Asymptotic behavior of the elastic form factor in two-dimensional scalar field theory of the bag model*

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(Received 25 September 1975)

In the framework of the two-dimensional scalar quantum theory of the bag model of Chodos *et al.* a definition of the physical field and a general scheme for constructing a physical state are given. We expose some of the difficulties associated with such an approach. Expressions for the physical current and the elastic form factor are given. The calculation of the latter is restricted at first to the approximation in which the mapping from a bag of changing shape to a fixed domain is realized only by a term which is a diagonal, bilinear function of the creation and annihilation operators. This is done for the case of a one-mode and an infinite-mode bag theory. By computing the form factor in an exact one-mode bag model we show that the logarithmic falloff of the asymptotic term is the same as the one in the approximation. On the basis of this a form for the asymptotic behavior of the form factor is suggested which may be correct for the general two-dimensional scalar bag theory.

I. INTRODUCTION

The bag theory[1] is nonlinear, owing to the second-order boundary condition. For the two-dimensional scalar theory[2] there is a nonlinear change of variables $x=x(\sigma)$, where $x=(1/\sqrt{2})(t-z)$, which linearizes the problem and leads to an analytic solution for the field in terms of $\tau=(1/\sqrt{2})(t+z)$ and σ. Upon quantization $x(\sigma)$ becomes a function bilinear in the creation and annihilation operators, and the relation between the initial field in terms of the light-cone coordinates (τ, x) and the field in terms of (τ, σ) is rather complicated. In the general case for an infinite number of modes nobody has been able to solve the inverse problem $\phi(\tau,\sigma)\rightarrow\phi(\tau, x)$ and the mathematical difficulties that plague it stand in the way of resolving the question of causality and computing the form factor and the zero-point energy in the bag theory.

Here I wish to present a calculation for the elastic form factor in a simplified theory, where we have only the lowest modes. We shall evaluate the contributions from different conformal generators introduced by Chodos *et al.*,[1] in our case only $L_0, L_{\pm 2}$. The result is that only L_0, which represents coupling between particle and antiparticle, is important for the asymptotic term of the form factor. The L_0 approximation is shown to be translationally invariant, which makes acceptable the suggestion that this pattern holds even for an infinite number of modes.

The paper is organized as follows: Sec. II introduces the notations, the solutions for the fields in (τ,σ) space, and some of the commutation relations that are used throughout the paper. Readers familiar with Ref. 1 may ignore this section. In Sec. III we give the definition of the physical field and the general scheme for constructing a physical state. We separate the center-of-mass motion and find that the difficulties arise from a Bogoliubov transform. We write an equation for the matrix elements of this transformation, the solution of which we hope will resolve the problems of causality, form factors, and zero-point energy for the two-dimensional scalar bag theory. In Sec. IV we give the definition of the physical current and restrict ourselves to the lowest mode. A calculation for the form factor in the L_0 approximation is given. The latter is shown to be translationally invariant. This example is generalized to the case of an infinite number of modes. In Sec. V we take into account all possible conformal generators for the lowest mode $(L_0, L_{\pm 2})$. We find that up to a constant and a phase the asymptotic behavior of the form factor is the same as the asymptotic form of the L_0 calculation.

II. REVIEW OF THE SCALAR FIELD THEORY OF THE BAG MODEL IN TWO DIMENSIONS

We follow the notation in Ref. 1. ϕ_1, ϕ_2 are two Hermitian fields, which, inside the bag, satisfy the equation of motion

$$\partial^\mu\partial_\mu\phi_i=0 \quad (i=1,2). \tag{2.1}$$

The boundary conditions are

$$\phi_i=0, \tag{2.2a}$$

$$\partial^\mu\phi_i\partial_\mu\phi_i=2B. \tag{2.2b}$$

The metric tensor is $g^{00}=-1$, $g^{11}=1$. The solution of (2.1) in light-cone coordinates has the form

$$\phi_i(x,\tau)=f_i(\tau)+g_i(x). \tag{2.3}$$

The two points which bound the bag are given by $x_k(\tau)$, $k=0,1$. In these notations the boundary conditions are

$$\dot{f}_i(\tau)+\dot{x}_k(\tau)g_i'(x_k(\tau))=0 \quad (i=1,2) \tag{2.4a}$$

which we get from differentiating (2.2a), and

$$\sum_{i=1}^{2}\dot{f}_i(\tau)g_i'(x_k(\tau))=-B\,. \tag{2.4b}$$

The constants of motion are

$$P^+\equiv P=\int_{x_0(\tau)}^{x_1(\tau)}dx\sum_i[g_i'(x)]^2\,, \tag{2.5a}$$

$$P^-\equiv H=B(x_1(\tau)-x_0(\tau))\,, \tag{2.5b}$$

$$M^+\equiv M=H\tau-\int_{x_0(\tau)}^{x_1(\tau)}dx\,x\sum_i[g_i'(x)]^2\,. \tag{2.5c}$$

