Electrodynamics and the equations of motion for charged particles with spin

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We show the extent to which the equations of motion for charged particles with spin, neglecting the radiation-reaction forces, follow from Maxwell equations. The standard derivations depend on cutoff procedures and other prescriptions. The conclusion is that the Lorentz force and the external torque driving the spin are cutoff-independent but other external forces driving the particle are not.

I. INTRODUCTION

In this paper we show the extent to which the equations of motion for charged particles with spin, neglecting the radiation-reaction forces, actually follow from Maxwell equations. This is done exhaustively for charged particles without spin in Ref. 1, which hereafter we denote by T. Definitions and notation we use here are the same of T.

The equation of motion for the particle is derived in the same way as in the above-mentioned paper, but taking into account the existence of the spin to calculate the retarded field $F^{\mu\nu}$ of the electron. Furthermore, we neglect here radiation damping effects, that is, we do not consider the contribution P_e^μ to the total four-momentum P^μ. This equation is cutoff-dependent. The dependence on the cutoff is explicitly calculated. For an arbitrary cutoff the equation of motion for the particle turns out to be inconsistent.

As for the equation of motion of the spin, the invariance under Lorentz transformation ensures that the total angular momentum

$$M^{\mu\nu}=\int_\sigma (T^{\mu\lambda}x^\nu - T^{\nu\lambda}x^\mu)d\sigma_\lambda \tag{1}$$

is independent of the hypersurface σ (under the same assumptions given in T with respect to the conservation of P^μ). This conservation theorem cannot be applied to the electron without modification because $M^{\mu\nu}$ is divergent. As in the four-momentum conservation, integral (1) is evaluated outside the tube Σ, postulating for the case that the electron has no acceleration, that its contribution inside Σ is just $S^{\mu\nu}+m(z^\mu v^\nu - z^\nu v^\mu)$, where $S^{\mu\nu}$ is the electron spin tensor and the second term is its orbital angular momentum. When the electron is accelerated we take Thomas precession into account. The contribution $M_{\rm ext}^{\mu\nu}$ of $T_{\rm ext}^{\mu\nu}$ to $M^{\mu\nu}$ is conserved independently since $T^{\mu\nu}_{{\rm ext},\nu}=0$ and $T_{\rm ext}^{\mu\nu}$ is symmetric. Also we neglect the contribution $M_e^{\mu\nu}$ of $T_e^{\mu\nu}$ to $M^{\mu\nu}$. As in the derivation of the equation of motion for the particle, the integral

$$M_{\rm mix}^{\mu\nu}=\int_\sigma (T_{\rm mix}^{\mu\lambda}x^\nu - T_{\rm mix}^{\nu\lambda}x^\mu)d\sigma_\lambda$$

may be evaluated entirely on the tube Σ, by applying the Gauss integral theorem on $(T_{\rm mix}^{\mu\sigma}x^\nu - T_{\rm mix}^{\nu\sigma}x^\mu)_{,\sigma}=0$.

Anticipating the result (12) for $M_{\rm mix}^{\prime\mu\nu}$ and Eq. (15), the equation of evolution for the spin vector is

$$\frac{D_F S^\rho}{D\tau}=\frac{dS^\rho}{d\tau}+(S\cdot\dot v)v^\rho = h_\lambda^\rho f^{\lambda\mu}S_\mu ,$$

where $D_F/D\tau$ is the Fermi derivative. This equation does not depend on the cutoff and is the well-known equation of Bargmann, Michel, and Telegdi.[3]

Finally we point out that there exists a cutoff that makes the equation of motion for the electron consistent and automatically eliminates some problems in Eq. (13) for $S^{\mu\nu}$.

If we use this cutoff the equations of motion for the electron read

$$\frac{d}{d\tau}(mv^\mu)=ef^{\mu\nu}v_\nu - \tfrac{1}{2}\alpha f^{\alpha\beta,\mu}S_{\alpha\beta} + \tfrac{1}{2}\alpha\frac{d}{d\tau}(f_{\rho\nu}S^{(\mu\rho}v^{\nu)}) ,$$

where $\alpha=ge/2m$ and g is the gyromagnetic ratio of the electron. $S^{(\mu\rho}v^{\nu)}$ stands for $S^{\mu\rho}v^\nu + S^{\rho\nu}v^\mu + S^{\nu\mu}v^\rho$. These are the well-known Bhabha-Corben equations.[2,4]

These equations do not follow from electrodynamics unless extra simplifying assumptions are made. Only the Lorentz force and the external torque driving the spin are reliable since they are cutoff-independent.

