be done in the Breit frame (shown in Fig. 3) since this frame can be reached by a Lorentz transformation in the plane of $\hat{p}$ and $\hat{p}'$. For this transformation the individual terms of $\bar{M}$ are invariant.

From this it follows that $\bar{M}'$ (where primes identify quantities in the system moving along $\hat{n}_\perp$ with speed $\beta$) differs from $\bar{M}$ by a phase factor

$$\bar{M}' = 2\frac{A'\cdot\bar{d}'}{Q} + 2\lambda i Q\, A'\cdot\bar{n}'$$
$$= \bar{M}e^{i\phi} . \tag{24}$$

Performing the algebra one obtains

$$\phi = 2\lambda\beta\tan\theta_B . \tag{25}$$

This phase can be compared with the difference in azimuth around $\hat{p}'$ of two transformations, both of which rotate $\hat{p}$ into $\hat{p}'$. One of these is the rotation around $\hat{n}_\perp$ in the unprimed system. The other is Lorentz transformation to the primed system, rotation around $\hat{n}'_\perp$, and then Lorentz transformation back. One can follow the orientation of a plane which initially contains $\hat{p}$ and $\hat{n}_\perp$. The difference between the two sequences is a rotation through angle $\beta\tan\theta_B$ around $\hat{p}'$. The phase factor in (24) is equal to the phase change of the helicity state in that rotation around $\hat{p}'$.

This discussion has shown that the main for-

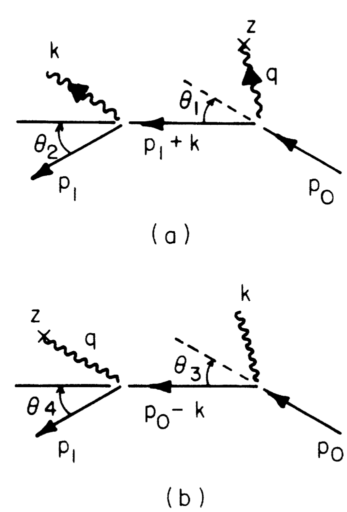


FIG. 4. Feynman diagrams for electron-nucleus bremsstrahlung.

mula (17) can be applied in any frame in a relativistically invariant way by proper treatment of the phase factors. It is necessary to keep track of a reference plane which initially contains the incident electron direction. For each diagram the orientation of this plane must be followed through the successive rotations. The final amplitude will contain a phase factor corresponding to the azimuthal difference between this reference plane and a fixed plane.

Keeping track of the reference azimuth is not difficult. At each vertex there is a rotation around $\hat{n}_\perp$. The azimuthal angle between the reference plane and the scattering plane is preserved in such a rotation.

## V. APPLICATION TO ELECTRON NUCLEUS BREMSSTRAHLUNG

### A. Elastic scattering from a massive point charge (Mott scattering)

Since there is only one diagram (Fig. 1) the phase factor does not matter. We have

$$A^\mu = \left[\frac{(4\pi)^{1/2} Z e}{q^2}, 0\right], \tag{26}$$

$$p' = p\,,$$
$$M_N = \frac{4\pi e^2 Z}{q^2}\, 2(pp')^{1/2} \cos\tfrac{1}{2}\theta'\,. \tag{27}$$

Including the phase-space factor gives

$$\frac{d\sigma}{d\Omega'} = \frac{1}{16\pi^2} |M_N|^2$$
$$= \frac{4Z^2 e^4 p^2}{q^4} \cos^2\tfrac{1}{2}\theta'\,. \tag{28}$$

This agrees with the Mott formula in the high-energy limit.

### B. Bremsstrahlung (helicity-conserving part)

There are several purposes in our discussion of this process. It will make clear that formula (17) is a simple generalization of the well-known formula[3] for the emission of soft photons. It will permit study of the accuracy of the method. It will yield formulas applicable as radiative corrections to inelastic electron scattering. Finally it will yield formulas for different polarization contributions, which, when added together, can be checked against the Bethe-Heitler formula.