As found in Ref. 1 there is a nonlinear transformation which defines a new space parameter $\sigma(x)$ and allows a linearization of the problem. In the case of a charged scalar field this is done by the differential equation

$$\frac{dx}{d\sigma}=\frac{1}{P}\sum_{i=1}^{2}[\tilde{g}_i'(\sigma)]^2\,, \tag{2.6}$$

where a new field $\tilde{g}_i(\sigma)$ has been introduced, $\tilde{g}_i(\sigma)\equiv g_i(x(\sigma))$, and P is the light-cone momentum. Now the boundary conditions are

$$\dot{f}_i(\tau)+\dot{\sigma}_k(\tau)\tilde{g}_i'(\sigma_k(\tau))=0\,, \tag{2.7a}$$

$$\sum_{i=1}^{2}\dot{f}_i(\tau)\tilde{g}_i'(\sigma_k(\tau))=-\frac{B}{P}\sum_{j=1}^{2}[\tilde{g}_j'(\sigma_k(\tau))]^2\,. \tag{2.7b}$$

From the expression for the light-cone momentum (2.5a) we find that $\sigma_1(\tau)-\sigma_0(\tau)=1$ and from (2.7a) and (2.7b) the boundary in σ space is

$$\begin{aligned}\sigma_0(\tau)&=B\tau/P\,,\\ \sigma_1(\tau)&=1+B\tau/P\,.\end{aligned} \tag{2.8}$$

It follows from (2.7a) that $\tilde{g}_i(\sigma)$ is periodic in the interval $[\sigma_0,\sigma_1]$ and the solution for the fields in (σ,τ) space can be written. We define $\phi=(1/\sqrt{2})(\phi_1+i\phi_2)$, and the result is

$$\phi=\frac{-i}{(4\pi)^{1/2}}\sum_{n=1}^{\infty}\left[\frac{b_n}{n}(e^{-2\pi inB\tau/P}-e^{-2\pi in\sigma})-\frac{a_n^\dagger}{n}(e^{2\pi inB\tau/P}-e^{2\pi in\sigma})\right], \tag{2.9a}$$

$$\phi^*=\frac{-i}{(4\pi)^{1/2}}\sum_{n=1}^{\infty}\left[\frac{a_n}{n}(e^{-2\pi inB\tau/P}-e^{-2\pi in\sigma})-\frac{b_n^\dagger}{n}(e^{2\pi inB\tau/P}-e^{2\pi in\sigma})\right], \tag{2.9b}$$

where

$$\begin{aligned}b_n&=\frac{1}{\sqrt{2}}(a_n^{(1)}+ia_n^{(2)})\,,\\ a_n&=\frac{1}{\sqrt{2}}(a_n^{(1)}-ia_n^{(2)})\,.\end{aligned} \tag{2.10}$$

The upper index of $a_n^{(i)}$ refers to the Hermitian fields. Note also that $a_n^{(i)\dagger}=a_{-n}^{(i)}$. Upon quantization b_n, a_n become the annihilation operators of charge ±1 particle in state n. Their commutation relations are

$$[a_k,a_n^\dagger]=[b_k,b_n^\dagger]=k\delta_{kn}\,,\quad\text{otherwise zero}\,. \tag{2.11}$$

The light-cone coordinate x is related to σ

$$x(\sigma)=\bar{x}_0+\frac{\Gamma(\sigma)}{P}=\bar{x}_0+\frac{1}{P}\sum_{n=-\infty}^{\infty}c_nL_n\,, \tag{2.12}$$

where

$$c_0=2\pi(\sigma-\tfrac{1}{2})\,,\quad c_n=\frac{i}{n}e^{-2\pi in\sigma} \tag{2.13}$$

and

$$L_n=\frac{1}{2}\sum_{i=1}^{2}\sum_{m=-\infty}^{\infty}a_{-m}^{(i)}a_{m+n}^{(i)}\,, \tag{2.14}$$

which can be expressed in terms of a_n, $a_n^\dagger$, b_n, $b_n^\dagger$.

All conformal generators in (2.12) are understood to be in normal order and

$$L_0=:L_0:+\frac{m_0^{\,2}}{4\pi B}. \tag{2.15}$$

m_0 is the mass of the empty bag (zero-point energy).

The operators L_n obey the algebra

$$[L_n,a_m^{(i)}]=-ma_{n+m}^{(i)}\,, \tag{2.16}$$

$$[L_n,L_m]=(n-m)L_{n+m}+\tfrac{1}{12}n(n^2-1)\delta_{n,-m}\,. \tag{2.17}$$

$\bar{x}_0$ is an operator associated with the center of mass of the bag. $\bar{x}_0$ and P commute with the creation (annihilation) operators and

$$[\bar{x}_0,P]=-i\,. \tag{2.18}$$

H is the light-cone energy and it commutes with $\bar{x}_0,P$; however,

$$[H,b_n^\dagger]=\frac{2\pi nB}{P}b_n^\dagger\,,\quad\text{the same for }a_n^\dagger\,. \tag{2.19}$$

