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Local compensation of quantum numbers and the phase of many-particle production amplitudes*

Don Weingarten
Department of Physics and Astronomy, University of Rochester, Rochester, New York 14627
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We assume that two-body scattering processes occur only as the shadow of a class of many-particle production events and that the probability distribution of this class of events obeys the requirement of local compensation of quantum numbers. The first of these assumptions implies that a certain functional integral yields the strongest bound on the energy dependence of two-body quantum-number exchange which can be obtained from inelastic production probabilities. Using the second assumption we evaluate this bound and show that recent data imply that the result for charge exchange is probably much weaker than the energy dependence actually observed. We therefore conclude the energy dependence of charge exchange should be generated mainly by the phase of many-particle production. Then we reformulate local compensation of quantum numbers as a condition on many-particle production amplitudes, including their phases. We show that this assumption implies the leading energy dependence of two-body quantum-number exchange is given by Regge poles with factorizable residues. The Regge trajectories we obtain are expected to be analytic functions of both charge and momentum transfer and can be extrapolated to provide a number of semiquantitative conclusions concerning exotic meson trajectories.

I. INTRODUCTION

In a recent series of papers it is shown that numbers which can be extracted from many particle production probabilities yield upper bounds on the energy dependence of two-body processes exchanging either charge,[1-7] transverse momentum,[3-6] or both at once[3,6] if, primarily, two assumptions are introduced: First, it is assumed that the probability distribution of a certain class of many-particle production events obeys some version of the requirement of local compensation of quantum numbers; and second, it is assumed all other scattering processes occur only as the shadow of this class of events.[8] Roughly speaking, local compensation of an additively conserved quantum number requires that each particle carrying the value q must almost always be surrounded by a small collection of particles carrying a total of $-q$ located nearby in rapidity space.

Since the completion of the work of Refs. 1–8 sufficient data have been collected from inelastic pp events at 69 GeV/c,[9] 104 GeV/c, and 400 GeV/c[10] to provide some degree of support for the assumption of local compensation of charge. Using these data we will estimate the probable value of the bound on charge exchange given in Refs. 3 and 5. The limit we obtain turns out to be significantly weaker than the observed energy dependence of charge-exchange cross sections.

Now the derivations given in Refs. 3 and 5 proceed by showing that a ratio of charge-exchange cross sections to elastic cross sections is smaller than a one-dimensional integral over production probabilities. Then this quantity is evaluated and proved to fall as a certain power of s, where $\sqrt{s}$ is the total c.m. system energy. The one-dimensional integration considered in Refs. 3 and 5, however, is not the strongest limit on the energy dependence of quantum-number exchange which can be obtained from production probabilities. The strongest inequality determined solely by inelastic production probabilities actually leads to an infinite-dimensional integration. But we will show that this integral also falls as a power of s and its exponent is likely to be almost identical to the result gotten from the one-dimensional integration of Refs. 3 and 5. We will conclude that if two-body scattering amplitudes are expressed as the shadow of many-particle production, then the energy dependence of the two-body charge-exchange amplitude which results should not, for the most part, be a consequence of the behavior of the modulus of many-particle amplitudes (which is all that enters the form of local compensation of quantum numbers considered in Refs. 1–8), but should instead be determined largely by variations of the phase of the amplitude for particle production.

How does the phase of particle production generate the energy dependence of two-body charge exchange? One possibility is that the phase varies in some complicated way which does not have any useful theoretical description short of a complete theory of strong interactions. An alternative to this, however, can be gotten by reformulating the assumption of local compensation of quantum numbers as a restriction on many-particle amplitudes including their phases. We will show that this condition implies the asymptotic behavior of the amplitude for two-body quantum-number exchange is given by a set of Regge poles with factorizable residues. To obtain this result we will use a collection of theorems borrowed from one-dimensional statistical mechanics. The Regge trajectories we derive are given by a single analytic function of both charge and momentum which, in principle, yields predictions for an infinite family of exotic mesons composed of arbitrarily large numbers of quark-antiquark pairs. Combined with the observed intercepts and slopes of the Pomeron and ρ trajectories, this formula suggests that the intercept of the leading $qq\overline{q}\overline{q}$ trajectory is roughly -0.6,[11] and that the slopes of exotic trajectories are not a universal constant, but instead vary with the number of quark-antiquark pairs.