For all of these purposes it is sufficient to use small-angle approximations. This means that we can again ignore the complication of the phase factors. That is because the commutator of two rotations through small angles is proportional to the product of the two small angles.

For bremsstrahlung there are two diagrams, labeled (a) and (b) in Fig. 4. $q$, $k$, $p_0$, and $p_1$ represent the 4-momenta of the nuclear recoil, the produced photon, and the incident and scattered electron, respectively. $\hat{\epsilon}$ is a unit photon polarization vector normal to $\hat{k}$ and

$$\hat{\epsilon}_\perp = \hat{k} \times \hat{\epsilon}.$$

Angles $\theta_1, \ldots, \theta_4$ are shown in Fig. 4. We will differentiate between helicity-flip and nonflip by the symbols $F$ and $N$.

Application of formula (17) yields

$$M_N^{(a)} = \frac{4\pi Z e^2}{q^2}\, 2[p_0(p_1+k)]^{1/2} \cos\tfrac{1}{2}\theta_1 \frac{e(4\pi)^{1/2}[(p_1+k)p_1]^{1/2}}{(p_1+k)^2 - m^2}\left[-\frac{\hat{\epsilon}\cdot(\hat{n}_{\vec{p}_1+\vec{k}} + \hat{p}_1)}{\cos\frac{1}{2}\theta_2} + \frac{2\lambda i \hat{\epsilon}\cdot\hat{p}_1 \times \hat{n}_{\vec{p}_1+\vec{k}}}{\cos\frac{1}{2}\theta_2}\right], \tag{29}$$

where $\hat{n}_{\vec{p}_1+\vec{k}} = \vec{p}_1 + \vec{k}/|\vec{p}_1 + \vec{k}|$. For small angles this can be simplified:

$$M_N^{(a)} = \frac{(4\pi)^{3/2} Z e^3 (p_0 p_1)^{1/2}}{q^2 (p_1\cdot k)}\, \hat{p}_1 \circ [-(p_0 + p_1)\hat{\epsilon} + 2\lambda i k \hat{\epsilon}_\perp]. \tag{30}$$

Similarly for diagram (b):

$$M_N^{(b)} = \frac{(4\pi)^{3/2} Z e^3 (p_0 p_1)^{1/2}}{q^2 (p_0\cdot k)}\, \hat{p}_0 [(p_0 + p_1)\hat{\epsilon} - 2\lambda i k \hat{\epsilon}_\perp]. \tag{31}$$

Adding (30) and (31) we get

$$M_N = \frac{(4\pi)^{3/2} Z e^3 (p_0 p_1)^{1/2}}{q^2}\left(\frac{\hat{p}_0}{p_0\cdot k} - \frac{\hat{p}_1}{p_1\cdot k}\right)[(p_0 + p_1)\hat{\epsilon} - 2\lambda i k \hat{\epsilon}_\perp]. \tag{32}$$

For small $k$ this reduces to

$$M_N = M_N(\text{Mott}) e (4\pi)^{1/2}\left(\frac{\vec{p}_0\cdot\hat{\epsilon}}{p_0\cdot k} - \frac{\vec{p}_1\cdot\hat{\epsilon}}{p_1\cdot k}\right), \tag{33}$$

which is a well-known[3] expression by which an elastic amplitude is modified for soft-photon emission.

Formula (32), on the other hand, remains valid at large $k$. In it the photon-polarization and electron-helicity dependence is explicit. Since we do not wish to discuss polarization effects we will average over polarizations.

