Null-coordinate gravidynamics and the spin coefficients

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By choosing a very natural tetrad we show that the null-energy density, found previously by two of us from a first-order action principle, is four times the modulus squared of the Newman-Penrose coefficient τ. Moreover, we compute in terms of the usual tensorial field variables of gravity the more important three tetradic scalars of the conformal Weyl tensor. Consequently, we are able to write down the mass formulas in terms of the field variables. Also given is a null-energy formula in terms of line and surface integrals. All the results are given for fields which are asymptotically flat and asymptotically "uniformly smooth."

I. INTRODUCTION

The dynamics of the Einstein field, either in the vacuum or in the presence of a reasonable matter field (for instance, an Abelian massive vector field), shows many interesting features when it is analyzed by starting from a first-order action (with the usual variables $g_{\mu\nu}, \Gamma_{\mu\nu}{}^{\alpha}$) and a set of coordinate conditions is chosen which imply that one of them (u) (at least) is null, i.e., $g^{uu}=0$ and that one (r) of the three remaining coordinates (r, x^1, x^2) is the affine parameter along the light rays contained in each of the null hypersurfaces.

For instance, it has been shown by Aragone and Chela-Flores (ACh)[1] that after the reduction process is accomplished, taking as the evolution coordinate the null one (u), one is left with an action depending upon the minimum number of physical variables allowed for the pure helicity-2, massless, self-interacting field: 2. This is half the number of independent variables needed in the canonical 3+1 analysis of Arnowitt, Deser, and Misner,[2] even if in this frame one does not need to make an *a priori* choice of coordinates.

Another relevant property of the 2+2 null dynamics is the simplicity of the differential constraints. Instead of being real, coupled, partial differential equations mixing democratically the derivatives with respect to the three internal coordinates of the null hypersurfaces, they are now mostly[3] ordinary differential equations with the affine parameter r as the independent variable.

The third interesting property that Aragone and Restuccia (AR)[4] have shown is that, after solving the differential vectorial constraint, a reduced action can always[5] be reached which explicitly shows a very simple structure $\sim \dot{q}q' - J^u$: It has a dynamical germ of the null-canonical type $\sim \dot{q}q'$ minus an explicitly non-negative quantity $J^u \geq 0$ which was called the total null-energy density of the system. This non-negative quantity $J^u = J^u_G + J^u_M$ is the sum of the pure gravitational quantity J^u_G, the null energy of gravity, and J^u_M (a non-negative quantity also) which provides the contribution of the matter-independent dynamical variables to the null energy of the interacting system.[6]

It was also shown[6] that for linearized gravity[7] $\int J^u_G d^2\vec{x}\, dr$ goes to the flat-space generator P_u. In spite of this, many other questions related to the physical significance of J^u_G could also be raised.

For instance, we can ask ourselves whether the integration of the null energy over a null hypersurface, assuming some definite asymptotic behavior of the field, might give a constant. Or, also, one can look for the connection between the mass of the system (in the sense of the definition given by Newman and Unti[8]) and J^u_G. Or, even more generally: Is there some connection between the spin-coefficient approach developed by Newman and Penrose[9] (where the null coordinates play a substantial role too) and the simpler dynamical approach developed in Refs. 1 and 4?

In order to give an answer to some of these questions, we shall give in the next section the minimum nontrivial review of the results already obtained for pure gravity in the 2+2 tensorial approach. Therefore, we shall recall some of the definitions introduced by Newman and Penrose (NP)[9] in their spin-coefficient approach, either for representing the dynamics of the local tetrad or for analyzing the behavior of the conformal Weyl tensor.[10]

After doing that, in Sec. III, we evaluate the NP scalars and the Newman-Unti mass formula in terms of the tensorial variables, showing very simple results.

Throughout this paper we shall assume that we are concerned with asymptotically flat and asymptotically 2-dimensional "uniformly smooth" fields, in the same sense given by NP.[11]

Thereafter, in Sec. IV, we give an expression for the null energy in terms of line and surface integrals. Finally, in Sec. V, we discuss the results presented here and some remarks are given.

II. NULL-COORDINATE DYNAMICS FROM THE PALATINI ACTION AND THE NEWMAN-PENROSE APPROACH

In the following we shall make all the calculations in a special gauge, the same gauge adopted by Robinson and Trautman,[12] by Newman and Penrose,[9] and very recently by Root,[13] Aragone and Chela-Flores,[1] and recently by Kaku,[14] in order to give a ghost-free formulation of quantum gravity. Because of its geometrical meaning we shall call it the ray gauge.