Although the new version of local compensation has the drawback that it can not be subjected to direct experimental tests, as could the original form in Refs. 1–8, there are indirect arguments which do support the new assumptions. In particular, they yield the original form of local compensation, for which there is some experimental evidence, as a logical consequence and are fulfilled by a number of variations of the multiperipheral models, such as Ref. 12. The assumptions we will describe are weaker than those of the multiperipheral model, however, and perhaps stand a somewhat better chance of actually being fulfilled.

Throughout most of the following discussion we will assume for simplicity that all hadrons are pions.

II. PROBABILITY FORMULATION OF LOCAL COMPENSATION OF QUANTUM NUMBERS

Let us begin by briefly reviewing a slightly altered form of the definitions, assumptions, and results of Ref. 3. Consider a scattering event involving a collection of hadrons $h_1, h_2, \ldots,$

$$h_1 + h_2 \to h_3 + \cdots . \tag{2.1}$$

Let $\sqrt{s}$ be the total center-of-mass–system (c.m.s.) energy, y_i be the c.m.s. rapidity of h_i, p_{1i} the component of momentum of h_i in the direction $\hat{e}_1$ perpendicular to the c.m.s. momentum of h_1 and h_2, p_{2i} the component of momentum of h_i in the direction $\hat{e}_2$ perpendicular to $\hat{e}_1$ and to the c.m.s. momentum of h_1 and h_2, and let p_{3i} be the charge of h_i. Assume $y_1 < y_2$. Thus, $y_1 = -Y/2$ and $y_2 = Y/2$ where

$$Y = \ln(s) + 2\ln\left[\frac{1}{2m} + \left(\frac{1}{4m^2} - \frac{1}{s}\right)^{1/2}\right],$$

and the mass of a pion is given by m. A three-component function $Z(y) = [Z_1(y), Z_2(y), Z_3(y)]$, which we will call a zone graph, can be defined by the equation

$$Z_j(y) = -\sum_{i=1,2} \theta(y - y_i) p_{ji} + \sum_{i\geq 3} \theta(y - y_i) p_{ji}, \tag{2.2}$$

where $\theta(y)$ as usual is given by 0 for negative arguments and 1 elsewhere. Each of the components $-Z_j(y)$ is the value of a corresponding quantum number transferred across the rapidity boundary y. For example, $-Z_3(0)$ is the conventional hemisphere charge transfer often represented by u. Since the probability is zero that any particle in the final state will have a transverse momentum of precisely zero, it follows that by taking the derivatives of $Z_1(y)$ and $Z_2(y)$ we can determine both the set of rapidities at which final-state particles are located, y_i, $i > 3$, and the transverse momentum components of each particle, p_{1i}, p_{2i}, $i > 3$. The derivative of $Z_3(y)$ yields the charge of each particle, p_{3i}. Thus, $Z(y)$ completely determines the final state from which it was obtained.

Now divide the class of all final configurations which could appear in (2.1) into two subsets, one called dense and the other diffuse. Dense configurations are those which include no rapidity gaps between successive pairs of particles greater than a certain increasing function $\Delta(s)$, $\Delta(s) \ll Y$. Diffuse configurations are the remainder. We will not need to specify $\Delta(s)$ precisely; roughly speaking, however, we wish to choose $\Delta(s)$ in such a way that those processes which can be considered "diffractive" almost always yield diffuse configurations, while "nondiffractive" processes almost always yield dense configurations. Let $\langle \cdots \rangle$ be the operation of averaging over the collection of dense final states. A set of functions called zone moments is defined by

$$C_{j_1 \cdots j_m}(y_1, \ldots, y_m) = \langle Z_{j_1}(y_1) \cdots Z_{j_m}(y_m) \rangle . \tag{2.3}$$

From these another set $\{c_{j_1 \cdots j_m}(y_1, \ldots, y_m)\}$, called zone correlations, is obtained by cluster decomposition. Using zone correlations the assumption of local compensation of quantum numbers in dense final states can be stated as follows:

(A) We assume that as $s \to \infty$ the functions $c_{j_1 \cdots j_m}(y_1, \ldots, y_m)$ approach energy-independent limits.

(B) If the variables $y_1, \ldots, y_m$ are divided into two nonempty sets S_1 and S_2 such that $|y_k - y_l| > d$ for all $y_k \in S_1$, $y_l \in S_2$, and d is made progressively larger, then $c_{j_1 \cdots j_m}(y_1, \ldots, y_m)$ rapidly approaches 0 once d is larger than a certain energy-independent correlation length λ.