The sum over photon polarizations can be performed using the result

$$\sum_{\hat{\epsilon}} |\vec{V}\cdot(A\hat{\epsilon}+iB\hat{\epsilon}_{\perp})|^2 = V_P^2(A^2+B^2), \tag{34}$$

where $\vec{V}$ is any real vector, $V_P$ is the length of its projection on a plane perpendicular to $\hat{k}$, and $A$ and $B$ are real. Hence we get

$$\sum_{\hat{\epsilon}} |M_N|^2 = \frac{(4\pi)^3 Z^2 e^6 p_0 p_1}{q^4}\left|\frac{\hat{p}_0\times\hat{k}}{p_0\cdot k}-\frac{\hat{p}_1\times\hat{k}}{p_1\cdot k}\right|^2 2(p_0^{\,2}+p_1^{\,2}). \tag{35}$$

The dependence on electron helicity has vanished in the average over photon polarization. Including the phase-space factor

$$2\pi\,\frac{1}{8p_0p_1k}\,\frac{d^3p_1k^2d\Omega_k}{(2\pi)^6}, \tag{36}$$

we get for the nonhelicity-flip cross section

$$\frac{d\sigma_N}{d^3p_1d\Omega_k} = \frac{2Z^2e^2r_e^{\,2}}{\pi^2}\,\frac{p_0^2+p_1^2}{kq^4}\left|mk\left(\frac{\hat{p}_0\times\hat{k}}{2p_0\cdot k}-\frac{\hat{p}_1\times\hat{k}}{2p_1\cdot k}\right)\right|^2. \tag{37}$$

Since the electron radius $r_e$(2.818 F) has been explicitly exhibited the momenta can be in any units. We shall see below, for values of electron momentum transfer large compared to the electron mass, that formula (37) dominates the Bethe-Heitler cross section. To make this comparison more satisfactory we will first evaluate the helicity-flip part which makes up the rest of the cross section.

### C. Helicity-flip part

For particle energies in the GeV range it is bad luck that neglect of the electron mass is not better justified. The fact that it is not is due to cancellations which cause the individual terms of (37) to vanish in the forward direction owing to a factor of $\theta$ in each amplitude. This allows terms with a factor $m/p$ to be competitive. To calculate these terms it is legitimate to assume, for factors in the numerator, that $\hat{p}_0$, $\hat{p}_1$, and $\hat{k}$ are collinear and to keep only terms linear in $m$. Terms neglected will have extra factors of either $\theta$ or $m$.

In evaluating the matrix element it is necessary to retain the second terms in expressions (7) and (8) for free spinors. Using the normal rules the amplitude for diagram (a) is given by

$$M_F^{(a)} = \frac{Ze^3(4\pi)^{3/2}}{q^2}\sqrt{p_1}\left(1+\frac{m}{2p_1}\gamma^0\right)(-\hat{\epsilon}\cdot\vec{\gamma})\frac{(p_1+k)\cdot\gamma+m}{2p_1\cdot k}\gamma^0\sqrt{p_0}\left(1+\frac{m}{2p_0}\gamma^0\right). \tag{38}$$

$M_F^{(b)}$ is given by a similar expression. Keeping only the term linear in $m$ and assuming that $\hat{\epsilon}$ is normal to $\hat{p}_0$ and $\hat{p}_1$ we get

$$M_F = \frac{Ze^3(4)^{3/2}}{q^2}\gamma^0\hat{\epsilon}\cdot\vec{\gamma}(p_0p_1)^{1/2}\left(\frac{1}{p_0}\frac{mk}{2p_0\cdot k}-\frac{1}{p_1}\frac{mk}{2p_1\cdot k}\right). \tag{39}$$

Summation over polarizations and inclusion of phase-space factors yields

$$\frac{d\sigma_F}{d^3p_1d\Omega_k} = \frac{2Z^2e^2r_e^{\,2}k}{\pi^2q^4}\left(\frac{1}{p_0}\frac{m^2k}{2p_0\cdot k}-\frac{1}{p_1}\frac{m^2k}{2p_1\circ k}\right)^2. \tag{40}$$

## VI. INTEGRATED CROSS SECTIONS AND ESTIMATION OF ERROR

In the approximation under discussion one would keep the nonflip amplitude (37) and neglect the flip amplitude (40). Since these cross sections are highly peaked it is more meaningful to compare integrals over photon directions rather than the differential cross sections (37) and (40) in estimating the error.

