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Theory of fermion exchange in massive quantum electrodynamics at high energy. VI. Summation of diagrams*

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We conclude our study of fermion exchange in massive quantum electrodynamics at high energy by summing the leading contributions which have been computed in perturbation theory in the four preceding papers.

I. INTRODUCTION

In the four preceding papers[1-4] we have examined in detail the high-energy behavior in massive quantum electrodynamics of Compton scattering near the backward direction and of pair annihilation. It remains to sum the leading logarithms which we have computed in each order of perturbation theory.

To study this summation of Feynman diagrams we define the positive-signature amplitude,

$$\mathfrak{M}_+(s,r_1) = \tilde{\mathfrak{M}}(s,r_1) + \bar{\mathfrak{M}}(s,r_1), \tag{1.1a}$$

and the negative-signature amplitude,

$$\mathfrak{M}_-(s,r_1) = \tilde{\mathfrak{M}}(s,r_1) - \bar{\mathfrak{M}}(s,r_1), \tag{1.1b}$$

where we recall that[3]

$$\tilde{\mathfrak{M}}_{\mu\nu}(s,r_1) = \bar{u}(r_3+r_1)\gamma_\mu\tilde{\mathfrak{M}}(s,r_1)\gamma_\nu u(r_2-r_1) \tag{1.2a}$$

is the backward Compton amplitude and

$$\bar{\mathfrak{M}}_{\mu\nu}(s,r_1) = \bar{v}(r_3+r_1)\gamma_\mu\bar{\mathfrak{M}}(s,r_1)\gamma_\nu u(r_2-r_1) \tag{1.2b}$$

is the pair-annihilation amplitude (see Fig. 1).

In Sec. II we will study the summation of diagrams for $\mathfrak{M}_+$. In Sec. III we will study the summation of diagrams for $\mathfrak{M}_-$. The physics of these results has already been discussed in the first paper of the series.[5]

II. DIAGRAM SUMMATION IN THE POSITIVE-SIGNATURE CHANNEL

In Fig. 2 we display graphically the results of the second- through twelfth-order perturbation theory for the positive signature. From this figure the behavior of the general term is obvious. In writing down these diagrams and the integrals which correspond to them, we have used the set of rules (B) of Sec. IV of the preceding paper with the modification that the factor $(-1)^{n+1}$ of (4.12) has been incorporated into the pairs (m_1, m_2).

We sum the diagrams by introducing the Mellin transform

$$\tilde{\mathfrak{M}}_\pm(\zeta) = \int_0^\infty ds\, s^{-\zeta-1}\mathfrak{M}_\pm(s). \tag{2.1}$$

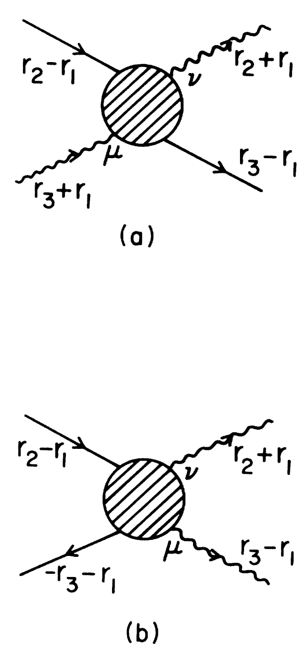

FIG. 1. (a) Kinematics for the process of Compton scattering near the backward direction. (b) Kinematics for the process of pair annihilation into 2 photons.

If we let

$$m_{\pm}(s)=\begin{cases}\dfrac{1}{n!}\ln^n s, & 1\le s\\ 0, & 0\le s<1\end{cases} \tag{2.2}$$

we have

$$\frac{1}{n!}\int_1^\infty ds\, s^{-\zeta-1}\ln^n s=\zeta^{-(n+1)}. \tag{2.3}$$

Therefore, since in s space the symbol (m_1,m_2) is defined for $(2n+2)$ order as

$$(m_1,m_2)=\frac{1}{n!}\left[m_1\ln^n s+m_2(\ln^n s-n\pi i\ln^{n-1}s)\right] \tag{2.4}$$