(C) The functions $c_{j_1 \cdots j_m}(y_1, \ldots, y_m)$ are translationally invariant in the region in which $y_1, \ldots, y_m$ are many correlation lengths from the boundaries of rapidity space: $c_{j_1 \cdots j_m}(y_1, \ldots, y_m) = c_{j_1 \cdots j_m}(y_1 + y, \ldots, y_m + y)$.

(D) If $y_1, \ldots, y_m$ are many correlation lengths from the boundaries of rapidity space, $c_{j_1 \cdots j_m}(y_1, \ldots, y_m)$ is the same for all choices of h_1 and h_2. Once s is sufficiently large that Y is much greater than λ, the value of $c_{j_1 \cdots j_m}(y_1, \ldots, y_m)$ for $y_1, \ldots, y_m$ near h_k depends only on h_k and not on h_l where $k=1$ and $l=2$ or $k=2$ and $l=1$.

Next let M be the component of the T matrix taking diffuse initial configurations to diffuse final configurations, and let N be the component taking diffuse initial configurations to dense final configurations. Unitarity of the S matrix combined with parity and time-reversal invariance imply that for any diffuse initial state I and diffuse final state F

$$\mathrm{Im}\langle F|M|I\rangle = \tfrac{1}{2}\langle F|N^\dagger N|I\rangle + \tfrac{1}{2}\langle F|M^\dagger M|I\rangle. \tag{2.4}$$

We then interpret the notion that diffuse states are produced only as the shadow of dense production to mean that the right-hand side of (2.4) can be adequately approximated by neglecting the second term:

$$\mathrm{Im}\langle F|M|I\rangle = \tfrac{1}{2}\langle F|N^\dagger N|I\rangle. \tag{2.5}$$

If most events yield dense configurations, as we would expect, then (2.5) should be reasonably accurate. In any case, once having obtained a first approximation to $\mathrm{Im}\langle F|M|I\rangle$ from (2.5), the real part of $\langle F|M|I\rangle$ can be restored by solving a dispersion relation, then the full amplitude $\langle F|M|I\rangle$ used to evaluate the second term in (2.4) and obtain a second approximation to $\mathrm{Im}\langle F|M|I\rangle$. If this iteration is continued, an exact solution to (2.4) can be generated. With reasonable technical assumptions it can be shown for two-body scattering that the value of $\langle F|M|I\rangle$ produced by a dispersion relation will have the same asymptotic energy dependence as $\mathrm{Im}\langle F|M|I\rangle$ and the amplitudes contributed by the second term in (2.4), which correspond roughly speaking to "multiple Regge exchange," will differ from the first approximation to the leading term at most by factors of $\ln(s)$ and small powers of s which go to zero smoothly as the exchanged transverse momentum goes to zero.

Consider the scattering process $h_1 + h_2 \to h_1' + h_2'$, where h_1' and h_2' are hadrons with transverse momenta and charges given by $p_{i1}' = p_{i1} + q_i$, $p_{i2}' = p_{i2} - q_i$, $i = 1, 2, 3$, for three numbers q_1, q_2, q_3 the last of which is an integer. It is shown in Ref. 3 that (2.5) combined with the Cauchy-Schwarz inequality and certain regularity conditions implies

$$\left|\frac{\mathrm{Im}\,T(h_1 + h_2 \to h_1' + h_2')}{[\mathrm{Im}\,T(h_1 + h_2 \to h_1 + h_2)\,\mathrm{Im}\,T(h_1' + h_2' \to h_1' + h_2')]^{1/2}}\right| \le \int d^3V\,[P(V)P(V - Yq)]^{1/2}, \tag{2.6}$$

where $P(V)$ is the probability distribution of the three-component random variable $V = (V_1, V_2, V_3)$

$$V_j = \int_{-Y/2}^{Y/2} dy\, Z_j(y), \tag{2.7}$$

in dense states produced by $h_1 + h_2$. By assumption D, $P(V)$ is also the probability distribution for V in dense states produced by $h_1' + h_2'$. Evaluating the right-hand side of (2.6) using A–D and one additional regularity condition leads to the bound

$$\left|\frac{\mathrm{Im}\,T(h_1 + h_2 \to h_1' + h_2')}{[\mathrm{Im}\,T(h_1 + h_2 \to h_1 + h_2)\,\mathrm{Im}\,T(h_1' + h_2' \to h_1' + h_2')]^{1/2}}\right| \le b(q)s^{a(q)},$$
$$a(q) = -\frac{q_1^2 + q_2^2}{8c_{20}} - \frac{q_3^2}{8c_{02}} + \frac{(q_1^2 + q_2^2)^2 c_{40}}{384 c_{20}^4} + \frac{q_3^4 c_{04}}{384 c_{02}^4} + \frac{(q_1^2 + q_2^2) q_3^2 c_{22}}{64 c_{20}^2 c_{02}^2} \cdots, \tag{2.8}$$

where

$$c_{mn} = \int_{-Y/2}^{Y/2} dy_2 \cdots \int_{-Y/2}^{Y/2} dy_{m+n}\, c_{1 \cdots 13 \cdots 3}(0, y_2, \ldots, y_{m+n}). \tag{2.9}$$

