Symmetries of electrodynamics with magnetic monopoles and the Hertz tensor

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We discuss discrete symmetries of the Maxwell equations with Dirac poles, written in manifestly duality-invariant form by means of the Hertz tensor. It is shown that in this framework the theory may give rise to parity and time-reversal violation only for systems without electric charges. When both electric and magnetic charges are present, the pseudovector character of the magnetic current—insuring parity and time-reversal invariance of the Maxwell equations—is a direct consequence of the vector transformation properties of the electric current.

I. INTRODUCTION

In the past, many authors[1-5] have considered the problem of discrete symmetries in electrodynamics with magnetic monopoles.[6,7] Different answers[1-5] were given to the question of whether the existence of magnetic charges may give rise to violation of parity and time-reversal invariance. In this paper, we want to re-examine the problem of the invariance properties of the Maxwell equations, when they are written in terms of the so-called Hertz tensor.[8-10]

The Hertz tensor is an antisymmetric tensor of rank two, whose nonzero components are the components of the so-called electric Hertz vector (introduced by Hertz in 1889) and the magnetic Hertz vector (introduced by Sommerfeld[9] in 1935). By means of the Hertz tensor, Han and Biedenharn[10] succeeded in writing the Maxwell equations with Dirac poles in a form that is manifestly "duality invariant,"[10] i.e., invariant under the "generalized duality transformations"[11] of both fields and sources. The most interesting consequence of this duality-invariant form of electromagnetism is that it automatically yields the Cabibbo-Ferrari[2] equation for the electromagnetic field tensor in terms of two vector potentials, contemporarily providing a physical basis[12] to the Cabibbo-Ferrari theory, by explaining the origin of the "gauge-mixing transformations" between the two four-potentials, and the absence of new degrees of freedom for the photon.[10]

In Sec. II we briefly review the duality-invariant formulation of electromagnetism. In Sec. III we discuss the behavior of the Maxwell equations, written in terms of the Hertz tensor, under parity, time-reversal, and charge conjugation, in the cases when (a) only electric charges, (b) only magnetic charges, and (c) both electric and magnetic charges are present. We show that *only in case (b)* may the theory be affected by symmetry violations. In case (c), the Maxwell equations are both parity and time-reversal invariant, and the pseudovector character[13] of the magnetic four-current g^μ is automatically required by the vector transformation properties of the electric current j^μ.

II. DUALITY-INVARIANT FORM OF MAXWELL EQUATIONS

Let us briefly review[14] the Han-Biedenharn formulation of electromagnetism.[10] The manifestly duality-invariant form of the Maxwell equations with magnetic poles is

$$\Box \mathcal{H}^{\mu\nu} = \mathcal{S}^{\mu\nu} \tag{1}$$

($\Box \equiv \partial^2/\partial t^2 - \nabla^2$), where the *Hertz tensor* $\mathcal{H}^{\mu\nu}$ is defined by

$$\mathcal{H}^{\mu\nu} \equiv \begin{bmatrix} 0 & \pi_x & \pi_y & \pi_z \\ & 0 & \Sigma_z & -\Sigma_y \\ & & 0 & \Sigma_x \\ & & & 0 \end{bmatrix}, \quad \mathcal{H}^{\mu\nu} = -\mathcal{H}^{\nu\mu}, \tag{2}$$

and the *source tensor* $\mathcal{S}^{\mu\nu}$ by

$$\mathcal{S}^{\mu\nu} \equiv \begin{bmatrix} 0 & P_x & P_y & P_z \\ & 0 & Q_z & -Q_y \\ & & 0 & Q_x \\ & & & 0 \end{bmatrix}, \quad \mathcal{S}^{\mu\nu} = -\mathcal{S}^{\nu\mu}. \tag{3}$$

Three-vectors $\vec{\pi}, \vec{\Sigma}$ and $\vec{P}, \vec{Q}$ are the so-called electric and magnetic Hertz vectors and polarization and magnetization vectors, respectively.