III. SOME GENERAL RESULTS IN THE TWO-DIMENSIONAL BAG THEORY

Here we shall sketch the general approach[3] in constructing a physical field in (τ,x) space from the solution in (τ,σ) space and expose some of the difficulties mentioned in the Introduction. Classically, the physical field is defined throughout space-time by

$$\phi(x,\tau)=\int_{x_0(\tau)}^{x_1(\tau)} dx'\delta(x-x')\phi(x',\tau)$$
$$=\int_{\sigma_0(\tau)}^{\sigma_1(\tau)} d\sigma \frac{dx'(\sigma)}{d\sigma}\delta(x-x'(\sigma))\phi(x'(\sigma),\tau).$$

After an integration by parts we get

$$\phi=\int_{B\tau/P}^{1+B\tau/P} d\sigma\, \theta(x-x'(\sigma))\tilde{g}'(\sigma).$$

The last expression does not make sense when P is an operator, so we shift σ to $\hat{\sigma}=\sigma-B\tau/P$. In a quantum theory of the bag the physical field is

$$\phi^{\text{phys}}(x,\tau)=\int_0^1 d\hat{\sigma}\, \theta(x-x'(\hat{\sigma}+B\tau/P))\tilde{g}'(\hat{\sigma}+B\tau/P). \tag{3.1}$$

Using the exponential representation for the θ function and (2.9a) we can write more explicitly

$$\phi(x,\tau)=\frac{-i}{(4\pi)^{1/2}}\int_0^1 d\sigma\int_{-\infty}^{\infty}\frac{dq}{q-i\epsilon}e^{i\alpha x}e^{-iq(\bar{x}_0+(1/P)\Gamma(\sigma+B\tau/P))}\sum_{n=1}^{\infty}(b_n e^{-2\pi i n(\sigma+B\tau/P)}+a_n^\dagger e^{2\pi i n(\sigma+B\tau/P)}). \tag{3.2}$$

With $\Gamma(\sigma)$ defined in (2.12) and the algebra of the conformal generators (2.17) one finds

$$e^{i\alpha L_0}\Gamma(\sigma)e^{-i\alpha L_0}=\Gamma\left(\sigma+\frac{\alpha}{2\pi}\right)-\alpha L_0. \tag{3.3}$$

Using (2.18) it is easy to prove that

$$e^{i\alpha L_0/P}[\bar{x}_0+\Gamma(\sigma)/P]e^{-i\alpha L_0/P}=\bar{x}_0+\frac{1}{P}\Gamma(\sigma+\alpha/2\pi P). \tag{3.4}$$

This relation enables us to prove translational invariance in time τ for any physical operator. From (2.16) we get

$$[L_0,b_n]=-nb_n,\quad [L_0,a_n^\dagger]=na_n^\dagger \tag{3.5}$$

which leads to

$$e^{-i\beta L_0}b_n e^{i\beta L_0}=e^{in\beta}b_n,$$
$$e^{-i\beta L_0}a_n^\dagger e^{i\beta L_0}=e^{-in\beta}a_n^\dagger. \tag{3.6}$$

If we use in (3.2) the results from (3.4) and (3.6) for $\alpha=2\pi B\tau$, $\beta=2\pi B\tau/P$ one obtains

$$\phi(x,\tau)=e^{2\pi iB\tau L_0/P}\phi(x,0)e^{-2\pi iB\tau L_0/P}.$$

The expression for the light-cone energy (see Ref. 1) is

$$H=2\pi B\frac{L_0}{P}, \tag{3.7}$$

which gives time translation for the physical field

$$\phi(x,\tau)=e^{iH\tau}\phi(x,0)e^{-iH\tau}. \tag{3.8}$$

Let us denote by $|\Omega_q\rangle$ the vacuum in the bag model, i.e., an empty bag of light-cone momentum q. Then a state of momentum q containing n_1 positively charged particles in state n and k_1 negatively charged in state k will be written as

$$|1\rangle\equiv(a_n^\dagger)^{n_1}(b_k^\dagger)^{k_1}|\Omega_q\rangle. \tag{3.9}$$

To construct a physical state we have to find how $\phi(x,\sigma)$ acts on $|1\rangle$. The core of the problem is to determine the action of the exponential operator $e^{-iq[\bar{x}_0+\Gamma(\sigma)/P]}$ on an arbitrary state. As we shall see this operator performs a boost and a unitary transformation of the state. First we separate the center-of-mass motion, which leads to the boost. Let us define

$$f_q(q_1)=e^{-iq[\bar{x}_0+\Gamma(\sigma)P]}|q_1\rangle. \tag{3.10}$$

We differentiate with respect to q and taking into account (2.18) we have the representation

$$\bar{x}_0|q\rangle=i\frac{\partial}{\partial q}|q\rangle. \tag{3.11}$$

Now we multiply both sides by $\exp[-i\Gamma(\sigma)\ln(q_1/m_0)]$ and find

$$\left(\frac{\partial}{\partial q}-\frac{\partial}{\partial q_1}\right)\exp\left[-i\Gamma(\sigma)\ln\left(\frac{q_1}{m_0}\right)\right]f_q(q_1)=0.$$

Using (3.10) as an initial condition for $q=0$ one obtains

$$f_q(q_1)=\exp\left[i\Gamma(\sigma)\ln\left(\frac{q_1}{q+q_1}\right)\right]|q+q_1\rangle. \tag{3.12}$$