The index 1 occurs m times and 3 occurs n times in (2.9). The terms omitted in (2.8) involve either c_{mn}, $m+n \ge 6$, or higher powers of c_{mn}, $m+n=4$. Rotational invariance combined with charge-conjugation invariance and D imply $c_{mn} = 0$ if either m or n is odd.

Since the values of c_{02} given by the data for pp scattering in Ref. 10 change only slightly from 104 to 400 GeV/c, we will assume that c_{02} in this energy range is already an adequate approximation to its asymptotic value. We arrive at $c_{02} \approx 2.5$, consistent with the estimate given in

Ref. 3. The exponent of s obtained in the bound on forward charge exchange is ≈ -0.05. For comparison the exponent given by the ratio of πp charge exchange to πp elastic scattering is ≈ -0.4. Thus, the leading term of (2.8) is weaker than the observed energy dependence by a factor of 8.

It may be useful to point out that the correlation function $c_{33}(y_1, y_2)$ of Ref. 10 is gotten from averages taken over all final configurations rather than from dense configurations alone. Following arguments of Ref. 3, however, $c_{j_1 j_2}(y_1, y_2)$ obtained from all configurations should be a rather accurate approximation to the result obtained from an average including only dense configurations.

Can a significantly stronger bound be obtained if higher-order terms in the series in (2.8) are included? The leading correction depends on c_{04}. But from the relation given in Ref. 3 between zone correlations and density correlations,[13] it follows that the ratio c_{04}/c_{02}^2 should be roughly of the same order of magnitude as the ratio γ_4/γ_2^2, where γ_m is the mth cumulant of the multiplicity distribution of charged particles. In pp scattering at 300 GeV/c, $\gamma_4/\gamma_2^2 \approx 0.7$.[13] Thus, we expect c_{04} to be of the order of 2. Another estimate of the value of c_{04} can be gotten by examining the probability distribution of $-Z_3(0)$, the charge transfer between hemispheres. The fourth cumulant of this distribution is equal to $c_{3333}(0,0,0,0)$, while the second cumulant is $c_{33}(0,0)$. We would expect the order of magnitude of the ratio c_{04}/c_{02} to be roughly the same as the ratio $c_{3333}(0,0,0,0)/c_{33}(0,0)$. Fitting pp data at 104 GeV/c,[14] we obtain approximately $c_{3333}(0,0,0,0)/c_{33}(0,0) \approx 2$. Then c_{04} should be of the order of 5. Combining these results it seems likely c_{04} is positive and smaller than 10. If we had $c_{04} = 10$ the value of the term proportional to c_{04} in (2.8) would be 6.6×10^{-3}, which actually weakens the bound by a tiny amount. Conceivably the effect of still higher correlations should be considered. But again the mth zone correlation is linearly related to the mth density correlation. The results of Ref. 13 suggest that higher-order density correlations are quite small. Thus, it would seem that an accurate estimate of the strength of (2.8) actually is given by the leading term.

Still another estimate which suggests that the effect of higher correlations in (2.8) is quite small can be obtained as follows. The mth cumulant of the probability distribution of $Z_3(0)$ is

$$g_m = c_{3\ldots 3}(0,\ldots,0) \tag{2.10}$$

where the subscript 3 and argument 0 are both repeated m times. Thus, it follows from the definition of cumulants that the probability distribution of $Z_3(0)$, $P[Z_3(0)]$, is given by

$$P[Z_3(0)] = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\alpha \exp\left[-iZ_3(0)\alpha + \sum_{m=2}^{\infty}\frac{(i\alpha)^m}{m!}g_m\right]. \tag{2.11}$$

Therefore, the relative importance of higher correlations in the quantity

$$\sum_{Z_3(0)} \{P[Z_3(0)]P[Z_3(0)-1]\}^{1/2} \tag{2.12}$$

should be of the same order of magnitude as their relative importance in (2.8). If (2.12) is evaluated using the probability distribution $P[Z_3(0)]$ gotten from pp data at 104 GeV/c,[14] the result is 0.87. On the other hand, if (2.12) is evaluated using a distribution obtained from (2.11) by setting g_2 to its observed value and replacing all g_m, $m>2$ with 0, then the result becomes 0.89. Thus, once again we would expect (2.8) to be nearly given by its leading terms.

