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Comments and Addenda

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Scattering of quantized solitary waves in the cubic Schrödinger equation*

L. Dolan
Laboratory for Nuclear Science and Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139
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The quantum mechanics for N particles interacting via a δ-function potential in one space dimension and one time dimension is known. The second-quantized description of this system has for its Euler-Lagrange equations of motion the cubic Schrödinger equation. This nonlinear differential equation supports solitary wave solutions. A quantization of these solitons reproduces the weak-coupling limit to the known quantum mechanics. The phase shift for two-body scattering and the energy of the N-body bound state is derived in this approximation. The nonlinear Schrödinger equation is contrasted with the sine-Gordon theory in respect to the ideas which the classical solutions play in the description of the quantum states.

Quantization of the solitary wave solutions of classical equations of motion has been investigated to describe the corresponding quantum field theory. This paper shows that for a system of N indistinguishable particles interacting via a δ-function potential, a quantization[1] of the solitary wave solutions to the second-quantized description reproduces the weak-coupling limit of the known quantum-mechanical result.

We derive from the classical solutions the weak-coupling approximation to the phase shift for two-body scattering, as well as the energy of the N-body bound state.[2,3] The quantum mechanics for an N-body δ-function interaction has been solved.[4-6] The Hamiltonian

$$H=\frac{-\hbar^2}{2m}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i{}^2}-\kappa\sum_{i<j}\delta(x_i-x_j)$$

can be expressed for N identical particles in terms of complex scalar fields in the second-quantized form

$$\mathcal{H}(x)=-\mathcal{L}(x)+\frac{i\hbar}{2}\left(\phi^*\frac{\partial}{\partial t}\phi-\phi\frac{\partial}{\partial t}\phi^*\right), \tag{1}$$

where

$$\mathcal{L}(x)=\frac{i\hbar}{2}\left(\phi^*\frac{\partial}{\partial t}\phi-\phi\frac{\partial}{\partial t}\phi^*\right)-\frac{\hbar^2}{2m}\nabla\phi\nabla\phi^*+\frac{\kappa}{2}(\phi^*\phi)^2$$

and (2)

$$N=\int_{-\infty}^{\infty}dx\,\phi^*\phi\,.$$

We consider a two-dimensional field theory. The Euler-Lagrange equation of motion for ϕ is the cubic Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t}\phi=-\frac{\hbar^2}{2m}\nabla^2\phi-\kappa(\phi^*\phi)\phi\,. \tag{3}$$

For $\kappa>0$ this nonlinear differential equation admits a family of "envelope" solitary wave solutions.[7]

The one-soliton solution ϕ_1 is given for arbitrary initial position x_0, phase χ, and momentum $\hbar^3|a|v/\kappa m$:

$$\phi_1=\frac{a\hbar}{\sqrt{\kappa m}}\frac{\exp(ivx/2-i\hbar v^2t/8m+i\hbar a^2t/2m+i\chi)}{\cosh a(x-x_0-\hbar vt/2m)}\,. \tag{4}$$

Here a is a parameter proportional to the number of particles. This follows from the Bohr-Sommerfeld quantization condition[1]

$$S_T + ET = n\pi \tag{5}$$

and from Eq. (2). In Eq. (5), $S_T = \int_0^T dt \int_{-\infty}^{\infty} dx\, \mathcal{L}(x)$, and T is the half period of the periodic solition solution being considered. Evaluation of Eq. (5) leads to $2\hbar^2|a|/\kappa m = n$, which together with Eq. (2) yields $|a| = \kappa m N/2\hbar^2$. Computation of the energy gives

$$E = \int_{-\infty}^{\infty} dx\, \mathcal{H}(x) = -\frac{\kappa^2 m N^3}{24\hbar^2}. \tag{6}$$

This is the leading term of the exact result $[E = (-\kappa^2 m/24\hbar^2)(N^3 - N)]$ for the bound state of N particles. Derivation of the full answer requires a more sophisticated quantization procedure.[8,9] We are thus led to treat the solitons as bound states of the fundamental particles, drawing in general on the N-soliton solution with $2\hbar|a_i|/\kappa m$ equal to the number of particles bound together in each of the N bound states.

For two-meson scattering, one particle goes into each of the two "parts" of the two-soliton solution. (Two particles in one part and no particles in the other part, i.e., $2\hbar^2|a_1|/\kappa m = 2$ and $2\hbar^2|a_2|/\kappa m = 0$ corresponds to two bound mesons moving with the same velocity and therefore does not describe two-body scattering.)