The ray gauge is defined in terms of the contravariant metric tensor $g^{\mu\nu}$ by

$$g^{\mu u} = -\delta^{\mu}_{r} . \tag{1}$$

This choice allows the representation of both $g_{\mu\nu}$ and $g^{\alpha\beta}$, where $g_{\mu\nu}g^{\nu\sigma}=\delta^{\sigma}_{\mu}$, in terms of the six independent gravity variables g_{ij}, $g^{ir}\equiv N^i$, and $g^{rr}\equiv -2n$, giving for the line element and for the gradient-squared form, respectively,

$$ds^2 = g_{ij}(dx^i + N^i du)(dx^j + N^j du) - 2du(dr - ndu), \tag{2}$$

$$(\partial f)^2 = g^{ij}\,\partial_i f\,\partial_j f - 2(\partial_u f - N^l\partial_l f + n\partial_r f)\partial_r f ,$$

where g^{ij} can be shown to be the 2-dimensional inverse of g_{ij}: $g_{ij}g^{jl}=\delta^l_i$.

The dynamical analysis of ACh started from the first-order Palatini action (in the ray gauge)

$$dA = g^{1/2} g^{\mu\nu} R_{\mu\nu}(\Gamma) d^4x , \tag{3}$$

where g_{ij}, N^i, n, and $\Gamma_{\mu\nu}{}^{\alpha}$ were the initial 46 independent variables, and $g=\det g_{ij}$.

After performing the reduction process the Γ's disappeared and ACh obtained the new unconstrained action $A_*(g^T_{ij}, \Gamma^{Tu}_{iu})$,

$$dA_*(g^T_{ij}, \Gamma^{Tu}_{iu}) = [\tfrac{1}{2}g^{1/2}(g^{il}g^{jm} - g^{ij}g^{lm})g'_{lm}\dot{g}_{ij} - \tfrac{1}{2}g^{1/2}g_{ij}N^{i\prime}N^{j\prime}]d^2x\,du\,dr , \tag{4}$$

where $N^{i\prime}\equiv\partial_r N^i$ is a shortened notation for the solution of the ordinary differential equation

$$C_i \equiv (g^{1/2}g_{ij}N^{j\prime})' = g^{1/2}D_j(g'^j{}_i - \delta^j{}_i g'^l{}_l), \quad g'^j{}_i \equiv g^{jl}g'_{li} \tag{5a}$$

and the tensor g_{ij} is restricted by the scalar contraint equation

$$C_r \equiv g^{ij}g''_{ij} - \tfrac{1}{2}g'^i{}_j g'^j{}_i = 0 . \tag{5b}$$

The null-energy density J^u is, by definition, the non-negative quantity

$$J^u \equiv \tfrac{1}{2}g^{1/2}g_{ij}N^{i\prime}N^{j\prime} \geq 0 , \tag{6}$$

evaluated on the field equations. It is easy to see that J^u is the generator of the u evolution of gravity. The only thing that J^u cannot control is the initial input (or the final output) on a world tube Σ_r of the type $r=$ constant (or $r=+\infty$).

Even if they are not quantities related to the "coordinates" g_{ij} through a dynamical equation, AR defined the associated momenta π^{ij} by

$$2\pi^{ij} \equiv g^{1/2}(g^{il}g^{jm} - g^{ij}g^{lm})g'_{lm} \equiv 2g^{il}g^{1/2}K^j_l . \tag{7}$$

The quantities $K^j_l \equiv g_{li}g^{-1/2}\pi^{ij}$ constitute a new set which will turn out to be very useful in stating NP results in terms of the action-field variables.

In a different way to the action approach, in NP the field equations are obtained either by projecting the equations of motion of a certain orthonormal tetrad upon this tetrad itself, or by projecting the Bianchi identities on the basic tetrad in order to analyze the dynamical behavior of the tetradic components Ψ_A of the Weyl conformal tensor $C_{\alpha\beta\gamma\delta}$. So, in order to be able to evaluate both the spin coefficients and the components of the Weyl tensor, we have to pick up a specific tetrad.