Now we come to the difficult task to define properly the formal unitary operation $e^{i\alpha\Gamma(\sigma)}$, which is related to the internal motion of the bag. $e^{i\alpha\Gamma(\sigma)}$ is a linear transformation of the creation (annihilation) operators and is an example of a Bogoliubov transformation,

$$e^{i\alpha\Gamma}a_n^{(i)}e^{-i\alpha\Gamma}\equiv a_n^{(i)}(\alpha)=\sum_m U_{nm}(\alpha)a_m^{(i)}. \tag{3.13}$$

We go back to the amplitudes of the Hermitian fields, because of the simplicity of (2.16). By differentiating with respect to α and using

$$[a_n^{(i)},\Gamma(\sigma)]$$
$$=2\pi(\sigma-\tfrac{1}{2})na_n^{(i)}-i\sum_{l\neq n}\frac{n}{n-l}e^{2\pi i(n-l)\sigma}a_l^{(i)} \tag{3.14}$$

one gets

$$i\frac{da_n^{(i)}(\alpha)}{d\alpha} = 2\pi(\sigma-\tfrac{1}{2})na_n^{(i)}(\alpha) - i\sum_{l\neq n}\frac{n}{n-l}e^{2\pi i(n-l)\sigma}a_l^{(i)}(\alpha). \tag{3.15}$$

For the matrix elements $U_{nm}(\alpha)$ from (3.13) and (3.15) we have the equation

$$i\frac{dU_{nm}(\alpha)}{d\alpha} = 2\pi(\sigma-\tfrac{1}{2})nU_{nm}(\alpha) - i\sum_{l\neq n}\frac{n}{n-l}e^{2\pi i(n-l)\sigma}U_{lm}(\alpha). \tag{3.16}$$

The difficulties associated with solving (3.16) are what made us look for approximations to the general theory. Formally the solution of this equation can be written in an exponential form

$$U_{nm}(\alpha) = [\exp(-i\alpha M)]_{nm}, \tag{3.16$'$}$$

where M is given by

$$M_{nm} = 2\pi(\sigma-\tfrac{1}{2})n\delta_{nm} - i\frac{n}{n-m}e^{2\pi i(n-m)\sigma}(1-\delta_{nm}). \tag{3.16$''$}$$

If we put the physical field $\phi(x,0)$ between two arbitrary states with initial (final) light-cone momentum q_1 (q_2) and take into account the separation of the center-of-mass motion [see (3.12)] we get for q in (3.2) from orthogonality $q=q_2-q_1$ and the field can be written in the form that exhibits translational invariance

$$\phi(x,\tau) = e^{i(H\tau+Px)}\phi(0,0)e^{-i(H\tau+Px)}. \tag{3.17}$$

To show that microcausality holds we need again the solution of Eq. (3.16) to find if

$$[\phi(x,\tau),\phi(x',\tau')]=0 \quad\text{when } (x-x')(\tau-\tau')<0. \tag{3.18}$$

At the present time this is an open question for the two-dimensional scalar bag theory.

IV. PHYSICAL CURRENT AND ELASTIC FORM FACTOR. THE L_0 APPROXIMATION FOR A ONE-MODE AND AN INFINITE-MODE BAG THEORY

In analogy with the discussion in Sec. III we can define classically the physical current as

$$J_\mu(x,\tau) = \int_{x_0(\tau)}^{x_1(\tau)} dx'\delta(x-x')J_\mu(x',\tau) = \int_{B\tau/P}^{1+B\tau/P} d\sigma\frac{dx'(\sigma)}{d\sigma}\delta(x-x'(\sigma))J_\mu(x'(\sigma),\tau).$$

When P is an operator we shift again to $\hat\sigma=\sigma-B\tau/P$:

$$J_\mu^{\text{phys}}(x,\tau) = \int_0^1 d\hat\sigma\frac{dx'(\hat\sigma+B\tau/P)}{d\hat\sigma}\delta(x-x'(\hat\sigma+B\tau/P))J_\mu^B(\hat\sigma,\tau), \tag{4.1}$$

which is the definition of the physical current in a quantum theory of the bag model. I shall refer to $J_\mu^B(\hat\sigma,\tau)$ as the bag current, which is given in the usual form

$$J_\mu^B(\hat\sigma,\tau) = \frac{i}{2}[\phi^*(\partial_\mu\phi)-(\partial_\mu\phi^*)\phi+(\partial_\mu\phi)\phi^*-\phi(\partial_\mu\phi^*)], \tag{4.2}$$

where ϕ and ϕ^* are given by (2.9a) and (2.9b).