Using small-angle approximations and with angles defined in the spherical triangles shown in Fig. 5, it can be shown[2] that (37) and (40) reduce to

$$\frac{d\sigma_N}{d^3p_1 d\Omega_k} = \frac{2Z^2e^2r_e^2}{\pi^2}\frac{p_0^2+p_1^2}{k^5}(-F+G), \tag{41}$$

$$\frac{d\sigma_F}{d^3p_1 d\Omega_k} = \frac{2Z^2e^2r_e^2}{\pi^2}\frac{1}{k^3}F, \tag{42}$$

where $F$ and $G$ are given by

$$F = \frac{1}{(\alpha^2+\alpha_0^2)^2}\left(\frac{\theta_0^2}{\theta^2+\theta_0^2} - \frac{\theta_0'^2}{\theta'^2+\theta_0'^2}\right)^2, \tag{43}$$

$$G = \frac{k^2\theta_0^2\theta_0'^2\alpha^2}{(\alpha^2+\alpha_0^2)^2(\theta^2+\theta_0^2)(\theta'^2+\theta_0'^2)}. \tag{44}$$

$\alpha$ is the polar angle of the photon relative to the "virtual-photon direction" (deep-inelastic electron scattering terminology). This direction is defined by the vector $\vec{p}_2 = \vec{p}_0 - \vec{p}_1$. $\theta$ and $\theta'$ are the photon's polar angles relative to the incident and scattered electron, respectively. In these formulas the small angles $\theta_0$, $\theta_0'$, and $\alpha_0$ are given by

$$\theta_0 = \frac{m}{p_0}, \tag{45}$$

$$\theta_0' = \frac{m}{p_1}, \tag{46}$$

$$\alpha_0 = \frac{m^2}{2p_0p_1}\left(1+\frac{p_0p_1}{k^2}\frac{Q^2}{m^2}\right), \tag{47}$$

where $Q^2 = p_0p_1\theta_1^2$. $Q^2$ is the square of the 4-momentum transfer of the scattered electron. Since we are interested in the case $Q \gg m$, $\alpha_0$ is given by

$$\alpha_0 \cong \frac{Q^2}{2k^2}. \tag{48}$$

Since the cross section is dominated by $G$ (as will be seen) the character of the photon distribution can be inferred from (44). There are peaks of characteristic widths $\theta_0$ and $\theta_0'$ along the incident and scattered electron directions and a peak (wide-angle bremsstrahlung) of characteristic width $\alpha_0$ along the "virtual-photon direction." As an aside we comment that it is the third peak which is dominant at low $Q^2$ [see (47)] and hence dominates the total cross section. At large $Q^2$ the other two peaks tend to dominate.

We wish to integrate over these three peaks and compare the contributions from the flip and nonflip amplitudes. These three contributions can be separated as shown in Fig. 6, where regions I, II, and III are defined by circles centered on $p_2$. Each region contains one peak, although naturally the placement of the boundaries is somewhat arbitrary. In performing radiative corrections these cross sections fall off from their peak values more slowly than one might wish and the separation is ambiguous. For a qualitative discussion this ambiguity is unimportant.

In a given region the expressions for $F$ and $G$ given in (43) and (44) can be simplified. For example, in region III, $\alpha_0$ and $\theta_0$ can be set to zero. Neglecting various small terms (down by $m^2/Q^2$) the integrated cross sections are given by

$$\int_{\mathrm{I}} F\,d\Omega_k = 0, \tag{49}$$

$$\int_{\mathrm{I}} G\,d\Omega_k = \frac{\pi k^2}{Q^4}\left(\frac{k^2}{p_0p_1}\right)^2\left[\ln\left(1+\frac{\bar{\alpha}^2}{\alpha_0^2}\right) - \frac{1}{1+\alpha_0^2/\bar{\alpha}^2}\right], \tag{50}$$

$$\int_{\mathrm{II}} F\,d\Omega_k = \frac{\pi k^2}{Q^4}\left(\frac{k^2}{p_0p_1}\right)\frac{p_0}{p_1}, \tag{51}$$