The two-soliton solution in the center-of-mass frame with relative velocity $2v$ is ϕ_2:

$$\begin{aligned}\phi_2 = \frac{2\hbar a}{\sqrt{\kappa m}} \exp\left(-\frac{i\hbar v^2 t}{8m} + \frac{i\hbar a^2 t}{2m}\right) \Bigg\{ & e^{ivx/2}\left[\exp\left(-ax + \frac{\hbar a v t}{2m}\right) + \frac{v^2}{(v-2ia)^2}\exp\left(-3ax - \frac{\hbar a v t}{2m}\right)\right] \\ & + e^{-ivx/2}\left[\exp\left(-ax - \frac{\hbar a v t}{2m}\right) + \frac{v^2}{(v+2ia)^2}\exp\left(-3ax + \frac{\hbar a v t}{2m}\right)\right]\Bigg\} \\ & \times \left[1 + 2e^{-2ax}\cosh\frac{\hbar a v t}{m} - 8a^2 e^{-2ax}\,\mathrm{Re}\left(\frac{e^{ivx}}{(v+2ia)^2}\right) + \frac{v^4}{(v^2+4a^2)^2}e^{-4ax}\right]^{-1}.\end{aligned} \tag{7}$$

The reference positions and phase have been set equal to zero. The number parameters a_1 and a_2 have been set equal to each other, i.e., a. Taking the limit $t \to \pm\infty$, we find the classical time delay for soliton-soliton scattering:

$$\Delta t = \frac{-2m}{\hbar a v} \ln \frac{v^2 + 4a^2}{v^2}. \tag{8}$$

For two-meson scattering, $a = \kappa m/2\hbar^2$. In terms of the energy[10] of the two-soliton system $E = (2\hbar^4 a/m^2\kappa)(\frac{1}{4}v^2 - \frac{1}{3}a^2)$, the time delay is

$$\Delta t = \frac{-2\hbar^2}{\kappa[mE(1+\kappa^2 m/12\hbar^2 E)]^{1/2}} \times \ln\left(\frac{1+\kappa^2 m/3\hbar^2 E}{1+\kappa^2 m/12\hbar^2 E}\right). \tag{9}$$

Equation (9) can be expanded for weak coupling to give

$$\Delta t = \frac{-\kappa\sqrt{m}}{2E^{3/2}}\ O(\kappa^2). \tag{10}$$

The quantum-mechanical phase shift for two identical particles scattering via a δ-function potential is

$$2\delta(E) = -2\tan^{-1}\frac{\kappa}{2\hbar}\left(\frac{m}{E}\right)^{1/2}. \tag{11}$$

Using the relation between the phase shift and the classical time delay[1]

$$\Delta t = \hbar \frac{\partial}{\partial E} 2\delta(E),$$

we find from Eq. (11) that $\Delta t = -\kappa\sqrt{m}/2E^{3/2} + O(\kappa^2)$. Furthermore, the integrated relation $2\delta(E) = (\kappa/\hbar)(m/E)^{1/2} + O(\kappa^2)$ is also established from the classical solutions by directly integrating the action (see Ref. 1). It has thus been verified in this example that the semiclassical quantization procedure applied to the classical solutions does give the weak-coupling approximation to the exact result.

In addition,[11] it is noted that the expression for the time delay is small for weak coupling. This is in contrast to the case of the sine-Gordon theory in which Δt is inversely proportional to the interaction strength.[1] In the cubic Schrödinger equation, the N-soliton solution describes the scattering of N bound states of mesons. In particular, it describes N-meson scattering, i.e., scattering of the fundamental fields which one assumes to be weak in perturbation theory. The sine-Gordon soliton scattering, however, describes scattering of states orthogonal to the fundamental meson states. These orthogonal (baryon) states are thought to be fundamental in a massive Thirring model whose coupling is inversely proportional to that of the sine-Gordon theory.[12] Thus, if the baryons interact weakly in the Thirring model, we expect the scattering to be strong in the sine-Gordon description and are heartened to see this behavior reproduced in the semiclassical approximation.

This paper gives support to the quantization program for the scattering of solitary waves by calculating its predictions for an exactly solvable model. We have shown that this method gives the weak-coupling approximation for two-body scattering with a δ-function interaction. Furthermore, the qualitative dependence of the phase shift on the coupling constant, in this approximation, in both the sine-Gordon theory and the Schrödinger model is consistent with the respective roles which the classical solutions play in the description of the quantum states.

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[1]R. Jackiw and G. Woo, Phys. Rev. D 12, 1643 (1975).

[2]The bound-state energy has been computed exactly in Ref. 9 and is also given in Ref. 8. The leading term was derived in Ref. 3. We reproduce the leading term in this paper using the semiclassical methods of Ref. 1.

[3]D. J. Kaup, J. Math. Phys. 16, 2036 (1975).

[4]H. Bethe, Z. Phys. 71, 205 (1931).

[5]C. N. Yang, Phys. Rev. 168, 1920 (1968).

[6]J. B. McGuire, J. Math. Phys. 5, 622 (1964).

[7]V. E. Zakharov and A. B. Shabat, Zh. Eksp. Teor. Fiz. 61, 118 (1971) [Sov. Phys.—JETP 34, 62 (1972)].

[8]L. D. Faddeev, Princeton report (unpublished).

[9]C. Nohl, Princeton report (unpublished).

[10]Note that the energy is negative for zero momentum. This is due to the approximation since it only produces the leading (N^3) term for the single soliton energy.

[11]This point was made in conversation with R. Jackiw.

[12]Sidney Coleman, Phys. Rev. D 11, 2088 (1975).