In the present paper we shall try to make the more natural choice[15] at our disposal for the null tetrad, assuming one started from the action principle. It turns out to be

$$\begin{aligned} l &\equiv e_- = \partial_r , \quad n \equiv e_+ = \partial_u + n\partial_r - N^l\partial_l , \\ m &\equiv m^i\partial_i \\ &\equiv \tfrac{1}{2}h^{-1/2}e^{-\alpha}(e^{\beta} + ie^{-\beta})\partial_1 \\ &\quad + \tfrac{1}{2}h^{-1/2}e^{\alpha}(-e^{\beta} + ie^{-\beta})\partial_2 , \end{aligned} \tag{8}$$

where $h\equiv g^{1/2}$ and α, β came from a parametrization of g_{ij} which factorizes the determinant of g_{ij},

$$(g_{ij}) \equiv h\begin{pmatrix} e^{2\alpha}\cosh 2\beta & \sinh 2\beta \\ \sinh 2\beta & e^{-2\alpha}\cosh 2\beta \end{pmatrix} . \tag{9}$$

It is completely straightforward to verify that the tetrad (8) is null orthonormal, i.e., that using the metric (2) one has $[\overline{m}^{\mu}\equiv(m^{\mu})^*]$

$$\begin{aligned} &l^2 = n^2 = m^2 = 0, \quad l\cdot m = n\cdot m = 0 , \\ &-l\cdot n = 1 = m\cdot\overline{m} = g_{\mu\nu}m^{\mu}\overline{m}^{\nu} . \end{aligned} \tag{10}$$

At this point it is worth pointing out that the tetrad given in Eqs. (8) corresponds in the NP notation to the choice $U\equiv n$, $\omega\equiv 0$, $X^i\equiv -N^i$, and

$\xi^i \equiv m^i$. It is noteworthy that this is not the choice made by NP and NU in order to give the asymptotic behavior of the physical quantities in vacuum universes. They required that their $\hat{m}^\mu$ and $\hat{n}^\mu$ be parallel propagated along the basic null vector l^α, which at the level of the complex scalars implied[16]

$$\hat{\pi} = \hat{\epsilon} = 0 . \tag{11}$$

In general these equations do not allow the vanishing of ω. Their choice of $\hat{l}$, $\hat{n}$, and $\hat{m}$ can be expressed in terms of ours in the following way:

$$\begin{aligned} &\hat{l} = l, \quad \hat{m} = e^{i\varphi} m + \omega l , \\ &\hat{n} = n + \omega\bar{\omega} l + e^{i\varphi}\bar{\omega} m + e^{-i\varphi}\omega \bar{m} , \end{aligned} \tag{12}$$

where ω and φ are determined in such a way that Eq. (11) is satisfied. As both the spin coefficients $\kappa, \pi, \epsilon, \rho, \sigma, \tau, \alpha, \beta$ and the Weyl scalars Ψ_A are not invariant under the transformation (12) linking the two tetrads, it is convenient to calculate how they are related.

It turns out that the transformation laws for some of the spin coefficients are

$$\hat{\kappa} = e^{i\varphi}\kappa , \tag{13a}$$

$$\hat{\rho} = \rho + e^{i\varphi}\bar{\omega}\kappa , \tag{13b}$$

$$\hat{\sigma} = e^{2i\varphi}\sigma + e^{i\varphi}\omega\kappa , \tag{13c}$$

$$\hat{\tau} = e^{i\varphi}\tau + e^{2i\varphi}\bar{\omega}\sigma + \omega\rho + e^{i\varphi}\omega\bar{\omega}\kappa , \tag{13d}$$

$$\hat{\pi} = e^{-i\varphi}\pi + 2\bar{\omega}\epsilon - i\bar{\omega} D\varphi - D\bar{\omega} + e^{i\varphi}\bar{\omega}^2\kappa , \tag{13e}$$

$$\hat{\epsilon} = \epsilon - \tfrac{1}{2} i D\varphi + e^{i\varphi}\bar{\omega}\kappa , \tag{13f}$$

where $\kappa \equiv l_{\mu;\nu} m^\mu l^\nu$, $\rho \equiv l_{\mu;\nu} m^\mu \bar{m}^\nu$, $\sigma \equiv l_{\mu;\nu} m^\mu m^\nu$, $\tau \equiv l_{\mu;\nu} m^\mu n^\nu$, $\pi \equiv -n_{\mu;\nu}\bar{m}^\mu l^\nu$, $\epsilon \equiv \frac{1}{2}(l_{\mu;\nu} n^\mu l^\nu - m_{\mu;\nu}\bar{m}^\mu l^\nu) = -\frac{1}{2} m_{\mu;\nu}\bar{m}^\mu l^\nu$.