$$\int_{\mathrm{II}} G\,d\Omega_k = \frac{\pi k^2}{Q^4}\left(\frac{k^2}{p_0p_1}\right)\left(\frac{p_0}{p_1}\ln\frac{Q^2}{m^2} + \frac{k^2}{p_0p_1}\ln\frac{\alpha^2}{\bar{\alpha}^2} + \frac{k}{p_1}\ln\frac{p_1}{k} - \frac{k}{p_0}\ln\frac{p_0^2}{kp_1}\right), \tag{52}$$

$$\int_{\mathrm{III}} F\,d\Omega_k = \frac{\pi k^2}{Q^4}\left(\frac{k^2}{p_0p_1}\right)\frac{p_1}{p_0}, \tag{53}$$

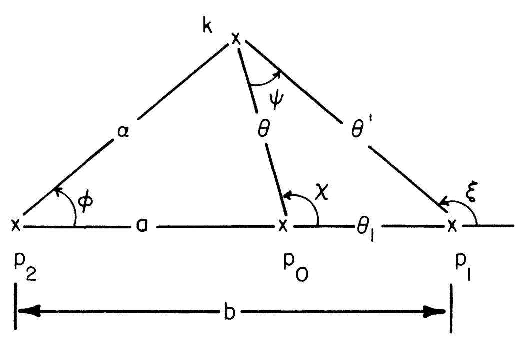


FIG. 5. Spherical triangles defining geometric variables. The sides represent polar angles while the angles represent azimuthal angles.

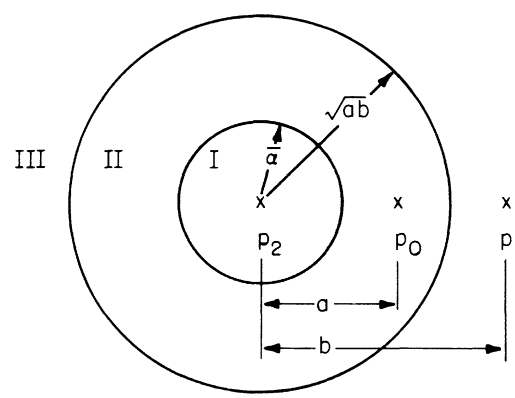


FIG. 6. Photon directions are assigned to mutually exclusive zones, I, II, and III, centered on $\hat{p}_2$, each containing one bremsstrahlung peak.

$$\int_{\text{III}} G\,d\Omega_k = \frac{\pi k^2}{Q^4}\left(\frac{k^2}{p_0 p_1}\right)\left(\frac{p_1}{p_0}\ln\frac{Q^2}{m^2} + \frac{k^2}{p_0 p_1}\ln\frac{p_0}{k}\right). \tag{54}$$

We are finally in a position to discuss the validity of helicity conservation. For wide-angle bremsstrahlung (region I) the error is of order $m^2/Q^2$ and hence is negligible. For region II the leading term in (52) is dominant. It is larger than the contribution from $F$ by a factor $\ln Q^2/m^2$. Hence from (41) and (42) the flip and nonflip contributions are related by

$$\frac{d\sigma_N/d^3p_1}{d\sigma_F/d^3p_1} \simeq \frac{p_0{}^2 + p_1{}^2}{k^2}\ln\frac{Q^2}{m^2}. \tag{55}$$

For $Q^2 = 1$ GeV the value of the logarithm is 15. Neglect of $\sigma_F$ would incur an error of 7% for very high energy photons and less than that otherwise.

For small values of $Q^2$ (of order $m^2$) it can be shown[1] that the factor $\ln Q^2/m^2$ in (55) is replaced by a factor near 2.