The elastic form factor is simply related to the matrix elements

$$F_\pm^n = \langle q_2,n|J_\pm(0,0)|q_1,n\rangle, \tag{4.3}$$

where $|q_1,n\rangle$, $|q_2,n\rangle$ are one-particle states of light-cone momenta q_1,q_2. For example, we take $|q_1,n\rangle\equiv b_n^\dagger|\Omega_{q_1}\rangle$, with $|\Omega_{q_1}\rangle$ being the state of the empty bag with momentum q_1 and $b_n^\dagger$ the creation operator for a positive quark in state n. From current conservation we find the following relation between F_+ and F_-:

$$\frac{4\pi nB+m_0^{\,2}}{2q_1q_2}F_-^n = F_+^n. \tag{4.4}$$

If p_1,p_2 are the ordinary two-dimensional momenta, the form factor is defined as

$$F_\pm^n = -(p_1+p_2)_\pm F^n \tag{4.5}$$

and in terms of light-cone momenta we have

$$F_-^n = (q_1+q_2)F^n. \tag{4.6}$$

To find the form factor for the general theory we have to evaluate the unitary transformation $e^{i\alpha\Gamma(\sigma)}$, with $\Gamma(\sigma)$ given by (2.12), between two arbitrary states

$$\psi_{kn}(\alpha) = \langle\Omega_q|b_ke^{i\alpha\Gamma(\sigma)}b_n^\dagger|\Omega_q\rangle.$$

This is apparent from the definition of the physical current and the discussion in Sec. III. To compute this we need to find the matrices of the Bogoliubov transform, which has not been done to the present day. That is why I restrict myself to the one-mode theory, which leads to a simplified form for $\Gamma(\sigma)$ and a one-dimensional Bogoliubov transform. In our case the only nonzero amplitudes are $a_{\pm1}^{(i)}$. The solution for the fields in (τ,σ) space is given by (2.9a) and (2.9b) where only the $n=1$ term in the sum is retained. The only conformal generators present are L_0 and $L_{\pm 2}$ and

$$x(\sigma)=\bar{x}_0+\frac{1}{P}(c_0L_0+c_1L_2+c_2L_{-2}), \tag{4.7}$$

where $c_1=c_2^*=(i/2)e^{-4\pi i\sigma}$, c_0 is given by (2.13), and

$$L_0=a^\dagger a+b^\dagger b+\frac{m_0^2}{4\pi B}, \tag{4.8}$$

$$L_2=(L_{-2})^\dagger=ab. \tag{4.9}$$

Even though we have a simplified version with only one mode we keep the zero-point constant m_0^2 of the general theory. The commutation relations between conformal generators and creation (annihilation) operators are

$$[L_0,b^\dagger]=b^\dagger,\quad [L_0,b]=-b,\quad \text{the same for } a,a^\dagger \tag{4.10}$$

$$[L_2,b^\dagger]=a,\quad [L_2,a^\dagger]=b,\quad \text{otherwise zero}. \tag{4.11}$$

The relations for L_{-2} are obtained from the Hermitian conjugate of (4.11). The algebra of the conformal generators is different due to the fact that their number is finite

$$[L_{\pm 2},L_0]=\pm 2L_{\pm 2},\quad [L_2,L_{-2}]=L_0+1. \tag{4.12}$$

We shall call the L_0 approximation to the theory the transformation from σ to x space, where only the L_0 operator is kept in the sum:

$$x(\sigma)=\bar{x}_0+\frac{c_0L_0}{P}. \tag{4.13}$$

This applies also in the case of an infinite number of modes. For our simplified theory, using the definition of the current (4.1) and (4.2) we obtain

$$J_-(x,\tau)=\frac{1}{2\pi}\int_0^1 d\sigma\int_{-\infty}^{\infty}dq\,e^{iqx}\exp\left[-iq\left(\bar{x}_0+\frac{c_0L_0}{P}+\frac{2\pi B\tau}{P^2}L_0\right)\right][1-\cos(2\pi\sigma)](b^\dagger b-a^\dagger a). \tag{4.14}$$

As in Sec. III, we can show that

$$e^{2\pi i(B\tau/P)L_0}\left(\bar{x}_0+\frac{c_0L_0}{P}\right)e^{-2\pi i(B\tau/P)L_0}=\bar{x}_0+\frac{c_0L_0}{P}+\frac{2\pi B\tau}{P^2}L_0. \tag{4.15}$$

If we put $J_-(x,\tau)$ between arbitrary initial (final) states with light-cone momentum q_1 (q_2) from orthogonality we get again $q=q_2-q_1$ and this gives [see (3.7)]

$$J_-(x,\tau)=e^{i(Px+H\tau)}J_-(0,0)e^{-i(Px+H\tau)}, \tag{4.16}$$

which shows that the L_0 approximation is translationally invariant. We expect, however, that microcausality will be destroyed by the approximation [see also (3.18)].

We evaluate the exponential operator symmetrically between the initial and final states [see (3.12)]:

$$\left\langle q_2\left|\exp\left[-iq\left(\bar{x}_0+\frac{c_0L_0}{P}\right)\right]\right|q_1\right\rangle=\left\langle q_2-\tfrac{1}{2}q\left|\exp\left[ic_0L_0\ln\left(\frac{q_2-\frac{1}{2}q}{q_2}\right)\right]\exp\left[ic_0L_0\ln\left(\frac{q_1}{q_1+\frac{1}{2}q}\right)\right]\right|q_1+\tfrac{1}{2}q\right\rangle, \tag{4.17}$$

where $|q_i\rangle=b^\dagger|\Omega_{q_i}\rangle$, $i=1,2$. From the last equation using (4.8) and (4.10) we can find that

$$\left\langle q_2\left|\exp\left[-iq\left(\bar{x}_0+\frac{c_0L_0}{P}\right)\right]\right|q_1\right\rangle=\exp\left\{ic_0\left(1+\frac{m_0^2}{4\pi B}\right)\left[\ln\left(\frac{q_2-\frac{1}{2}q}{q_2}\right)+\ln\left(\frac{q_1}{q_1+\frac{1}{2}q}\right)\right]\right\}\langle\Omega_{q_2-q/2}|\Omega_{q_1+q/2}\rangle.$$