In particular, owing to the choice of the ray gauge, one has for the l, n, m tetrad

$$\kappa = 0, \quad \rho = \bar{\rho}, \quad \tau = \bar{\alpha} + \beta, \quad \operatorname{Re}\epsilon = 0. \tag{14}$$

Consequently, the transformation laws (13) become very simple:

$$\begin{aligned} &\hat{\kappa} = 0, \quad \hat{\rho} = \rho = \operatorname{Re}\rho , \\ &\hat{\sigma} = e^{2i\varphi}\sigma, \quad \operatorname{Re}\hat{\epsilon} = 0 , \\ &\hat{\tau} = e^{i\varphi}\tau + e^{2i\varphi}\bar{\omega}\sigma + \omega\rho , \\ &\operatorname{Im}\hat{\epsilon} \equiv \hat{\epsilon}_2 = \epsilon_2 - \tfrac{1}{2} D\varphi , \\ &\hat{\pi} = e^{-i\varphi}\pi + 2i\bar{\omega}(\epsilon_2 - \tfrac{1}{2} D\varphi) - D\bar{\omega} , \end{aligned} \tag{15}$$

showing, for instance, that $\hat{\epsilon}_2$ and $\hat{\pi}$ can be constrained to vanish (the definition of the parallel-propagated NP tetrad $\hat{l}, \hat{n}, \hat{m}$) by taking $\varphi = \int^r 2\epsilon_2 dr$ and $\omega = \int^r e^{i\varphi}\pi\, dr$. These Eqs. (15) can also be very useful because the knowledge of the asymptotic behavior of $\hat{\sigma}, \hat{\tau}, \hat{\rho}, \omega$ will allow us to deduce the asymptotic behavior of σ, τ, ρ which can easily be evaluated in terms of the action variables.

For the first three components of the Weyl tensor the transformation law looks like

$$\begin{aligned} &-C_{\alpha\beta\gamma\delta}\hat{l}^\alpha \hat{m}^\beta \hat{l}^\gamma (\hat{m}^\delta)^* \equiv \hat{\Psi}_0 = e^{2i\varphi}\Psi_0 , \\ &-C_{\alpha\beta\gamma\delta}\hat{l}^\alpha \hat{n}^\beta \hat{l}^\gamma \hat{m}^\delta \equiv \hat{\Psi}_1 = e^{i\varphi}\Psi_1 + e^{2i\varphi}\bar{\omega}\Psi_0 + \tfrac{1}{2}\omega(\Psi_0 + \bar{\Psi}_0) , \end{aligned} \tag{16}$$

$$\tfrac{1}{2} C_{\alpha\beta\gamma\delta}[\hat{l}^\alpha \hat{n}^\beta \hat{m}^\gamma (\hat{m}^\delta)^* - \hat{l}^\alpha \hat{n}^\beta \hat{l}^\gamma \hat{n}^\delta] \equiv \hat{\Psi}_2 = \Psi_2 + 2e^{i\varphi}\bar{\omega}\Psi_1 + \tfrac{1}{2}\omega\bar{\omega}(\Psi_0 + \bar{\Psi}_0) ,$$

which can easily be inverted if one wishes to determine the asymptotic behavior of Ψ_0, Ψ_1, Ψ_2 in terms of the asymptotic behavior of $\hat{\Psi}_0$, $\hat{\Psi}_1$, $\hat{\Psi}_2$, and ω.

The last introductory remark we want to make is that the conditions $\hat{\pi} = 0 = \hat{\epsilon}$ which hold in the tetrad $\hat{l}, \hat{n}, \hat{m}$ are now in a certain sense substituted, when we adopt the tetrad l, n, m, by another set of two complex equations (equivalent to three real scalar equations). These equations result from two of the NP metric equations for the commutators of the scalar function $\varphi(x^i, u, r) \equiv r$:

$$\begin{aligned} &[\delta, D] r = 0 = \bar{\alpha} + \beta - \bar{\pi} \rightleftarrows \bar{\pi} = \bar{\alpha} + \beta , \\ &[\delta, \bar{\delta}] r = 0 = \bar{\mu} - \mu \rightleftarrows \operatorname{Im}\mu = 0 , \end{aligned} \tag{17}$$

where $D \equiv \partial_r$, $\Delta \equiv -N^l \partial_l + \partial_u + n\partial_r$, and $\delta \equiv m^i \partial_i$.