## VII. TRIDENT PRODUCTION

It is not our intention to give detailed formulas for this process but rather to show the extreme simplicity with which such formulas can be obtained. This process has been considered by Henry[2] and also by Bjorken and Chen,[2] whose notation we follow. There are eight diagrams of which diagram 1 is shown in Fig. 7. The incident electron 4-momentum is $p$, the outgoing electron 4-momenta are $p_1$ and $p_2$, and the outgoing positron 4-momentum is $p_+$. Other momenta are the internal line

$$p' = p - p_1 - p_2 \tag{56}$$

and the target recoil

$$q = p - p_1 - p_2 - p_+ . \tag{57}$$

If $\lambda_1$ and $\lambda_2$ are the electron helicities (assuming helicity conservation) let us confine ourselves, for diagram 1, to the amplitude proportional to $\lambda_1$ and $\lambda_2$.

The cross section is given by

$$d\sigma = \frac{(4\pi\alpha)^4}{q^4}\,\frac{1}{2^4}\,\frac{2\pi}{p p_1 p_2 p_+}\,\frac{1}{(2\pi)^9} \times \frac{dp_1 dp_2 d\Omega_1 d\Omega_2 d\Omega_+}{(p-p_1)^2(p'^2 - m^2)}\left|\sum_{i=1}^{8} M_i\right|^2, \tag{58}$$

where, directly from (17),

$$M_1 = \frac{-(\vec{p}\times\vec{p}_1)\cdot(\vec{p}_2\times\vec{p}\,')}{\cos\frac{1}{2}\theta_1\cos\frac{1}{2}\theta_2'}\left(\frac{p_+}{p p_1 p_2}\right)^{1/2} 2\cos\tfrac{1}{2}\theta_+ . \tag{59}$$

Using conventional projection operators and traces Bjorken and Chen obtain (for the same amplitude)

$$M_1' = \frac{2}{(p_+\cdot\tilde{p}_2 p_1\cdot\tilde{p})^{1/2}}\begin{vmatrix} p_+ & p_1\cdot\tilde{p}_+ & p\cdot\tilde{p}_+ \\ p_2 & p_1\cdot p_2 & p\cdot p_2 \\ p-p_1 & p\cdot p_1 & -p\cdot p_1\end{vmatrix}, \tag{60}$$

where all symbols are 4-vectors and the other factors of (58) are common to the two methods. The expression $p_+\cdot\tilde{p}$ stands for $p_+p + \vec{p}_+\cdot\vec{p}$. $M_1$ and $M_1'$ are not quite indentical but they are very close. For small angles, $M_1'$ can be manipulated to the form

$$M_1' = -2\left(\frac{p_+}{p p_1 p_2}\right)^{1/2}(\vec{p}\times\vec{p}_1)\cdot(\vec{p}_2\times\vec{p}\,')\left(1 + O\left(\frac{q}{p}\right)\right), \tag{61}$$

in agreement with (59). Furthermore, direct numerical evaluation at randomly selected values of $\vec{p}_1$, $\vec{p}_2$, and $\vec{p}_+$ inside 5° always gave agreement to within 1% accuracy. The small discrepancy is presumably due to our neglect of the mass of the virtual electron.

In passing we comment that the assumption of helicity conservation can be expected to break down somewhat when all outgoing particles are very near the forward direction. Since this region contributes significantly to the total cross section one cannot integrate formula (58) to obtain the total cross section but should use it only when one of the particles has transverse momentum large compared to the electron mass. Neither Bjorken and Chen nor Henry appears to make this