The normalization of the states is

$$\langle\Omega_{q_2}|\Omega_{q_1}\rangle=2\pi 2q_1\delta(q_1-q_2). \tag{4.18}$$

Now we can remove the q integration and perform the σ integration using (4.14) for $x=\tau=0$, and the result for the elastic form factor is

$$F=\frac{\sin(\pi a)}{\pi a(1-a^2)}, \tag{4.19}$$

where

$$a=\left(1+\frac{m_0^2}{4\pi B}\right)\ln\left(\frac{q_1}{q_2}\right). \tag{4.20}$$

The relation between $q_1/q_2=x$ and the momentum transfer $t=(p_2-p_1)^2$, where p_1,p_2 are the ordinary momenta, is

$$\frac{t}{2(4\pi B+m_0^2)}=1-\frac{1+x^2}{2x} \tag{4.21}$$

and is linear for x large ($x\gg 1$). $J_+(0,0)$ which is consistent with current conservation [see (4.4)] has the form

$$J_+(0,0)=\int_0^1 d\sigma \frac{2\pi L_0}{P}\delta(x(\sigma))\frac{B}{P}(1-\cos 2\pi\sigma)(b^\dagger b-a^\dagger a)\,. \tag{4.22}$$

The calculation of the form factor in this section can be carried through with no difficulties for an infinite number of modes. In this case $J_-(0,0)$ is

$$J_-(0,0)=\frac{1}{2}\int_0^1 d\sigma\,\delta(x(\sigma))\sum_{m,k}\frac{1}{m}[(a_k^\dagger a_m-b_k^\dagger b_m)e^{2\pi i k\sigma}(1-e^{-2\pi i m\sigma})+(a_m^\dagger a_k-b_m^\dagger b_k)e^{-2\pi i k\sigma}(1-e^{2\pi i m\sigma})$$
$$+(a_m b_k-b_m a_k)e^{-2\pi i k\sigma}(1-e^{-2\pi i m\sigma})+(a_m^\dagger b_k^\dagger-b_m^\dagger a_k^\dagger)e^{2\pi i k\sigma}(1-e^{2\pi i m\sigma})]\,. \tag{4.23}$$

Of course, $x(\sigma)$ is understood in the L_0 approximation. We calculate the diagonal matrix elements of J_- between one-particle states

$$|1\rangle\equiv b_n^\dagger|\Omega_{q_1}\rangle\,,\quad \langle 2|\equiv\langle\Omega_{q_2}|\,b_n\,,$$

$$\langle 2|J_-(0,0)|1\rangle=(q_1+q_2)\int_0^1 d\sigma[1-\cos(2\pi n\sigma)]\exp\left[ic_0\left(n+\frac{m_0^2}{4\pi B}\right)\ln\left(\frac{q_1}{q_2}\right)\right], \tag{4.24}$$

which gives for the form factor

$$F^n=\frac{\sin(\pi a_n)}{\pi a_n(1-a_n^2)}\,, \tag{4.25}$$

where

$$a_n=\left(n+\frac{m_0^2}{4\pi B}\right)\ln\left(\frac{q_1}{q_2}\right).$$

V. THE ELASTIC FORM FACTOR IN ONE-MODE THEORY OF THE BAG MODEL. ASYMPTOTIC FORM.

Now we shall calculate the exact form factor for the one-mode theory. Following the discussion in Sec. III [see (3.12)] we get for F_-

$$F_-(\alpha)=\frac{1}{2\pi}\int_0^1 d\sigma(1-\cos 2\pi\sigma)e^{ic_0\alpha m_0^2/4\pi B}$$
$$\times\langle\Omega|\,be^{i\alpha\Gamma(\sigma)}b^\dagger|\,\Omega\rangle\,, \tag{5.1}$$

with $\alpha=\ln(q_1/q_2)$ and $\Gamma(\sigma)=c_0L_0+c_1L_2+c_2L_{-2}$ [see (4.7)], where L_0 is normal ordered, i.e., $L_0|\Omega\rangle=0$. To compute the matrix element in (5.1) we have to find the one-dimensional Bogoliubov transform

$$e^{i\alpha\Gamma}b^\dagger e^{-i\alpha\Gamma}=f_1(\alpha)b^\dagger+f_2(\alpha)a\,. \tag{5.2}$$