The observed value of c_{02}, the estimate for c_{20} given in Ref. 3, and the arguments of the preceding paragraphs also yield an upper bound for c_{22}, and, therefore, limit the capacity of (2.8) to account for the observed difference between the momentum transfer dependence of elastic scattering and the momentum transfer dependence of charge-exchange scattering. Using the random variables V_1 and V_3 defined by (2.7) we can form the expression $\langle [\xi(V_1^2 - \langle V_1^2\rangle) + V_3^2 - \langle V_3^2\rangle]^2\rangle$, which must be positive for all values of ξ. From this restriction and the relation between moments and cumulants[13,15] we obtain

$$(c_{04} + 2c_{02}^2)(c_{40} + 2c_{20}^2) \geq c_{22}^2 .$$

The value of c_{20} guessed in Ref. 3 is 0.8 (GeV/c)^2. Thus, if we again suppose that c_{40}/c_{20}^2 is of the order of γ_4/γ_2^2, the largest we would expect for c_{40} is ≈ 1 (GeV/c)^4. Then with $c_{04} \approx 10$ we find $|c_{22}| \lesssim 8$ (GeV/c)^2. Therefore, in (2.8) the largest the absolute value of the coefficient of $(q_1^2 + q_2^2)q_3^2$ can become is ≈ 0.03 (GeV/c)$^{-2}$. To account for the entire difference observed between the momentum transfer dependence of elastic scattering and the momentum transfer dependence of charge exchange the absolute value of this coefficient would have to be roughly 0.7 (GeV/c)$^{-2}$. Thus, (2.8) most probably cannot account for the magnitude of the difference which has been observed. A similar conclusion is reached in Ref. 6.

III. FUNCTIONAL INTEGRATION

Now the bound given by (2.8), even including all higher-order terms, is not the most stringent result which follows from A–D and (2.5), since Eq. (2.6) itself is not the strongest inequality determined by production probabilities. The strongest bound follows from (2.5) by using the Cauchy-Schwarz inequality and can be written symbolically

$$\left|\frac{\operatorname{Im}T(h_1+h_2\to h_1'+h_2')}{[\operatorname{Im}T(h_1+h_2\to h_1+h_2)\operatorname{Im}T(h_1'+h_2'\to h_1'+h_2')]^{1/2}}\right|\leqslant\prod_y\left[\int\int dZ_1(y)dZ_2(y)\sum_{Z_3(y)}\right]\{P[Z(y)]P'[Z(y)]\}^{1/2}, \tag{3.1}$$

where $P[Z(y)]$ is the "differential probability" with respect to $\prod_y dZ_1(y)dZ_2(y)$ that the zone graph $Z(y)$ will be produced by h_1+h_2, and $P'[Z'(y)]$ is the "differential probability" with respect to $\prod_y dZ_1(y)dZ_2(y)$ that the zone graph $Z'(y)$ will be produced by the initial state $h_1'+h_2'$. The graph $Z'(y)$ itself is a function of $Z(y)$ evaluated by taking the derivative of $Z(y)$ thereby extracting the final state $h_3+\cdots$ from which $Z(y)$ came, then calculating the graph $Z'(y)$ which would have been produced if this state came from $h_1'+h_2'$. Thus, (3.1) is just a particular notation for the usual overlap integral with the pahses of amplitudes removed. If (2.5) holds, (3.1) is the strongest bound on quantum-number exchange determined solely by inelastic production probabilities.

Is the observed energy dependence of quantum-number exchange significantly closer to the limit given by (3.1) than it is to the bound we obtained from (2.8)? We will show that it most probably is not. To arrive at this result, however, requires a somewhat lengthy calculation. The first step will be to introduce a series of approximations which convert the functional integration in (3.1) to a simpler finite dimensional expression.