II. EQUATION OF MOTION FOR THE ELECTRON

In this section we find the equation of motion for the electron using the same method of Sec. III of T, but taking into account the existence of spin.

We have to calculate

$$P^{\mu}_{\mathrm{mix}} = -\int_{\Sigma} T^{\mu\nu}_{\mathrm{mix}} d\Sigma_{\nu}\,, \tag{2}$$

where the integral has to be evaluated on the tube Σ, as discussed in T.

The limits of integration are from the infinite past up to σ. When Σ shrinks to the electron world line (EWL) $f^{\mu\nu}$ and $f^{\mu\nu,\lambda}$ contribute to P_{mix} on the EWL only. (See Fig. 1.)

The retarded field is obtained from the Liénard-Wiechert potentials,

$$A^{\mu} = \frac{ev^{\mu}}{4\pi\kappa} + \frac{\partial}{\partial x^{\nu}}\left(\frac{J^{\mu\nu}}{4\pi\kappa}\right), \tag{3}$$

where $J^{\mu\nu}$ is the dipole moment of the current associated with the electron. $J^{\mu\nu}$ is necessarily skew-symmetric because of charge conservation.

We assume for simplicity that $J^{\mu\nu}v_{\nu} = 0$, that is, the electron has zero electric moment in its rest system, and the relation between $J^{\mu\nu}$ and the spin tensor $S^{\mu\nu}$,

$$J^{\mu\nu} = \alpha S^{\mu\nu}\,, \tag{4}$$

where α is given in the Introduction.

Let $\epsilon > 0$ be a distance scale such that when $\epsilon \to 0$ the world tube Σ degenerates to the EWL. We choose its equation to be

$$\kappa = \Sigma(\tau, \theta, \phi; \epsilon)\,, \tag{5}$$

and assume the following regularity conditions:

$$\Sigma(\tau, \theta, \phi; \epsilon) = \gamma(\tau, \theta, \phi)\epsilon + O(\epsilon^2)\,. \tag{6}$$

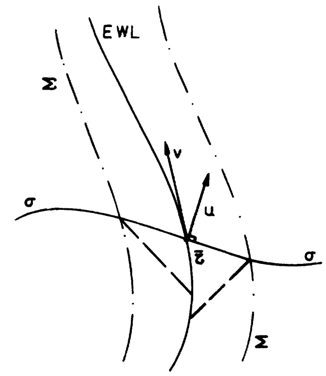

FIG. 1. The world diagram describing the hypersurfaces used in evaluating P^{μ}_{mix} and M^{μ}_{mix}. σ is a spacelike hypersurface cutting the electron world line (EWL) at proper time $\bar{\tau}$. Σ is a timelike tube surrounding the EWL.

The volume element $d\Sigma^{\nu}$ for the tube Σ is given by Eq. (A3):

$$d\Sigma^{\nu} = \kappa^2 d\tau d\Omega_0 \left[k^{\nu}(1 + \Sigma_{\tau} - \kappa(\dot{v}\cdot k)) - v^{\nu} + \frac{\Sigma_{\theta} k^{\nu}_{\theta}}{\Sigma k_{\theta}^{2}} + \frac{\Sigma_{\phi} k^{\nu}_{\phi}}{\Sigma k_{\phi}^{2}}\right].$$

In the limit in which the tube is shrunk to the EWL, the nonvanishing contribution to (2) is obtained by inserting for the electron field the equation

$$\begin{aligned}F^{\mu\nu} \simeq{}& \frac{e}{4\pi\kappa^2}(\kappa_{\perp}^{[\mu} v^{\nu]}) + \frac{\alpha}{4\pi\kappa^2}\{2[\dot{S}^{\mu\nu} - (k\cdot\dot{v})S^{\mu\nu}] + 3\bar{S}'^{[\mu}k^{\nu]} - 2\bar{S}'^{[\mu}v^{\nu]} + S'^{[\nu}k^{\mu]} + S'^{[\nu}k^{\mu]} + \dot{v}^{[\mu}\bar{S}^{\nu]} \\ &- 6(k\cdot\dot{v})\bar{S}^{[\mu}k^{\nu]} + 3(k\cdot\dot{v})\bar{S}^{[\mu}v^{\nu]}\} \\ &+ \frac{\alpha}{4\pi\kappa^3}(2S^{\mu\nu} + 3S^{[\mu}k_{\perp}^{\nu]})\,,\end{aligned} \tag{7}$$

where

$$a^{[\mu}b^{\nu]} \equiv a^{\mu}b^{\nu} - a^{\nu}b^{\mu}$$

and

$$\bar{S}^{\nu} \equiv S^{\nu\alpha}k_{\alpha}\,, \quad \bar{S}'^{\nu} \equiv \dot{S}^{\nu\alpha}k_{\alpha}\,, \quad S'^{\nu} \equiv \dot{S}^{\nu\alpha}v_{\alpha}\,,$$

which is obtained from Ref. 2 by picking terms of the order κ^{-2} and κ^{-3}.