In the general case (3.13) f_1 and f_2 are infinite matrices. Using (4.10) and (4.11) we find

$$[\Gamma(\sigma),b^\dagger]=c_1a+c_0b^\dagger$$

and

$$[\Gamma(\sigma),[\Gamma(\sigma),b^\dagger]]=(c_0^2-c_1c_2)b^\dagger\,.$$

For $f_1(\alpha)$ and $f_2(\alpha)$ we obtain

$$f_1(\alpha)=\cos[\alpha(c_0^2-c_1c_2)^{1/2}]$$
$$+\frac{ic_0}{(c_0^2-c_1c_2)^{1/2}}\sin[\alpha(c_0^2-c_1c_2)^{1/2}]\,, \tag{5.3}$$

$$f_2(\alpha)=\frac{ic_1}{(c_0^2-c_1c_2)^{1/2}}\sin[\alpha(c_0^2-c_1c_2)^{1/2}]\,. \tag{5.4}$$

Also for the transformation of a one gets

$$e^{-i\alpha\Gamma}ae^{i\alpha\Gamma}=f_1(\alpha)a+\frac{c_2}{c_1}f_2(\alpha)b^\dagger\,. \tag{5.5}$$

Using (5.2) we can write

$$\psi(\alpha)\equiv\langle\Omega|\,be^{i\alpha\Gamma}b^\dagger|\,\Omega\rangle$$
$$=f_1(\alpha)\langle\Omega|\,e^{i\alpha\Gamma}|\,\Omega\rangle+f_2(\alpha)\langle\Omega|\,bae^{i\alpha\Gamma}|\,\Omega\rangle\,.$$

From (5.5) this leads to

$$\psi(\alpha)=f_1(\alpha)\langle\Omega|\,e^{i\alpha\Gamma}|\,\Omega\rangle+\frac{c_2}{c_1}f_2^2(\alpha)\psi(\alpha)$$

and a straightforward calculation gives

$$\langle\Omega|\,be^{i\alpha\Gamma}b^\dagger|\,\Omega\rangle=\frac{1}{f_1(-\alpha)}\langle\Omega|\,e^{i\alpha\Gamma}|\,\Omega\rangle\,. \tag{5.6}$$

With c_1, $c_2\to 0$ we recover the result of the L_0 approximation. Let us denote by $\theta(\alpha)$

$$\theta(\alpha)=\langle\Omega|\,e^{i\alpha\Gamma}|\,\Omega\rangle\,. \tag{5.7}$$

If one differentiates with respect to α and uses (4.8) and (4.9) one gets

$$\theta'(\alpha)=ic_1\langle\Omega|\,L_2e^{i\alpha\Gamma}|\,\Omega\rangle$$
$$=ic_2\langle\Omega|\,e^{i\alpha\Gamma}L_{-2}|\,\Omega\rangle\,. \tag{5.8}$$

Recalling the algebra of the conformal generators (4.12) one can prove that

$$4(c_0^{\,2}-c_1c_2)[\Gamma,L_2]=[\Gamma,[\Gamma,[\Gamma,L_2]]]\,,$$

from which it is easy to get

$$e^{-i\alpha\Gamma}L_2e^{i\alpha\Gamma}|\,\Omega\rangle=\left(\frac{ic_2\sin(\alpha\beta)}{\beta}-\frac{2c_2}{\beta^2}[1-\cos(\alpha\beta)](c_0+c_2L_{-2})\right)|\,\Omega\rangle\,, \tag{5.9}$$

where

$$\beta=2(c_0^{\,2}-c_1c_2)^{1/2}=\{[4\pi(\sigma-\tfrac{1}{2})]^2-1\}^{1/2}\,.$$

This leads to the following differential equation for $\theta(\alpha)$

$$\frac{\theta'(\alpha)}{\theta(\alpha)}=\frac{(-i/\beta)c_0c_1c_2\{(2/\beta)[1-\cos(\alpha\beta)]-(i/c_0)\sin(\alpha\beta)\}}{1+(2c_1c_2/\beta^2)[1-\cos(\alpha\beta)]}\,. \tag{5.10}$$

This expression can easily be integrated and gives

$$\theta(\alpha)=e^{-ic_0\alpha}\left(1+\frac{2c_1c_2}{\beta^2}[1-\cos(\alpha\beta)]\right)^{-1/2}\exp\left\{i\arctan\left[\frac{2c_0}{\beta}\tan\left(\frac{\alpha\beta}{2}\right)\right]\right\}\langle\Omega_q|\,\Omega_q\rangle\,.$$

As in Sec. IV $|\,\Omega\rangle$ has light-cone momentum $(q_1+q_2)/2$ and after removing the q integration by using (4.18) we obtain

$$F_-(\alpha)=(q_1+q_2)\int_0^1 d\sigma\,\frac{(1-\cos2\pi\sigma)[1+(1/\beta^2)\sin^2(\frac{1}{2}\alpha\beta)]^{-1/2}}{\cos(\frac{1}{2}\alpha\beta)-(2ic_0/\beta)\sin(\frac{1}{2}\alpha\beta)}\exp\left\{i\arctan\left[\frac{2c_0}{\beta}\tan\left(\frac{\alpha\beta}{2}\right)\right]\right\}\exp\left[i\alpha c_0\left(\frac{m_0^{\,2}}{4\pi B}-1\right)\right]. \tag{5.11}$$