From $\mathcal{H}^{\mu\nu}$, one may obtain two four-potentials through the relations

$$A^\mu = \partial_\lambda \mathcal{H}^{\lambda\mu} \tag{4}$$

and

$$B^\mu = \partial_\lambda \tilde{\mathcal{H}}^{\lambda\mu} \tag{5}$$

$\tilde{\mathcal{K}}^{\mu\nu}$ is the dual of $\mathcal{K}^{\mu\nu}$, according to the definition

$$\tilde{\mathcal{K}}^{\mu\nu} \equiv \tfrac{1}{2}\epsilon^{\mu\nu\lambda\sigma}\mathcal{K}_{\lambda\sigma}\,, \tag{6}$$

where $\epsilon^{\mu\nu\lambda\sigma}$ is the totally antisymmetric Ricci tensor, normalized so that $\epsilon^{0123}=1$. A^μ, given by Eq. (4), is the usual electromagnetic four-potential, whereas it may be shown[10] that B^μ is the second four-potential entering the Cabibbo-Ferrari relation

$$\mathfrak{F}^{\mu\nu} = \partial^\nu A^\mu - \partial^\mu A^\nu + \epsilon^{\mu\nu\lambda\sigma}\partial_\lambda B_\sigma \tag{7}$$

that holds when both electric and magnetic charges are present.[2]

Electric and magnetic currents, j^μ and g^μ, are derived from the source tensor $\mathcal{S}^{\mu\nu}$ in the following way:

$$j^\mu = \partial_\lambda \mathcal{S}^{\lambda\mu}\,, \tag{8}$$

$$g^\mu = \partial_\lambda \tilde{\mathcal{S}}^{\lambda\mu} \tag{9}$$

$[j^\mu = (\rho, \vec{j});\ g^\mu = (\sigma, \vec{g})]$. Explicitly, Eqs. (8) and (9) read

$$\rho = -\vec{\nabla}\cdot\vec{P}\,,\quad \vec{j} = (\partial\vec{P}/\partial t) - \vec{\nabla}\times\vec{Q}\,, \tag{8$'$}$$

$$\sigma = -\vec{\nabla}\cdot\vec{Q}\,,\quad \vec{g} = (\partial\vec{Q}/\partial t) + \vec{\nabla}\times\vec{P}\,. \tag{9$'$}$$

From the four-potentials A^μ, B^μ one may define two electromagnetic tensors in the usual way:

$$F^{\mu\nu} = \partial^\nu A^\mu - \partial^\mu A^\nu\,, \tag{10}$$

$$G^{\mu\nu} = \partial^\nu B^\mu - \partial^\mu B^\nu\,. \tag{11}$$

Han and Biedenharn showed[10] that (i) $F^{\mu\nu}$ and $G^{\mu\nu}$ coincide in a source-free region; (ii) $F^{\mu\nu}$ is the appropriate electromagnetic tensor in the absence of magnetic charges, whereas tensor $G^{\mu\nu}$ must be used when no electric charge is present; (iii) the total field tensor

$$\mathfrak{F}^{\mu\nu} \equiv F^{\mu\nu} - \tilde{G}^{\mu\nu} = \tilde{F}^{\mu\nu} + G^{\mu\nu} \tag{12}$$

enters the Maxwell equations with *both* electric *and* magnetic charges.[15]

Specifying the physical case under consideration results in *different* equations for the electric and magnetic fields in terms of $\vec{\pi}$ and $\vec{\Sigma}$.[10] In fact, from Eqs. (2)–(5) and (10)–(12), one obtains the following:

(a) no magnetic charge

$$\vec{E} = \vec{\nabla}\times(\partial\vec{\Sigma}/\partial t + \vec{\nabla}\times\vec{\pi}) - \Box\vec{\pi}\,, \tag{13}$$

$$\vec{H} = \vec{\nabla}\times(\partial\vec{\pi}/\partial t - \vec{\nabla}\times\vec{\Sigma})\,; \tag{14}$$

(b) no electric charge

$$\vec{E}' = \vec{\nabla}\times(\partial\vec{\Sigma}/\partial t + \vec{\nabla}\times\vec{\pi})\,, \tag{15}$$

$$\vec{H}' = \vec{\nabla}\times(\partial\vec{\pi}/\partial t - \vec{\nabla}\times\vec{\Sigma}) + \Box\vec{\Sigma}\,; \tag{16}$$