In the same limit, it is sufficient to expand $f^{\mu\nu}$ about the EWL up to terms linear in κ, that is,

$$f_{\mu\nu}(\kappa) \simeq f_{\mu\nu}(0) + f_{\mu\nu,\lambda}(0)(\kappa k^{\lambda})\,. \tag{8}$$

After a simple but long calculation, by use of the partial integration formulas developed in Appendix A of T, the integrals involving $k_{\perp}$'s given in the same appendix, and Eqs. (B5) and (B6) developed in Appendix B, we get

$$\begin{aligned}P^{\mu}_{\mathrm{mix}}(\bar{\tau}) ={}& -\int_{-\infty}^{\tau} d\tau \left[ef^{\mu\nu}v_{\nu} - \frac{\alpha}{2}f^{\rho\nu,\mu}S_{\rho\nu} + \alpha\frac{d}{d\tau}(f_{\rho\nu}S^{(\mu\rho}v^{\nu)}(I + \tfrac{1}{3})) + 3\alpha\frac{d}{d\tau}(f_{\rho\nu}S^{\beta(\mu}v^{\rho}I^{\nu)}_{\beta}) + \alpha\dot{v}_{\beta}I^{\beta}f_{\rho\nu}S^{(\mu\rho}v^{\nu)}\right] \\ &+ \frac{\alpha u^{\lambda}}{u\cdot v}[f^{\mu\sigma}(S_{\sigma\lambda}(2I - \tfrac{1}{3}) + 3(S_{\sigma\nu}I^{\nu}_{\lambda} - S_{\lambda\nu}I^{\nu}_{\sigma})) + 2f_{\sigma\nu}(S^{(\mu\sigma}h^{\nu)}_{\lambda}(\tfrac{1}{3} + I) + 3S^{\beta\,(\mu}h^{\sigma}_{\lambda}I^{\nu)}_{\beta})]\,,\end{aligned} \tag{9}$$

where

$$I=\int(\ln\gamma)\frac{d\Omega_0}{4\pi}\,,$$

$$I^{\alpha}=\int(\ln\gamma)k_\perp^{\alpha}\frac{d\Omega_0}{4\pi}\,, \tag{10}$$

$$I^{\alpha\beta}=\int(\ln\gamma)k_\perp^{\alpha}k_\perp^{\beta}\frac{d\Omega_0}{4\pi}\,,$$

and u^λ denotes the unit normal to the hypersurface σ at the EWL. $S^{(\mu\rho}v^{\nu)}$ has been defined in the Introduction.

We point out that the divergent terms in (2) cancel each other. We see that the equation of motion for the electron

$$\frac{d}{d\tau}(mv^\mu+P^\mu_{\rm mix})=0$$

is cutoff-dependent.

If we neglect the cutoff-dependent terms, which is equivalent to using the Bhabha tube (i.e., $\kappa=$ constant), we get the equation

$$\frac{d}{d\tau}(mv^\mu)=ef^{\mu\nu}v_\nu-\frac{\alpha}{2}f^{\rho\beta,\mu}S_{\rho\beta}+\frac{\alpha}{3}\frac{d}{d\tau}(f_{\rho\beta}S^{(\mu\rho}v^{\beta)})\,,$$

which is not compatible with $v^\mu v_\mu=1$ unless the internal four-momentum mv^μ is redefined by adding terms dependent on the external field $f^{\mu\nu}$.

III. EQUATION OF MOTION FOR THE SPIN

In this section we find the equation of motion for the spin by computing $M^{\mu\nu}_{\rm mix}$ as was discussed in the Introduction.

However, since electrodynamics fails near the electron it is necessary to make some extra assumptions. In a good theory we expect that near the electron the field, no longer a solution of the Maxwell equations, gives a finite contribution to $M^{\mu\nu}$. We assume that the solution of this field, corresponding to a spinning electron with constant velocity, gives $S^{\mu\nu}+mz^{[\mu}v^{\nu]}$ for $M^{\mu\nu}$.