In Refs. 1 and 3–5 it is shown that for all values of y outside a pair of finite regions at the boundaries of rapidity space, $Z'(y)$ is nearly given by

$$Z_j'(y)=Z_j(y)-q_j. \tag{3.2}$$

But suppose $z(y)$ is gotten from $Z(y)$ by restricting y to the region many correlation lengths from the boundaries of rapidity space, and let $z'(y)$ be obtained from $Z'(y)$ by a similar process. Then D implies that the probability distribution on $z(y)$ produced by $P[Z(y)]$ and the probability distribution on $z'(y)$ produced by $P'[Z'(y)]$ will be nearly identical. Thus, when s is large, there is an s-independent function $c(q)$ such that we nearly have

$$\prod_y\left[\int\int dZ_1(y)dZ_2(y)\sum_{Z_3(y)}\right]\{P[Z(y)]P'[Z'(y)]\}^{1/2}=c(q)\prod_y\left[\int\int dZ_1(y)dZ_2(y)\sum_{Z_3(y)}\right]\{P[Z(y)]P[Z(y)-q]\}^{1/2}. \tag{3.3}$$

Then since each $Z(y)$ which contributes to the integral in (3.3) is a well-behaved step function, if we choose a finite, equally spaced lattice of points in rapidity space $\{y_i\}$, and require the interval δ between successive points to be sufficiently small, (3.3) can be accurately approximated using

$$\prod_y\left[\int\int dZ_1(y)dZ_2(y)\sum_{Z_3(y)}\right]\{P[Z(y)]P[Z(y)-q]\}^{1/2}=\prod_i\left[\int\int dZ_1(i)dZ_2(i)\sum_{Z_3(i)}\right]\{P[Z(i)]P[Z(i)-q]\}^{1/2}, \tag{3.4}$$

where we have introduced the abbreviation $Z(y_i)=Z(i)$.

Next we will need a sequence of equations which connect (3.4) to the set of zone correlations. Now it follows from the definition of zone correlations that $P[Z(i)]$ is given by[3,15]

$$\begin{aligned}P[Z(i)]&=\prod_i\left[\int\frac{d^3\alpha(i)}{(2\pi)^3}\right]\exp\left(-i\sum\alpha_j(i)Z_j(i)+\mathfrak{C}[\alpha(i)]\right),\\ \mathfrak{C}[\alpha(i)]&=\sum_{\substack{m=2\\ m\text{ even}}}^{\infty}\sum_{\substack{i_1,\ldots,i_m\\ j_1,\ldots,j_m}}(-1)^{m/2}\alpha_{j_1}(i_1)\cdots\alpha_{j_m}(i_m)\frac{c_{j_1\ldots j_m}(i_1,\ldots,i_m)}{m!},\end{aligned} \tag{3.5}$$

where $c_{j_1\ldots j_m}(i_1,\ldots,i_m)=c_{j_1\ldots j_m}(y_{i_1},\ldots,y_{i_m})$. The integrals with respect to $d\alpha_1(i)$ and $d\alpha_2(i)$ in (3.5) are from $-\infty$ to ∞, and those with respect to $d\alpha_3(i)$ are from $-\pi$ to π. These limits will also be used for integrals with respect to $d^3\alpha(i)$ in the remainder of the present discussion. Using $\mathfrak{C}[\alpha(i)]$ we can define a new series of moments

$$\begin{aligned}D&=\prod_i\left[\int\frac{d^3\alpha(i)}{(2\pi)^3}\right]\exp\{\mathfrak{C}[\alpha(i)]\},\\ D_{j_1\ldots j_m}(i_1,\ldots,i_m)&=D^{-1}\prod_i\left[\int\frac{d^3\alpha(i)}{(2\pi)^3}\right]\alpha_{j_1}(i_1)\cdots\alpha_{j_m}(i_m)\exp\{\mathfrak{C}[\alpha(i)]\},\end{aligned} \tag{3.6}$$

and from these extract a set of correlations $\{d_{j_1\cdots j_m}(i_1,\ldots,i_m)\}$ by cluster decomposition. If $P[Z(i)]$ is infinitely differentiable with respect to $Z_1(i)$ and $Z_2(i)$, all the integrals in (3.6) converge. Since $\mathcal{C}[\alpha(i)]$ is an even function of $\alpha(i)$, the odd moments in (3.6) vanish. We arrive at the expression

$$P[Z(i)]=D\exp\{\mathfrak{D}[Z(i)]\},\quad \mathfrak{D}[Z(i)]=\sum_{\substack{m=2\\ m\text{ even}}}\sum_{\substack{i_1,\ldots,i_m\\ j_1,\ldots,j_m}}(-1)^{m/2}Z_{j_1}(i_1)\cdots Z_{j_m}(i_m)\frac{d_{j_1\cdots j_m}(i_1,\ldots,i_m)}{m!},\tag{3.7}$$