(c) both electric and magnetic charge

$$\vec{\mathcal{E}} = -\Box\vec{\pi}\,, \tag{17}$$

$$\vec{\mathcal{K}} = -\Box\vec{\Sigma}\,. \tag{18}$$

III. SYMMETRIES OF THE THEORY

Having outlined the features of duality-invariant electromagnetism relevant to us, we are now able to discuss symmetry properties of the Maxwell equations in this framework. We shall deal with cases (a), (b), and (c) of the preceding section. As usual, we shall denote by C, P, and T charge-conjugation, parity, and time-reversal operations. Obviously, in all cases we must require ordinary transformation properties of electric and magnetic fields, i.e.,

$$P(\vec{x}\to-\vec{x}):\ \vec{E}\to-\vec{E}\,,\ \ \vec{H}\to\vec{H}\,; \tag{19}$$

$$T(t\to-t):\ \vec{E}\to\vec{E}\,,\ \ \vec{H}\to-\vec{H}\,; \tag{19$'$}$$

$$C(e\to-e):\ \vec{E}\to-\vec{E}\,,\ \ \vec{H}\to-\vec{H}\,. \tag{19$''$}$$

Such a requirement univocally determines the corresponding transformation laws for electric and magnetic Hertz vectors. Indeed, by taking into account Eqs. (13)–(18), one easily finds that, for all systems, the behavior of $\vec{\pi}$ and $\vec{\Sigma}$ under C, P, T is given by

$$P:\ \vec{\pi}\to-\vec{\pi}\,,\ \ \vec{\Sigma}\to\vec{\Sigma}\,; \tag{20}$$

$$T:\ \vec{\pi}\to\vec{\pi}\,,\ \ \vec{\Sigma}\to-\vec{\Sigma}\,; \tag{20$'$}$$

$$C:\ \vec{\pi}\to-\vec{\pi}\,,\ \ \vec{\Sigma}\to-\vec{\Sigma}\,. \tag{20$''$}$$

As regards transformation properties of sources, we need to analyze separately the physical systems of cases (a)–(c).

Case (a) corresponds to standard electromagnetism without Dirac poles. By imposing the usual transformation laws to the electric four-current j^μ, and taking into account Eqs. (8$'$), we obtain for the polarization and magnetization vectors the behavior expressed by the following equations:

$$P:\ \vec{P}\to-\vec{P}\,,\ \ \vec{Q}\to\vec{Q}\,; \tag{21}$$

$$T:\ \vec{P}\to\vec{P}\,,\ \ \vec{Q}\to-\vec{Q}\,; \tag{21$'$}$$

$$C:\ \vec{P}\to-\vec{P}\,,\ \ \vec{Q}\to-\vec{Q}\,. \tag{21$''$}$$

Obviously—as it must be in this case—transformations (20)–(20$''$) and (21)–(21$''$) together make the Maxwell equations invariant under C, P, and T separately.

Next, let us consider a system made of both electric and magnetic charges [so that we are dealing now with case (c) of Sec. II]. We again have to require standard behavior of the electric current under symmetry operations, i.e., Eqs.

(21)–(21″) still hold for vectors $\vec{P}$ and $\vec{Q}$. But, according to Eqs. (9′), *such a requirement univocally fixes the transformation properties of the magnetic four-current* g^μ, namely

$$P:\ \sigma\to-\sigma,\quad \vec{g}\to\vec{g}; \tag{22}$$

$$T:\ \sigma\to-\sigma,\quad \vec{g}\to\vec{g}; \tag{22′}$$

$$C:\ \sigma\to-\sigma,\quad \vec{g}\to-\vec{g}. \tag{22″}$$

Equations (22) and (22′) express the pseudovector character of g^μ with respect to space and time reflections. As regards Eq. (22′), one immediately realizes that, in this context, the operation we called "charge conjugation" actually means conjugation of both electric and magnetic charge.[16] It is well known[7] that a pseudovector behavior of magnetic current ensures C, P, and T invariance of electromagnetism for systems with both kinds of charges. We may therefore conclude that the Maxwell equations, when written in terms of the Hertz tensor, exhibit invariance under space, time, and charge reflection whenever both electric and magnetic charges are present. Notice that here "invariance" must be understood in the sense of Pintacuda's "weak invariance."[3]

Lastly, let us assume that only magnetic monopoles are present [case (b)]. Now, in stating transformation laws for $\vec{P}$ and $\vec{Q}$, we are not guided, as before, by the presence and vector properties of the electric current. Indeed, two possibilities are open in this case. The first is assuming Eqs. (21)–(21″) still hold for $\vec{P}$ and $\vec{Q}$ (whence the pseudovector character of g^μ follows): Such a choice clearly implies C, P, and T invariance of the Maxwell equations. The second one is given by the following equations:

$$P:\ \vec{P}\to\vec{P},\quad \vec{Q}\to-\vec{Q}; \tag{23}$$

$$T:\ \vec{P}\to-\vec{P},\quad \vec{Q}\to\vec{Q}; \tag{23′}$$

$$M:\ \vec{P}\to-\vec{P},\quad \vec{Q}\to-\vec{Q} \tag{23″}$$

(in the last equation, we replaced C by M to emphasize that now charge conjugation actually means magnetic pole conjugation[1,2,15]). One readily realizes that Eqs. (23)–(23″) imply the magnetic current behaving as a four-vector under space and time reflections. Thus, *this second choice of transformation laws for the components of the source tensor leads to noninvariance of the Maxwell equations* (for systems containing only magnetic charges) under parity and tine-reversal separately. Obviously, one may recover invariance introducing, in the place of P and T, the operations $T'\equiv TM$ and $P'\equiv PM$ (see Refs. 1, 2, and 7).