When the electron is accelerated we imagine its world line broken up into short uniform motions, as shown in Fig. 2. When the spin does not couple to the external field it is natural to assume that the fields, near the electron, for two consecutive uniform motions are related by a Lorentz transformation without rotation that transforms v_1^μ into v_2^μ, say (see Fig. 2). This implies that $S^{\mu\nu}$ is Fermi-Walker-transported along the EWL. S^μ precesses with Thomas angular velocity with respect to an inertial frame. When $S^{\mu\nu}$ couples to the external field, in addition to the precession, it acquires an extra variation. This amounts to the fact that the true variation of $S^{\mu\nu}$ in proper time

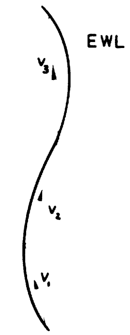

FIG. 2. The electron world line (EWL) is broken up to consider Thomas precession.

$d\tau$ is $(D_F S^{\mu\nu}/D\tau)d\tau$ instead of $(dS^{\mu\nu}/d\tau)d\tau$.[5] A more familiar example of this phenomenon is the relativistic definition of constant acceleration. Constant acceleration means $D_Fa^\mu/D\tau=da^\mu/d\tau+a^2v^\mu=0$ and not $da^\mu/d\tau=0$.[6]

Then from electrodynamics and this extra assumption we have

$$\frac{D_F}{D\tau}(S^{\mu\nu}+mz^{[\mu}v^{\nu]})=-\frac{d}{d\tau}(M^{\mu\nu}_{\rm mix})\,,$$

where

$$\frac{D_F}{D\tau}(S^{\mu\nu})=\dot S^{\mu\nu}+(S^{\sigma\nu}a_\sigma)v^\mu+(S^{\mu\sigma}a_\sigma)v^\nu$$

is the Fermi derivative of $S^{\mu\nu}$.

If we make $x^\mu=z^\mu+\kappa k^\mu$ in the above equation and use the Gauss integral theorem as discussed in the Introduction we have

$$\frac{D_F}{D\tau}(S^{\mu\nu})=-\frac{d}{d\tau}(M'^{\mu\nu}_{\rm mix})\,,$$

where

$$M'^{\mu\nu}_{\rm mix}=-\int_\Sigma T^{\lambda[\mu}_{\rm mix}(\kappa k^{\nu]})d\sigma_\lambda\,. \tag{11}$$

The limits of integration are from the infinite past up to σ. When Σ shrinks to the EWL only values of $f^{\mu\nu}$ on the EWL contribute to $M'^{\mu\nu}_{\rm mix}$.

In this limit the nonvanishing contribution to (11) is obtained from (7) by considering only the leading terms

$$F^{\mu\nu}\simeq\frac{\alpha}{4\pi\kappa^3}2S^{\mu\nu}+3(S^\mu k^\nu-S^\nu k^\mu)\,.$$

In the same limit, from (A3), we have

$$d\Sigma^\nu=\kappa^2d\tau d\Omega_0k^\nu+(\ln\gamma)_\theta\frac{k_\theta^\nu}{k_\theta^{\;2}}+(\ln\gamma)_\phi\frac{k_\phi^\nu}{k_\phi^{\;2}}\,.$$

After a simple calculation, by use of the partial integration formulas developed in Appendix A of T, and the integrals involving k's, we get

$$M'^{\mu\nu}_{\rm mix}(\tau) = -\int_{-\infty}^{\tau} d\tau[f^{\lambda[\mu}S^{\nu]}{}_{\lambda} + f_{\sigma\lambda}v^{\sigma}(\tfrac{2}{3}+2I)(S^{\lambda[\mu}v^{\nu]}) + f_{\sigma\lambda}S^{\sigma\lambda}I^{[\mu}v^{\nu]} + 2f_{\sigma\lambda}I^{\sigma}S^{\lambda[\mu}v^{\nu]} + S_{\sigma\lambda}I^{\sigma}f^{\lambda[\mu}v^{\nu]}$$
$$+ 3f_{\sigma\lambda}v^{\sigma}I^{\lambda}_{\alpha}S^{\alpha[\mu}v^{\nu]} + 3f_{\sigma\lambda}v^{\sigma}S^{\lambda\alpha}I^{[\mu}_{\alpha}v^{\nu]}] . \tag{12}$$

and therefore the equation of motion for the spin tensor is

$$\frac{D}{D\tau}(S^{\mu\nu}) = \dot{S}^{\mu\nu} + (S^{\mu\sigma}\dot{v}_{\sigma})v^{\nu} + (S^{\sigma\nu}\dot{v}_{\sigma})v^{\mu}$$
$$= -\frac{dM'^{\mu\nu}_{\rm mix}}{d\tau} . \tag{13}$$