and, therefore,

$$\{P[Z(i)]\}^{1/2}=\sqrt{D}\exp\{\mathfrak{D}[Z(i)]/2\}.$$

Continuing the process, let us define another set of moments

$$\begin{aligned}E&=\prod_i\Big[\int\int dZ_1(i)dZ_2(i)\sum_{Z_3(i)}\Big]\sqrt{D}\exp\{\mathfrak{D}[Z(i)]/2\},\\ E_{j_1\cdots j_m}(i_1,\ldots,i_m)&=E^{-1}\prod_i\Big[\int\int dZ_1(i)\,dZ_2(i)\sum_{Z_3(i)}\Big]Z_{j_1}(i_1)\cdots Z_{j_m}(i_m)\sqrt{D}\exp\{\mathfrak{D}[Z(i)]/2\},\end{aligned}\tag{3.8}$$

and again extract a set of correlations $\{e_{j_1\cdots j_m}(i_1,\ldots,i_m)\}$ by cluster decomposition. The integrals in (3.8) are finite as long as the original zone correlations $\{c_{j_1\cdots j_m}(i_1,\ldots,i_m)\}$ themselves are well defined. For $\{P[Z(i)]\}^{1/2}$ we obtain

$$\begin{aligned}\{P[Z(i)]\}^{1/2}&=E\prod_i\Big[\int\frac{d^3\alpha(i)}{(2\pi)^3}\Big]\exp\Big\{i\sum_{i,j}\alpha_j(i)Z_j(i)+\mathcal{E}[\alpha(i)]\Big\},\\ \mathcal{E}[\alpha(i)]&=\sum_{\substack{m=2\\ m\text{ even}}}^{\infty}\sum_{\substack{i_1,\ldots,i_m\\ j_1,\ldots,j_m}}(-1)^{m/2}\alpha_{j_1}(i_1)\cdots\alpha_{j_m}(i_m)\frac{e_{j_1\cdots j_m}(i_1,\ldots,i_m)}{m!},\end{aligned}\tag{3.9}$$

so that the right-hand side of (3.4) can be expressed as

$$\prod_i\Big[\int\int dZ_1(i)dZ_2(i)\sum_{Z_3(i)}\Big]\{P[Z(i)]P[Z(i)-q]\}^{1/2}=E^2\prod_i\Big[\int\frac{d^3\alpha(i)}{(2\pi)^3}\Big]\exp\Big\{i\sum_{i,j}\alpha_j(i)q_j+2\mathcal{E}[\alpha(i)]\Big\}.\tag{3.10}$$

Finally, if we define the moments

$$F_{j_1\cdots j_m}(i_1,\ldots,i_m)=\prod_i\Big[\int\frac{d^3\alpha(i)}{(2\pi)^3}\Big]\alpha_{j_1}(i_1)\cdots\alpha_{j_m}(i_m)E^2\exp\{2\mathcal{E}[\alpha(i)]\},\tag{3.11}$$

and let their cluster decomposition yield the correlations $\{f_{j_1\cdots j_m}(i_1,\ldots,i_m)\}$, we have the result

$$\begin{aligned}\prod_i\Big[\int\int dZ_1(i)dZ_2(i)\sum_{Z_3(i)}\Big]\{P[Z(i)]P[Z(i)-q]\}^{1/2}&=\exp\Big[\sum_{\substack{m=2\\ m\text{ even}}}^{\infty}\sum_{j_1\cdots j_m}(-1)^{m/2}q_{j_1}\cdots q_{j_m}\frac{f_{j_1\cdots j_m}}{m!}\Big],\\ f_{j_1\cdots j_m}&=\sum_{i_1,\ldots,i_m}f_{j_1\cdots j_m}(i_1,\ldots,i_m).\end{aligned}\tag{3.12}$$

For $q=0$ both sides of (3.10) must reduce to 1, thus the normalization of (3.11) is correct without additional factors. The integrals in (3.11) converge if $\{P[Z(i)]\}^{1/2}$ is an infinitely differentiable function of $Z_1(i)$ and $Z_2(i)$.