Comparing the first of Eqs. (17) with (14) we have for the nonvanishing value of π

$$\pi = \bar{\tau} . \tag{18}$$

At this point we are well prepared to look again at the dynamics of the gravity field.

III. EVALUATION OF THE NP SCALARS AND THE DYNAMICS OF GRAVITY

Using the values given in ACh for the affinity components $\Gamma_{\mu\nu}{}^\alpha$ we can afford the direct calculation of $\rho, \sigma, \epsilon_2, \tau$ [and obtain π from Eq. (18)] in the ray gauge. They yield ($K_i^i \equiv K$)

$$\begin{aligned} &\rho = -\tfrac{1}{2}\, lnh' = \tfrac{1}{2} K , \\ &\sigma = -\beta' + i\alpha' \cosh 2\beta = K_j^i\, m_i m^j , \\ &\tau = \tfrac{1}{2} m_j N^{j\prime} = \bar{\pi} , \\ &\epsilon_2 = -\tfrac{1}{2}\alpha' \sinh 2\beta \\ &\quad = \tfrac{1}{4}(m_j \bar{m}^{j\prime} - \bar{m}_j m^{j\prime}) . \end{aligned} \tag{19}$$

From these results, taking into account that $g_{ij} = m_i \bar{m}_j + \bar{m}_i m_j$, we can immediately calculate the modulus squared of the complex scalars σ and τ:

$$\begin{aligned} &\sigma\bar{\sigma} = \tfrac{1}{2} K_j^i K_i^j - \tfrac{1}{4} K^2 , \\ &|\tau|^2 = \tfrac{1}{8} g_{ij} N^{i\prime} N^{j\prime} = |\pi|^2 . \end{aligned} \tag{20}$$

From this last equation we have the possibility of recognizing the generator J^u of the dynamical evolution of the gravity field

$$J_u = 4g^{1/2}|\tau|^2 = 4g^{1/2}|\pi|^2 . \tag{21}$$

It is worth recalling that τ precisely describes how the direction of l changes as we move along n, while π has the converse meaning[9]: It describes the motion of n along l. [These variations are substantially contained in the 2-dimensional spacelike surface (x^1, x^2).]

We can also go further and estimate the first three components of the Weyl tensor, Ψ_0, Ψ_1, and Ψ_2. For Ψ_0 we obtain $(K_j^{i\,\prime} \equiv \partial_r K_j^i)$

$$-\Psi_0 = (K_j^{i\,\prime} - K'\delta_j^i + K_l^i K_j^l + K^2\delta_j^i - 2KK_j^i) m_i m^j , \tag{22a}$$

which means that Ψ_0 is exactly the traceless part of the symmetric tensor: $L_j^i \equiv K_j^{i\,\prime} - K'\delta_j^i + K_l^i K_j^l + K^2\delta_j^i - 2KK_j^i$. It is very remarkable that the vanishing of the trace of this tensor, L_i^i, is exactly equivalent to the constraint equation (5b):

$$L_i^i = L_j^i(m_i \overline{m}^j + \overline{m}_i m^j) = -K' + K_l^i K_i^l = \tfrac{1}{2}C_r . \tag{22b}$$

The second component of the Weyl tensor, Ψ_1, gives (on the field equations)

$$-\Psi_1 = (K_i{}^j|_j + \tfrac{1}{2}K_{ij}N^{j\,\prime})m^i . \tag{23}$$

The last quantity we shall calculate is the real part of Ψ_2. In fact, it is easy to see that, on the field equations,

$$2\Psi_2 = -(n'' + 6\tau\overline{\tau}) + \imath h^{-1}R_{21r}{}^r . \tag{24}$$

As the last of Eqs. (16) shows, one only needs the real part of Ψ_0 and Ψ_2 and the whole Ψ_1 to obtain $\mathrm{Re}\hat{\Psi}_2$ after an (ω, φ) change of tetrad. We are disregarding the remaining scalars Ψ_3, Ψ_4 because they do not have a simple structure in terms of g_{ij}, N^l, n and neither do they enter into the NU discussion of the initial-value problem.[17]