Let S^{ρ} be the dual of $S^{\mu\nu}$ in the flat space orthogonal to v^{μ}, i.e.,

$$S^{\mu\rho} = \epsilon^{\mu\rho\alpha\beta}v_{\alpha}S_{\beta}$$

and (14)

$$S^{\beta}v_{\beta} = 0 .$$

Then, from (12) and (13) we get the equation for the spin vector S^{ρ}

$$\frac{DS^{\rho}}{D\tau} = \dot{S}^{\rho} + (S\cdot\dot{v})v^{\rho} = S^{\mu}h^{\rho\lambda}f_{\lambda\mu} , \tag{15}$$

which is cutoff-independent and is the well-known equation of Bargmann, Michel, and Telegdi.[3]

From (15) we have that $S^2 \equiv S^{\rho}S_{\rho} = \text{const}$ and we see that this equation is compatible with $S^{\rho}v_{\rho} = 0$.

For an arbitrary cutoff (including the tube of Bhabha) eq. (13) is not consistent with the condition $S^{\mu\nu}v_{\mu} = 0$ that we impose at the beginning.

It is important to point out that there exists a cutoff (or possibly a family of cutoffs) that eliminates simultaneously the problems we have in the equation of motion for the electron and in the equation of motion for the spin. This cutoff is defined by the following:

(i) u^{μ} is parallel to v^{μ}, and
(ii) $\ln\gamma = -\frac{5}{4}(k\cdot S)^2/S\cdot S$.

In this case, by use of Eqs. (10) and those involving integrals of k's we get

$$I = \tfrac{5}{12} ,$$

$$I^{\alpha} = 0 ,$$

and

$$I^{\alpha\beta} = -\tfrac{1}{12}(h^{\alpha\beta} + 2S^{\alpha}S^{\beta}/S^2) ,$$

and the equation of motion for the electron becomes

$$\frac{d}{d\tau}(mv^{\mu}) = ef^{\mu\nu}v_{\nu} - \frac{\alpha}{2}f^{\alpha\beta,\mu}S_{\alpha\beta} + \frac{\alpha}{2}\frac{d}{d\tau}(f_{\alpha\beta}S^{(\mu\alpha}v^{\beta)}) , \tag{16}$$

which is the equation of Bhabha and Corben.[2,4]

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APPENDIX A

Here, we calculate the volume element for Σ. Let $d\Sigma_{\mu}$ be the volume element of Σ, that is,

$$d\Sigma_{\mu} = \epsilon_{\mu\nu\lambda\rho}d_{\tau}x^{\nu}d_{\theta}x^{\lambda}d_{\phi}x^{\rho} , \tag{A1}$$

where $\epsilon_{\mu\nu\lambda\rho}$ is the Levi-Civita permutation symbol and $d_{\tau}x^{\nu}$, $d_{\theta}x^{\lambda}$, and $d_{\phi}x^{\rho}$ are three independent displacements within Σ; $d_{\tau}x^{\nu}$ is the displacement when τ alone varies and so on.

From Eq. (T9) and (9) we have

$$d_{\tau}x^{\nu} = [v^{\nu} - \kappa(\dot{v}\cdot k)k^{\nu} + k^{\nu}\Sigma_{\tau}]d\tau ,$$

$$d_{\theta}x^{\lambda} = (\kappa k^{\lambda}_{\theta} + k^{\lambda}\Sigma_{\theta})d\theta ,$$

$$d_{\phi}x^{\rho} = (\kappa k^{\rho}_{\phi} + k^{\rho}\Sigma_{\phi})d\phi ,$$

which when replaced in (A1) gives us

$$d\Sigma_{\mu} = \kappa^2 d\tau d\theta d\phi\epsilon_{\mu\nu\lambda\rho}\Big[v^{\nu}k^{\lambda}_{\theta}k^{\rho}_{\phi} + v^{\nu}k^{\lambda}k^{\rho}_{\phi}\frac{\Sigma_{\theta}}{\Sigma} + v^{\nu}k^{\rho}k^{\lambda}_{\theta}\frac{\Sigma_{\phi}}{\Sigma}$$
$$+ [\Sigma_{\tau} - \kappa(k\cdot\dot{v})](k^{\lambda}_{\theta}k^{\rho}_{\phi}k^{\nu} + k^{\lambda}_{\theta}k^{\rho}_{\phi}v^{\nu})\Big] . \tag{A2}$$