TABLE I. The behavior of electromagnetic quantities under the C, P, and T operations.

Case		Symmetry	$\vec{P}$	$\vec{Q}$	$\vec{\Sigma}$	$\vec{\pi}$	σ	$\vec{g}$
(a)		P	−	+	+	−		
		T	+	−	−	+		
		C	−	−	−	−		
(b)	First choice	P	−	+	+	−	−	+
		T	+	−	−	+	−	+
		M	−	−	−	−	−	−
	Second choice	P	+	−	+	−	+	−
		T	−	+	−	+	+	−
		M	−	−	−	−	−	−
(c)		P	+	−	+	−	−	+
		T	−	+	−	+	−	+
		$C'=CM$	−	−	−	−	−	−

IV. CONCLUSIONS

The behavior of electromagnetic quantities under C, P, T operations—in the framework of the manifestly duality-invariant formulation of electrodynamics—is summarized in Table I. (We assumed, as in Sec. III, standard transformation properties for $\vec{E}, \vec{H}, j^\mu$.)

From the discussion of Sec. III and Table I, we conclude that, in this context, magnetic Dirac poles may give rise to parity and time-reversal violation *only* when one deals with systems without electric charges. In this case, indeed, there is a freedom in the choice of transformation laws for source tensor components, whence a vector, as well as a pseudovector, behavior of magnetic current can follow. On the contrary, no P or T violation may occur for systems with both kinds of charges, because the behavior of g^μ is determined by transformation properties of the electric current. Moreover, if one agrees to the view that the duality-invariant theory of magnetic poles physically supports the Cabibbo-Ferrari theory,[10] in this latter framework too no freedom is left in choosing transformation laws of magnetic current, and one has to attribute to it a pseudovector character,[17] rather than a vector one.

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[1]N. F. Ramsey, Phys. Rev. 109, 225 (1958).

[2]N. Cabibbo and E. Ferrari, Nuovo Cimento 23, 1147 (1962).

[3]N. Pintacuda, Nuovo Cimento 29, 216 (1963).

[4]F. Rohrlich, Phys. Rev. 150, 1104 (1966).

[5]J. M. Leinaas, Nuovo Cimento 15A, 740 (1973).

[6]P. A. M. Dirac, Proc. R. Soc. London A133, 60 (1931); Phys. Rev. 74, 817 (1948).

[7]For a comprehensive review, see, e.g., E. Amaldi, in *Old and New Problems in Elementary Particles*, edited by G. Puppi (Academic, New York, 1968), p. 1.

[8]See A. Nisbet, Proc. R. Soc. London A231, 250 (1955).

[9]A. Sommerfeld, in P. Frank and R. Von Mises, *Riemann-Webers Differentialgleichungen der Physik* (1935), Vol. 2, p. 789.

[10]M. Y. Han and L. C. Biedenharn, Nuovo Cimento 2A, 544 (1971).

[11]See Eqs. (44a) and (44b) of Ref. 7.

[12]A different physical meaning to the Cabibbo-Ferrari relation and the occurrence of a second potential may be given if one assumes that magnetic monopoles only exist as faster-than-light (electrically charged) particles. See R. Mignani and E. Recami, Nuovo Cimento 30A, 533 (1975).

[13]The theory is therefore "weakly invariant," in the sense of Ref. 3.

[14]In the following, we adopt natural units ($c=\hbar=1$) and the metric ($+---$). Greek indices run from 0 to 3.

[15]By the way, from Eqs. (10)–(11) it follows immediately that $\mathfrak{F}^{\mu\nu}$, Eq. (12), coincides with the Cabibbo-Ferrari tensor, Eq. (7).

[16]In fact, now C coincides with the operation denoted by $C'=CM$ in Refs. 1 and 2 (where M means conjugation of magnetic charge).

[17]This is contrary to what was assumed in Ref. 2.