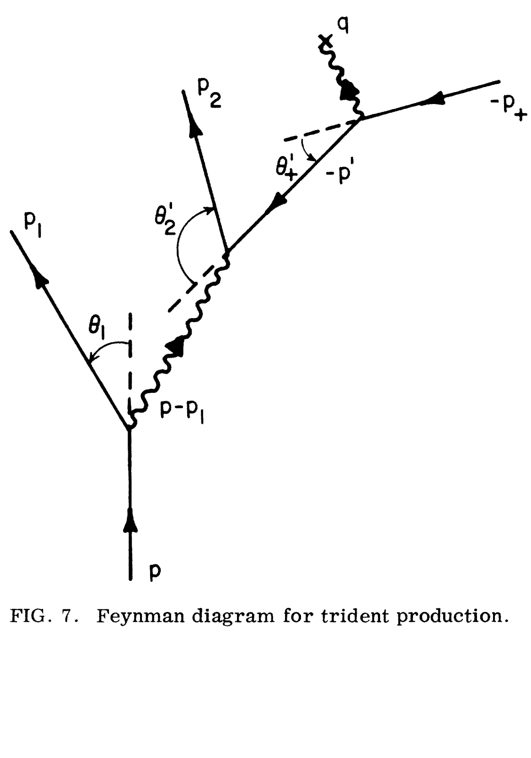


FIG. 7. Feynman diagram for trident production.

point. For bremsstrahlung this has been discussed elsewhere.[1] As with bremsstrahlung, to obtain the term proportional to $m$ one can, when evaluating the numerator, assume that $\vec{p}$, $\vec{p}_1$, $\vec{p}_2$, and $\vec{p}_+$ are collinear, as was done in formula (39).

## VIII. CONCLUSION

We have shown how, using formula (17), to write down a good approximation to any Feynman amplitude without spin complications. For the approximation to be valid the particle mass, $m$, must be negligible compared to external energies. Also further refinements are necessary for very small or very large angles. For small angles neglect of the electron mass becomes invalid. For bremsstrahlung this has been discussed in connection with formulas (40) and (55). For very large angles the neglect of internal 4-momenta squared relative to external energies becomes unjustified. Taking these effects into account is straightforward but naturally results in more complicated formulas.

As well as being easy to write down, the resulting amplitudes are somewhat free of the defect of being small differences between large numbers. This is illustrated for trident production, in the comparison between formulas (59) and (60). Formulas (59) are a product of two sine functions (each presumably small) which in (60) must be manufactured out of terms containing only cosines.

The formulas of QED (for example, relating to radiative corrections) are frequently so complicated as to make blunders likely. The present method is an effort to alleviate the situation. A preliminary version of this paper discussed the resulting radiative corrections in greater detail than has been presented here, but this material was rather specialized. A more systematic description of the application to radiative correction is in preparation. Here we call attention only to a couple of simple results. The angular distribution of emitted photons is given to a good approximation by (44). Using the peaking approximation the dependence near each of the peaks was first derived by Schiff[5] but not in the compact form of (44). This form is sufficiently simple to allow analytic integration over photon directions in (49)–(54). These can be compared with the somewhat more general, but much more complicated, formula (B5) of Mo and Tsai,[6] which have still to be integrated over one more variable ($\alpha$ in our notation). It must, however, be acknowledged that the effect of target size is to modify the integrand by a form factor depending on $\alpha$. For this and various similar reasons it is probably sound procedure to attack radiative corrections on two fronts, one using formulas as nearly exact as possible (e.g., Mo and Tsai[6]) and one using the simpler and more physical formulas following from from helicity conservation.

## ACKNOWLEDGMENTS

I am grateful for useful conversations with Don Yennie, Lou Hand, and Nari Mistry.

---

*Work supported in part by the National Science Foundation.

[1]A preliminary version of this paper with greater emphasis on radiative corrections was privately circulated as Cornell Report No. CLNS-297 (unpublished). A more detailed description of the application to radiative corrections is now in preparation.

[2]J. D. Bjorken and M. C. Chen, Phys. Rev. 154, 1335 (1967); 162, 1750(E) (1967); G. R. Henry, *ibid*. 154, 1534 (1967); 162, 1750(E) (1967).

[3]V. B. Berestetskii, E. M. Lifshitz, and L. P. Pitaevskii, *Relativistic Quantum Theory* (Pergammon, New York, 1971).

[4]E.g., Berestetskii *et al*., Ref. 3, p. 346.

[5]L. I. Schiff, Phys. Rev. 87, 750 (1952).

[6]L. W. Mo and Y. S. Tsai, Rev. Mod. Phys. 41, 205 (1969).