The first term in (A2) is orthogonal to $v^{\mu}, k^{\mu}_{\theta}$, k^{μ}_{ϕ}, and is therefore parallel to $k^{\mu}_{\perp}$. The following terms are parallel to k^{μ}_{θ}, k^{μ}_{ϕ}, v^{μ}, and $k^{\mu}_{\perp}$, respectively. The proportionality factors are calculated by contraction with $k^{\mu}_{\perp}$, k^{μ}_{θ}, k^{μ}_{ϕ}, v^{μ}, and $k^{\mu}_{\perp}$, respectively.

Finally,

$$d\Sigma^{\mu} = \kappa^2 d\tau d\Omega_0\Big[k^{\mu}[1 + \Sigma_{\tau} - \kappa(k\cdot\dot{v})] - v^{\mu} + \frac{\Sigma_{\phi}k^{\mu}_{\phi}}{\Sigma k_{\phi}{}^2} + \frac{\Sigma_{\theta}k^{\mu}_{\theta}}{\Sigma k_{\theta}{}^2}\Big] , \tag{A3}$$

where

$$d\Omega_0 = \epsilon_{\lambda\mu\nu\rho}v^{\lambda}k^{\mu}k^{\nu}_{\theta}k^{\rho}_{\phi}d\theta d\phi \tag{A4}$$

is the solid-angle element for the inertial frame with time axis v^{λ}.

From (A4) and (T8) we get

$$\frac{\partial}{\partial\tau}(d\Omega_0) = -2(k\cdot\dot{v})d\Omega_0 , \tag{A5}$$

which is used repeatedly along the calculations of Sec. II.

APPENDIX B

We explain now how the integral

$$J=\int_\Sigma d\tau\, d\Omega_0 \frac{f(\tau,\theta,\phi)}{\kappa} \tag{B1}$$

(which is divergent when $\epsilon\to 0$) can be decomposed in a convenient way for the calculation of $P^\mu_{\rm mix}$.

The limits of integration in (B1) are from the remote past up to σ. We assume that $f(\tau,\theta,\phi)/\kappa(\tau,\theta,\phi)$ admits expansion about $\bar\tau$, the proper time of the EWL at which σ cuts it.

We decompose J as follows:

$$\begin{aligned} J&=\int d\Omega_0 \int_{-\infty}^{\tau'(\Omega)} d\tau \frac{f(\tau,\theta,\phi)}{\kappa(\tau,\theta,\phi)} \\ &=\int_{-\infty}^{\bar\tau} d\tau \int d\Omega_0\, f(\tau,\theta,\phi)\frac{1}{\kappa} \\ &\quad+\int d\Omega_0 \int_{\bar\tau}^{\tau'(\Omega)} d\tau \frac{f(\tau,\theta,\phi)}{\kappa}\,. \end{aligned} \tag{B2}$$

In the limit $\epsilon\to 0$, in the second integral on the right-hand side of (B2) it is sufficient to replace

$$\frac{f}{\kappa}(\tau,\theta,\phi)\simeq\frac{f}{\kappa}(\bar\tau)\,, \tag{B3}$$

and then integrate over τ. Let u^μ denote the unit normal to σ in the EWL. Then, in the same limit

$$\bar\tau-\tau'\simeq\kappa\frac{k\cdot u}{v\cdot u}\,. \tag{B4}$$

If we make use of (B3) and (B4) in (B2) we finally get

$$J=\int_{-\infty}^{\bar\tau} d\tau \int d\Omega_0 \frac{f}{\kappa}(\tau,\Omega)-\int d\Omega_0 \frac{k\cdot u}{v\cdot u} f(\bar\tau)\,. \tag{B5}$$

Some integrals involving $k_\perp$'s are given in Appendix A of T. Here, for completeness we give the additional integral

$$\int k_\perp^\alpha k_\perp^\beta k_\perp^\mu k_\perp^\nu \frac{d\Omega_0}{4\pi}=\tfrac{1}{15}(h^{\alpha\beta}h^{\mu\nu}+h^{\alpha\mu}h^{\beta\nu}+h^{\alpha\nu}h^{\beta\mu})\,, \tag{B6}$$

where

$$h^{\mu\nu}=\eta^{\mu\nu}-v^\mu v^\nu$$

and

$$k_\perp^\mu=k^\mu-v^\mu\,.$$

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