After some trivial algebra and change of variables $x=4\pi(\sigma-\frac{1}{2})\alpha$ we get the following expression for the exact form factor in the one-mode bag theory:

$$F(\alpha)=\frac{1}{4\pi\alpha}\int_{-2\pi\alpha}^{2\pi\alpha}dx\,e^{x\gamma/2i}\left(1+\cos\frac{x}{2\alpha}\right)\left(1+\frac{\alpha^2}{x^2-\alpha^2}\sin^2\frac{(x^2-\alpha^2)^{1/2}}{2}\right)^{-3/2}$$
$$\times\exp\left\{2i\arctan\left[\frac{x}{(x^2-\alpha^2)^{1/2}}\tan\left(\frac{(x^2-\alpha^2)^{1/2}}{2}\right)\right]\right\}\,, \tag{5.12}$$

where

$$\gamma=1-m_0^{\,2}/4\pi B\,.$$

It is impossible to calculate analytically this integral, but one can find the asymptotic behavior of $F(\alpha)$ for large α, or according to (4.21) for large momentum transfer. Integration by parts leads to an expansion in inverse powers of α and the first nonvanishing term comes from differentiating twice $[1+\cos(x/2\alpha)]$. One gets the following result:

$$F^{\text{as}}(\alpha)=-\frac{1}{\pi}\frac{\sin(\pi\alpha\gamma)}{(\alpha\gamma)^3}\left(1+\frac{1}{(2\pi)^2-1}\sin^2\{(\alpha/2)[(2\pi)^2-1]^{1/2}\}\right)^{-3/2}\,. \tag{5.13}$$

Considering $1\gg1/[(2\pi)^2-1]$ we can write

$$F^{\text{as}}(\alpha)\simeq-\frac{1}{\pi}\frac{\sin(\pi\alpha\gamma)}{(\alpha\gamma)^3}\,. \tag{5.14}$$

Up to a constant and a phase factor this is the same as the asymptotic behavior, suggested by the L_0 approximation [see (4.19) and (4.25)]. The contribution of $L_{\pm2}$ is in the constant and the phase, but they do not change the logarithmic falloff of the form factor for large momentum transfer.

VI. DISCUSSION AND SUMMARY

A very different result is suggested by a recent version of the two-dimensional bag-model field theory.[4] There for the exact form factor one has the formula

$$F(t)=-\frac{1}{2-t/4\pi B}\int_{-\pi}^{\pi}\frac{d\theta}{2\pi}\left(2\cos\frac{\theta}{2}\right)^{2-t/4\pi B}\exp\left\{-i\theta\left[\frac{1}{4}\left(\frac{t}{4\pi B}\right)^2-\frac{t}{4\pi B}\right]^{1/2}\right\}\,, \tag{6.1}$$

which has a bad asymptotic behavior for large spacelike momentum transfer ($t<0$)

$$F^{\mathrm{as}}(t)\sim(-t)^{-3/2}2^{-t/4\pi B}\,,\quad \frac{|t|}{4\pi B}\gg 1\,. \qquad (6.2)$$

However, we think that this version is not equivalent to the bag model in Ref. 1 which we have used in the present paper. In Ref. 4 the theory is solved by a functional integration in the Euclidean domain. The assumptions for evaluating the Green's functions in Euclidean space hold true for ordinary perturbation theory, but there is no *a priori* reason to expect that this is valid for an extended nonlinear system like the bag. It is possible, I believe, that in the general case for an infinite number of modes the asymptotic behavior of the form factor is of the form suggested by (4.19), (4.25), and (5.14), namely

$$F^{\mathrm{as}}(t)=c\varphi(t)\left[\ln\left(\frac{-t}{4\pi B+m_0^{\,2}}\right)\right]^{-3}, \qquad (6.3)$$

where c is a constant and $\varphi(t)$ a phase. Certainly we do not have a proof to back this formula at the present time, because this would require solving Eq. (3.16) which is a very difficult problem. Such a solution would answer the question of causality and zero-point energy (m_0) as well as resolve the problem of the form factor.

However, the fact that the L_0 approximation is Lorentz invariant and the example of the one-mode case show that up to a constant and a phase factor maybe only the coupling between particle and antiparticle in L_0 is important for the asymptotic term.

Of course, all generators create L_0 in their commutation relations of the general problem (2.17)

$$[L_n, L_{-n}]=2nL_0+\tfrac{1}{12}n(n^2-1) \qquad (6.4)$$

and in this way affect the constant and the phase of the L_0 approximation.

Very often intuition about the nature of the result helps in proving it, and it is with the hope that this work will stimulate further investigations that the present paper was written.

ACKNOWLEDGMENT

I am most grateful to Professor K. Johnson for suggesting this problem to me and for continuous encouragement and advice. I would like to thank Dr. J. Kiskis for a critical reading of the manuscript, as well as Professor R. Jaffe, Dr. A. Chodos, and Dr. D. Shalloway for helpful discussions.

*This work is supported in part through funds provided by ERDA under Contract No. AT(11-1)-3069.

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[2]This is also true for the fermion case. The corresponding nonlinear transform from x to σ is $\tilde{g}(\sigma)=(dx/d\sigma)^{1/2}g(x(\sigma))$; see Sec. II and D. Shalloway, private communication.

[3]K. Johnson (unpublished).

[4]C. Rebbi, Nucl. Phys. B99, 287 (1975).