At this point we shall try to give a mass formula, intrinsically linked to the action-principle gravity variables. Let us assume that we are in a well-behaved system of null coordinates,[18] such that the following conditions hold:

$$\lim_{r\to\infty}[r^2\sigma]_{u=u_1} = 0 = \lim_{r\to\infty}[r^2\sigma]_{u=u_2} , \tag{25a}$$

and

$$\lim_{r\to\infty} r^2 g^{ij} = 2P^2(\vec{x})\delta^{ij} . \tag{25b}$$

We shall also assume that the field is asymptotically flat and asymptotically 2-dimensional "uniformly smooth" in the sense defined in NP, moreover, that $\Psi_0 \sim O(r^{-5})$ for r tending to infinity.

In these conditions we assert that the Newman-Unti mass formula[8] which has been proved for the parallel propagated tetrad $\hat{l}$, $\hat{n}$, $\hat{m}$, is also true in the closer-to-the-null-coordinates frame l, n, m; in other words, that the quantity $M(u)$ given by

$$8\pi M(u) \equiv \int_{\Sigma_{u\infty}} \lim_{r\to\infty}[-2r(\mathrm{Re}\Psi_2)g^{1/2}]d^2\vec{x} \tag{26}$$

is a nonincreasing function of the null-time u ($\Sigma_{u\infty}$ means the 2-dimensional closed and compact variety u = constant, $r = \infty$, x^1, x^2 variables).

We know that the formula similar to Eq. (26) already holds in the NP tetrad $\hat{l}$, $\hat{n}$, $\hat{m}$

$$8\pi M(u) \equiv \int_{\Sigma_{u\infty}} \lim_{r\to\infty}[-2r(\mathrm{Re}\hat{\Psi}_2)g^{1/2}]d^2\vec{x} , \tag{26$'$}$$

because conditions (25a) and (25b) together with the transformation laws (15) imply that

$$\begin{aligned}\lim_{r\to\infty}[r^2\hat{\sigma}]_{u=u_1} &= e^{2i\varphi}\lim[r^2\sigma]_{u=u_1}\\ &= 0\\ &= \lim_{r\to\infty}[r^2\hat{\sigma}]_{u=u_2}\end{aligned}$$

and Eq. (25b) does not depend upon the tetrad. Therefore, the problem lies in showing the consistency of definition (26$'$) of $M(u)$ with Eq. (26). One has to show that the limit inside the integral (26$'$) is equal to the limit inside the integral expression (26). Or, which is equivalent because $\lim g^{1/2}r^{-2} = \lim(r^2/g^{1/2})^{-1} = 2^{-1}P^{-2}$ independent of the tetrad one is using, one has to show that

$$\lim_{r\to\infty} -2r^3\mathrm{Re}\hat{\Psi}_2 = \lim_{r\to\infty} -2r^3\mathrm{Re}\Psi_2 . \tag{27a}$$

But according to NU, $\hat{\Psi}_0 \sim O(r^{-5})$, $\hat{\Psi}_1 \sim O(r^{-4})$, $\hat{\Psi}_2 \sim O(r^{-3})$, and, moreover, $\omega \sim O(r^{-1})$. Consequently, the first of Eqs. (16) indicates that $\Psi_0 = e^{-2i\varphi}\hat{\Psi}_0$ is of order

$$\Psi_0 = O(r^{-5}) . \tag{27b}$$

Feeding this information into the second of Eqs. (16), as $\omega \sim O(r^{-1})$, we obtain that

$$\Psi_1 \sim O(r^{-4}) . \tag{27c}$$

Finally, going to the last of Eqs. (16) we have that

$$\lim_{r\to\infty} 2r^3\hat{\Psi}_2 = \lim_{r\to\infty} 2r^3\Psi_2 , \tag{27d}$$

because the remaining two terms on the right-hand side of the third Eq. (16) are of orders $2e^{i\varphi}\overline{\omega}\Psi_1 \sim O(r^{-5})$ and $|\omega|^2\mathrm{Re}\Psi_0 \sim O(r^{-7})$ and therefore Eq. (26) is consistent with the definition (26$'$).