we find that in ζ space

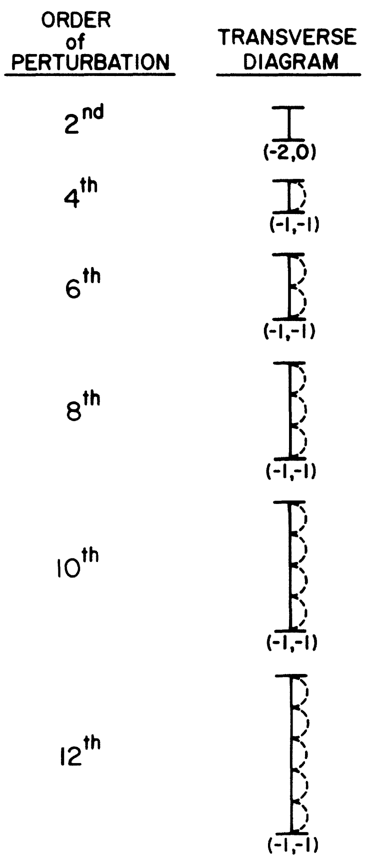

FIG. 2. The transverse diagrams which contribute to the positive-signature amplitude $\mathfrak{M}_+$ in second- through twelfth-order perturbation theory. The pair (m_1,m_2) is shown for each diagram.

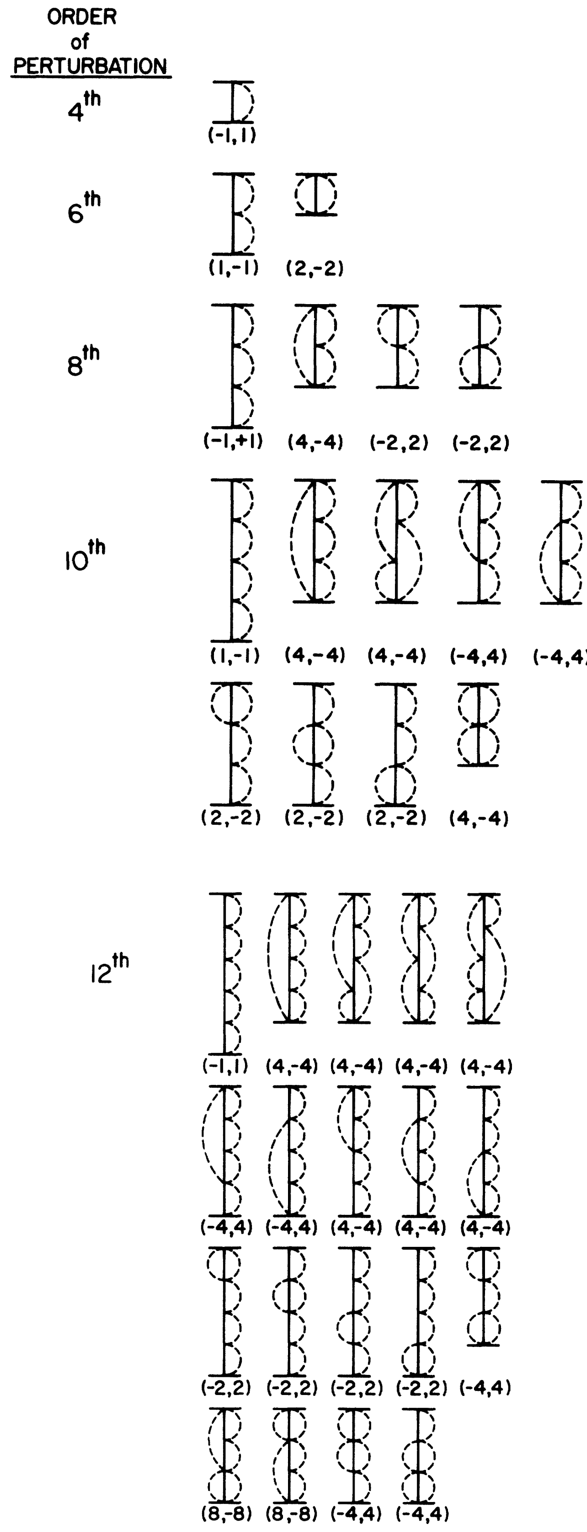

FIG. 3. The transverse diagrams which contribute to the negative-signature amplitude $\mathfrak{M}_-$ in fourth- through twelfth-order perturbation theory. The pair $(m,-m)$ is shown for each diagram.

$$(m_1, m_2) \to m_1 \zeta^{-(n+1)} + m_2(\zeta^{-(n+1)} - \pi i \zeta^{-n})$$

$$= (m_1 + m_2)\zeta^{-(n+1)} - m_2 \pi i \zeta^{-n}. \quad (2.5)$$

For convenience we define

$$\vec{\Delta} = 2\vec{r}_\perp . \quad (2.6)$$

Then we find from Fig. 2 that

$$\mathfrak{M}_+^{(2)}(\zeta) = 2g^2 \zeta^{-1} \frac{1}{\vec{\Delta} + m} \quad (2.7a)$$

and, for ζ near zero and $n \geq 1$,

$$\mathfrak{M}_+^{(2n+2)} \doteq (2g^2 \zeta^{-(n+1)} - \pi i g^2 \zeta^{-n})(\vec{\Delta}+m)^{-1} \alpha^n(\vec{\Delta}), \quad (2.7b)$$

where

$$\alpha(\vec{\Delta}) = \frac{g^2}{8\pi^3}(\vec{\Delta}+m) \int d^2 \vec{k}_\perp \frac{\vec{\Delta} + \vec{k}_\perp - m}{[(\vec{\Delta} + \vec{k}_\perp)^2 + m^2](\vec{k}_\perp{}^2 + \lambda^2)}. \quad (2.8)$$

We readily sum (2.7) over n to find

$$\mathfrak{M}_+(\zeta) = \sum_{n=0}^{\infty} \mathfrak{M}_+^{(2n+2)}(\zeta)$$

$$= \frac{g^2}{\vec{\Delta}+m} \frac{1}{1 - \zeta^{-1}\alpha(\vec{\Delta})}[2\zeta^{-1} - \pi i \alpha(\vec{\Delta})]. \quad (2.9)$$

Inverting the Mellin transform we find

$$\mathfrak{M}_+(s) \doteq \frac{g^2}{\vec{\Delta}+m} s^{\alpha(\vec{\Delta})}[2 - \pi i \alpha(\vec{\Delta})]. \quad (2.10)$$

III. DIAGRAM SUMMATION IN THE NEGATIVE-SIGNATURE CHANNEL

In Fig. 3 we draw the transverse diagrams, together with the pairs $(m, -m)$ for the negative-signature amplitude in fourth through twelfth order. There are clearly many more diagrams than for the positive-signature amplitude. From Fig. 3 it is clear what the generalization to $(2n+2)$ order is and we could write out an explicit formula. However, for the purpose of computing the sum of all diagrams it is more convenient to work with Fig. 3 as it stands.

The diagrams of Fig. 3 may be divided into two classes:

1. Those diagrams which do not factor into two or more parts and contain only three-particle intermediate states.

2. Those diagrams which either factor into two or more parts or contain at least one segment with a two-particle intermediate state.

We note that in terms of the variable ζ that in $(2n+2)$ order

$$(m, -m) \to \pi m i \zeta^{-n} \quad (3.1)$$

and we define

$$-g^2 \pi i \Sigma_3(\vec{\Delta};\zeta) = \text{the sum of the contributions from class-1 diagrams}. \quad (3.2)$$

Our first step is to express $\mathfrak{M}_-(\zeta)$ in terms of Σ_3 and

$$\Sigma_2 = \frac{1}{\vec{\Delta}+m} \sum_{n=1}^{\infty} [-\zeta^{-1}\alpha(\vec{\Delta})]^n$$

$$= \frac{-1}{\vec{\Delta}+m} \frac{\zeta^{-1}\alpha(\vec{\Delta})}{1 + \zeta^{-1}\alpha(\vec{\Delta})}. \quad (3.3)$$

From Fig. 3 we see that if a diagram contains several three-particle-state segments that the pair $(m, -m)$ is given as

$$(m, -m) = \left[\prod m_i, -\prod m_i\right], \quad (3.4)$$

where $(m_i, -m_i)$ is the contribution for the ith three-particle-state segment taken separately. We also see that the term which involves only two-body states is $-g^2\pi i \Sigma_2$ but that when two-body states can separate three-body states, the sum over the two-body segments yields $(\vec{\Delta}+m)\Sigma_2 + 1$ since we must include the term where the three-body segments directly adjoin each other. Consequently we find