Now we can make use of the value (24) which we obtained for Ψ_2 in order to write down the mass in terms of the field-action variables:

$$\lim_{r\to\infty} -2r^3\mathrm{Re}\Psi_2 = \lim_{r\to\infty} r^3 n'' + \lim_{r\to\infty} r^3|\tau|^2 . \tag{28}$$

However, because of the set of relations (15) and the asymptotic behavior of $\hat{\rho}$, $\hat{\sigma}$, $\hat{\tau}$ found in NP and NU,

$$\hat{\rho} \sim O(r^{-1}), \quad \hat{\sigma} = O(r^{-2}), \quad \hat{\tau} = O(r^{-3}), \tag{29a}$$

we have that τ goes to zero more slowly:

$$\tau \sim O(r^{-2}). \tag{29b}$$

Fortunately, the behavior shown in Eq. (29b) is enough to make the second term on the right-hand side of Eq. (28) vanish:

$$\lim_{r\to\infty} r^3|\tau|^2 = 0. \tag{30}$$

If we put down jointly the results stated by Eqs. (28) and (30), we can finally write

$$8\pi M(u) = \int_{\Sigma_{u,\infty}} \lim_{r\to\infty} r n'' g^{1/2} d^2\vec{x}. \tag{31}$$

Incidentally, Eq. (29b) still allows us to show that the null energy $\mathcal{P}_u$ is finite:

$$\mathcal{P}_u \equiv \int_{\Sigma_u} J_u d^2\vec{x}\, dr = 4\int_{\Sigma_u} |\tau|^2 g^{1/2} d^2\vec{x}\, dr < +\infty \tag{32}$$

because $\int |\tau|^2 g^{1/2} dr \sim O(r^{-1})$.

If, for instance, we take the Schwarzschild metric in the form given by Robinson and Trautman[12]

$$ds^2 = r^2(1+\tfrac{1}{4}\vec{x}^2)^{-2} d\vec{x}^2 - 2du\,dr + \left(\frac{2m}{r} - 1\right) du^2, \tag{33}$$

we find that $M(u) = m$ and $\mathcal{P}_u = 0$.

IV. EVALUATION OF THE NULL ENERGY BY SURFACE INTEGRALS

Because of the physical relevance of $\mathcal{P}_u$, it is worth trying to reduce its calculation to the evaluation of 2-dimensional surface integrals. Fortunately, we have the way to do that. We must use the so-called "trivial" Einstein equation $G^u_u = -g^{ij}\,{}_4R_{ij} = 0$, which was obtained in ACh by variation of $g_{\mu\nu} = -m^2$ in the reduced action $({}_2g^{1/2} = h)$

$$G^u_u = 0 \rightleftarrows J_u = 2(nh')' + 2\dot{h}\,' + h\,{}_2R - \partial_i(h^{-1}(h^2N^i)'), \tag{34a}$$

and recall[19] that $h\,{}_2R$ can be written as the 2-divergence of a 2-dimensional contravariant density, i.e.,

$$h\,{}_2R \equiv \partial_i \mathcal{R}^i(g_{lm}). \tag{35a}$$

Therefore, by substituting in Eq. (34a) $h\,{}_2R$ by its value (35) and using the fact that the NP scalar $\mu \equiv -n_{\mu;\nu} m^\mu \overline{m}^\nu$ verifies the relation

$$2\mu h = \dot{h} + nh' - \partial_i(hN^i), \tag{35b}$$

the null-energy density can be cast in the form of an exact 3-divergence on the hypersurface Σ_u (u = constant):

$$J_u = (4\mu h)' + \partial_i(\mathcal{R}^i + 2h\rho N^i). \tag{34b}$$

At this point it is useful to assume the same asymptotic behavior of the previous section, when the mass formula was commented on. This implies that $\mu h' \sim O(r)$ for r tending to infinity and therefore we cannot expect that after integration the divergence can be canceled out [remember that $J^u \sim O(r^{-2})$]. If we wish to calculate $\mathcal{P}_u$, we have to integrate Eqs. (34b) on a finite region R of Σ_u and then make R invade the whole Σ_u.

Let us imagine Σ_u such that Σ_u ($r=0$) degenerates into a point (the origin) and that for each value of r, $R(r)$ is a large 2-dimensional surface with the 1-dimensional boundary $\partial R(r)$. Moreover, let us assume that

$$\phi(u,r) \equiv \lim_{R(r)\to\Sigma_{ur}} \int_{R(r)} \partial_i(\mathcal{R}^i + 2h\rho N^i) d^2\vec{x} = \lim_{\partial R(r)\to\infty} \int \epsilon_{ij}(\mathcal{R}^i + 2h\rho N^i) dx^j \tag{36}$$

does exist and that it is finite for any value of r. Then, integrating Eq. (34b) between 0 and r and taking into account that h ($r=0$) = 0 we have

$$\mathcal{P}_u = \int_{\Sigma_u} J_u d^2\vec{x}\, dr = \lim_{r\to\infty}\left[\int_{\Sigma_{ur}} 4\mu h\, d^2\vec{x} + \int_0^r \phi(u,r) dr\right], \tag{37}$$

which is the final formula we wanted to exhibit.

V. DISCUSSION

We have shown that the most important of the Newman-Penrose variables are strongly connected to dynamical quantities which appear in the reduced null-action formulation of gravity.

Besides having shown that the null-energy density is four times the modulus squared of the NP scalar τ, in the evaluation of the first component of the Weyl tensor Ψ_0 there appears a tensor whose trace is precisely the scalar constraint C_r of the action formulation and, therefore, must vanish.

The remaining two Weyl components which enter into the Newman-Unti formulation of the initial-value problem, Ψ_1 and Ψ_2, have also very simple expressions in terms of g_{ij}, $N^{i\,\prime}$, and n''.

In order to be able to regain a mass formula in terms of the action variables, we studied the effect of the local transformation of the natural (induced by the coordinates) tetrad and the parallel propagated NP tetrad, both for the spinor coefficients $\rho, \sigma, \tau, \epsilon, \pi$ and for the Weyl components.

Finally, in the last section we exhibited a formula for evaluating the null energy $\mathcal{P}^u$ as a sum of a surface integral plus a line integral.

All the calculations have been made in the ray gauge, the same coordinate choice as Robinson and Trautman and Newman and Penrose. They show the ray gauge to be much more adequate to understand the physics contained in the theory than the Bondi gauge.

There still remain many points which deserve a careful investigation, for instance, whether g_{ij} could be taken as the set of variables to be given initially on Σ_u and $N^{i\prime}$ and the $\lim_{r\to\infty} r^3 n''$ the variables to be given on $\Sigma_{r=+\infty}$ in order to ensure the existence of one solution of the field equations, or whether these results can be extended in the presence of matter, especially the null energy and the mass integral formulas.

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[2]R. Arnowitt, S. Deser, and C. W. Misner, Phys. Rev. 120, 313 (1960).

[3]This is evident for the vectorial constraint $C_i = 0$ [Eq. (19) of Ref. 1]. More difficult is the scalar constraint $C_r = 0$ [Eq. (18) of Ref. 1], but it also becomes an ordinary differential equation if one decouples the determinant from $g_{ij}: g_{ij} \equiv h u_{ij}$, $\det u_{ij} = 1$.

[4]C. Aragone and A. Restuccia, preceding paper, Phys. Rev. D 13, 207 (1976).

[5]Actually it has been shown for gravitating bosons of spin not greater than 1. Fermions have to be dealt with with a different representation of gravity.

[6]For instance, in the case of matter being represented by the Maxwell field, the null-energy density of the system J_u consists of J_u^G plus $J_u^{\rm em} = \frac{1}{2}\kappa_2{}^2 g^{1/2} B^2 + \frac{1}{2}\kappa^2 g^{-1/2} E^2$, two terms depending upon the only two dynamical variables E, B of the classical helicity-1 massless field. See Ref. 4.

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[10]In the vacuum, $C_{\mu\nu\alpha\beta}$, the conformal Weyl tensor, coincides with the Riemann tensor $R_{\mu\nu\alpha\beta}$ on the field equations.

[11]See Ref. 9, p. 574 ff. for the precise meaning of asymptotically flat and asymptotically "uniformly smooth" fields.

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[15]In Ref. 1 there is some preliminary discussion on the geometrical meaning of the field variables (g_{ij}, N^i, n) which provides a hint for this choice.

[16]See especially Sec. IV of Ref. 9 (NP).

[17]For instance, Ψ_3 contains terms of the type $\sim \overline{m}_i \dot{N}^{i\prime}$ and Ψ_4 terms proportional to $\overline{m}^i \overline{m}^j \ddot{g}_{ij}$ which can only be found in the complicated subsidiary equation $G_{ui} = 0 = G_{uu}$.

[18]That such kind of coordinates do exist has been discussed in Ref. 8.

[19]In connection with this point see especially the Appendix of Ref. 4.