$$\mathfrak{M}_-(\zeta) = -\frac{g^2 \pi i}{\vec{\Delta}+m} \left\{ (\vec{\Delta}+m)\Sigma_2 + [(\vec{\Delta}+m)\Sigma_2 + 1] \sum_{n=1}^{\infty} [(\vec{\Delta}+m)\Sigma_3]^n [(\vec{\Delta}+m)\Sigma_2 + 1]^n \right\}$$

$$= -\frac{g^2 \pi i}{\vec{\Delta}+m} \frac{(\vec{\Delta}+m)\Sigma_2 + (\vec{\Delta}+m)\Sigma_3[(\vec{\Delta}+m)\Sigma_2 + 1]}{1 - (\vec{\Delta}+m)\Sigma_3[(\vec{\Delta}+m)\Sigma_2 + 1]}, \quad (3.5)$$

and thus, using (3.3), we have

$$\mathfrak{M}_-(\zeta) = -\frac{g^2\pi i}{\vec{\not\Delta}+m}\,\frac{-\zeta^{-1}\alpha(\vec{\Delta}) + (\vec{\not\Delta}+m)\Sigma_3(\vec{\Delta};\zeta)}{1+\zeta^{-1}\alpha(\vec{\Delta}) - (\vec{\not\Delta}+m)\Sigma_3(\vec{\Delta};\zeta)}\,. \tag{3.6}$$

It remains to find $\Sigma_3(\vec{\Delta};\zeta)$. This we study in exactly the same way we studied graphs with 3 particles in the t channel in ϕ^3 theory.[6] In $(2n+2)$ order let $-g^2\pi i\Sigma_3^{(n)}(\vec{\Delta};\zeta)$ be the sum of all three-particle graphs and let $f^{(n)}(\vec{k}_{1\perp},\vec{k}_{2\perp},\vec{k}_{3\perp};\zeta)$ be the function obtained by cutting these graphs on the bottom loop as illustrated in Fig. 4 and omitting the 3 propagators of the 3 cut lines and the numerator factor that goes with the upper vertex. In $f^{(n)}(\vec{k}_{1\perp},\vec{k}_{2\perp},\vec{k}_{3\perp};\zeta)$ $\vec{k}_{1\perp}$ and $\vec{k}_{2\perp}$ are the photon momenta and

$$\vec{k}_{1\perp}+\vec{k}_{2\perp}+\vec{k}_{3\perp}+\vec{\Delta}=0\,. \tag{3.7}$$

Then since from its definition we see that $f^{(n)}(\vec{k}_{1\perp},\vec{k}_{2\perp},\vec{k}_{3\perp};\zeta)$ must be of the form

$$f^{(n)}(\vec{k}_{1\perp},\vec{k}_{2\perp},\vec{k}_{3\perp};\zeta) = f_1^{(n)}(\vec{k}_{1\perp}) + f_2^{(n)}(\vec{k}_{2\perp})\,, \tag{3.8}$$

we obtain (suppressing the ζ)

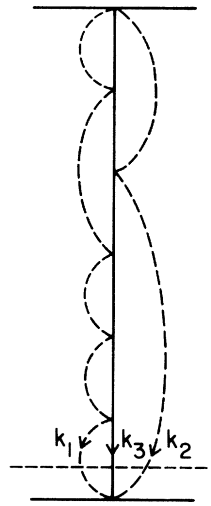

FIG. 4. A typical transverse diagram of eighteenth order which contributes to $\Sigma_3^{(18)}$, showing the cut used to construct $f_1^{(18)}(\vec{k}_{1\perp})$ and $f_2^{(18)}(\vec{k}_{2\perp})$.

$$-g^2\pi i\,\Sigma_3^{(n)} = g^4\zeta^{-1}\int\frac{d^2\vec{k}_{1\perp}}{8\pi^3}\frac{d^2\vec{k}_{2\perp}}{8\pi^3}(\vec{k}_{1\perp}{}^2+\lambda^2)^{-1}(\vec{k}_{2\perp}{}^2+\lambda^2)^{-1}\frac{\vec{\not k}_{1\perp}+\vec{\not k}_{2\perp}+\vec{\not\Delta}-m}{(\vec{k}_{1\perp}+\vec{k}_{2\perp}+\vec{\Delta})^2+m^2}$$
$$\times[(\vec{\not k}_{1\perp}+\vec{\not\Delta}+m)f_1^{(n)}(\vec{k}_{1\perp})+(\vec{\not k}_{2\perp}+\vec{\not\Delta}+m)f_2^{(n)}(\vec{k}_{2\perp})]\,. \tag{3.9}$$

Then, defining

$$f^{(n)}(\vec{k}_\perp) = f_1^{(n)}(\vec{k}_\perp) + f_2^{(n)}(\vec{k}_\perp) \tag{3.10}$$

and summing on n we find

$$-g^2\pi i\Sigma_3 = g^2\zeta^{-1}\int\frac{d^2\vec{k}_\perp}{8\pi^3}\frac{1}{\vec{k}_\perp^2+\lambda^2}\,\alpha(\vec{k}_\perp+\vec{\Delta})f(\vec{k}_\perp)\,, \tag{3.11}$$

where

$$\sum_{n=2}^{\infty} f^{(n)}(\vec{k}_\perp) = f(\vec{k}_\perp)\,. \tag{3.12}$$

We also find from Fig. 3 that

$$f^{(2)}(\vec{k}_\perp) = -\frac{2\pi i g^2\zeta^{-1}}{\vec{\not\Delta}+\vec{\not k}_\perp+m}\,. \tag{3.13}$$

It is now a simple matter to construct a recurrence relation for $f_i^{(n+1)}(\vec{k}_\perp)$ in terms of $f_i^{(n)}(\vec{k}_\perp)$ and we find for $n\geq 2$

$$\begin{bmatrix} f_1^{(n+1)}(\vec{k}_\perp) \\ f_2^{(n+1)}(\vec{k}_\perp)\end{bmatrix} = \zeta^{-1}\alpha(\vec{\Delta}+\vec{k}_\perp)\begin{bmatrix} f_1^{(n)}(\vec{k}_\perp) \\ f_2^{(n)}(\vec{k}_\perp)\end{bmatrix}$$
$$+\zeta^{-1}g^2\int\frac{d^2\vec{k}'_\perp}{8\pi^3}\frac{1}{\vec{k}'^2_\perp+\lambda^2}\frac{(\vec{\not\Delta}+\vec{\not k}_\perp+\vec{\not k}'_\perp)-m}{(\vec{\Delta}+\vec{k}_\perp+\vec{k}'_\perp)^2+m^2}(\vec{\not\Delta}+\vec{\not k}'_\perp+m)\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix} f_1^{(n)}(\vec{k}'_\perp) \\ f_2^{(n)}(\vec{k}'_\perp)\end{bmatrix}\,. \tag{3.14}$$

Adding these two equations together gives for $n\geq 2$

$$f^{(n+1)}(\vec{k}_\perp) = \zeta^{-1}\alpha(\vec{\Delta}+\vec{k}_\perp)f^{(n)}(\vec{k}_\perp)$$
$$+\zeta^{-1}g^2\int\frac{d^2\vec{k}'_\perp}{8\pi^3}\frac{1}{\vec{k}'^2+\lambda^2}\frac{\vec{\not\Delta}+\vec{\not k}_\perp+\vec{\not k}'_\perp-m}{(\vec{\Delta}+\vec{k}_\perp+\vec{k}'_\perp)^2+m^2}(\vec{\not\Delta}+\vec{\not k}'_\perp+m)f^{(n)}(\vec{k}'_\perp)\ . \tag{3.15}$$

Then if we sum on n from 2 to ∞ and use (3.12) and (3.13) we obtain the desired equation for $f(\vec{k}_\perp)$:

$$[1-\zeta^{-1}\alpha(\vec{\Delta}+\vec{k}_\perp)]f(\vec{k}_\perp) = -2\pi i g^2\zeta^{-1}\frac{1}{\vec{\not\Delta}+\vec{\not k}_\perp+m}$$
$$+\zeta^{-1}g^2\int\frac{d^2\vec{k}'_\perp}{8\pi^3}\frac{1}{\vec{k}'^2_\perp+\lambda^2}\frac{(\vec{\not\Delta}+\vec{\not k}_\perp+\vec{\not k}'_\perp)-m}{(\vec{\Delta}+\vec{k}_\perp+\vec{k}'_\perp)^2+m^2}(\vec{\not\Delta}+\vec{\not k}'_\perp+m)f(\vec{k}'_\perp)\ . \tag{3.16}$$

Equations (3.6), (3.10), and (3.16) taken together constitute our final result which is quoted in the first paper of this series.

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