Using (3.5)–(3.12) it can be shown that (3.4) falls as a power of s. If we are given the set of zone correlations $\{c_{j_1\cdots j_m}(i_1,\ldots,i_m)\}$ then from (3.6) we can, in principle, determine the set of correlations $\{d_{j_1\cdots j_m}(i_1,\ldots,i_m)\}$, from which by (3.8) we can obtain $\{e_{j_1\cdots j_m}(i_1,\ldots,i_m)\}$ and then by (3.11) derive $\{f_{j_1\cdots j_m}(i_1,\ldots,i_m)\}$. But if the quantity $\mathcal{C}[\alpha(i)]$, constructed from $\{c_{j_1\cdots j_m}(i_1,\ldots,i_m)\}$ by (3.5) is interpreted as the Hamiltonian of a one-dimensional array of spins $\{\alpha(i)\}$, then by (3.6) the correlations $\{d_{j_1\cdots j_m}(i_1,\ldots,i_m)\}$ may be thought of as the connected parts of the n-point functions obtained from a canonical ensemble. It is well known[16] that for a large class of short-range interactions in one dimension a canonical ensemble does not yield any critical points and the connected parts of the n-point functions fall rapidly for large separations between any pair of arguments. Thus, as a consequence of condition B requiring each $c_{j_1 \cdots j_m}(i_1, \ldots, i_m)$ to fall rapidly for large separations between its arguments, we would expect each $d_{j_1 \cdots j_m}(i_1, \ldots, i_m)$ to fulfill a similar restriction. In fact, if all $\alpha_j(i)$ are restricted to a certain large but finite number of values, and the integrals in (3.6) replaced by finite sums, then by using a result of Araki,[17] it can be shown rigorously that an appropriate precise interpretation of B imposed on the set $\{c_{j_1 \cdots j_m}(i_1, \ldots, i_m)\}$ implies that each $d_{j_1 \cdots j_m}(i_1, \ldots, i_m)$ must fall exponentially for large separations between its arguments. If each $d_{j_1 \cdots j_m}(i_1, \ldots, i_m)$ falls rapidly at large separations, then by repeating the reasoning we have just given applied now to (3.8), we would expect each $e_{j_1 \cdots j_m}(i_1, \ldots, i_m)$ to fall rapidly for large separations between arguments. One more repetition of this process yields the result for each $f_{j_1 \cdots j_m}(i_1, \ldots, i_m)$. In addition, translational invariance of all $c_{j_1 \cdots j_m}(i_1, \ldots, i_m)$, required by C, implies, in turn, translational invariance of each $d_{j_1 \cdots j_m}(i_1, \ldots, i_m)$, $e_{j_1 \cdots j_m}(i_1, \ldots, i_m)$, and $f_{j_1 \cdots j_m}(i_1, \ldots, i_m)$. Thus, if the spacing δ between successive points of the lattice $\{y_i\}$ is sufficiently small, we arrive at the conclusion that for large s, $f_{j_1 \cdots j_m}$ of (3.12) is nearly given by

$$f_{j_1 \cdots j_m} = a_{j_1 \cdots j_m} \ln(s) + b_{j_1 \cdots j_m}, \tag{3.13}$$

for a pair of constants $a_{j_1 \cdots j_m}, b_{j_1 \cdots j_m}$ with

$$a_{j_1 \cdots j_m} = \delta^{-1} \sum_{i_2, \ldots, i_m} f_{j_1 \cdots j_m}(0, i_2, \ldots, i_m). \tag{3.14}$$

Combining (3.12) and (3.13) yields

$$\prod_i \left[\int \int dZ_1(i) dZ_2(i) \sum_{Z_3(i)} \right] \{P[Z(i)] P[Z(i) - q]\}^{1/2} = b(q) s^{a(q)}, \tag{3.15}$$

$$a(q) = \sum_{\substack{m=2 \\ m \text{ even}}}^{\infty} (-1)^{m/2} q_{j_1} \cdots q_{j_m} \frac{a_{j_1 \cdots j_m}}{m!},$$

$$b(q) = \exp\left[\sum_{\substack{m=2 \\ m \text{ even}}}^{\infty} (-1)^{m/2} q_{j_1} \cdots q_{j_m} \frac{b_{j_1 \cdots j_m}}{m!} \right]. \tag{3.16}$$

Finally, let us calculate $a(q)$ and compare the result with (2.8). By combining (3.6), (3.8), and (3.11), each $f_{j_1 \cdots j_m}(i_1, \ldots, i_m)$ may be considered a function of the set $\{c_{j_1 \cdots j_m}(i_1, \ldots, i_m)\}$. But as we already mentioned in Sec. III, each $c_{j_1 \cdots j_m}(i_1, \ldots, i_m)$ is a linear combination of density correlations of order m. Since it has been shown in Ref. 13 that density correlations of order $m \geq 4$ appear to be rather small, we would expect each $c_{j_1 \cdots j_m}(i_1, \ldots, i_m)$, $m \geq 4$, to be rather small. Thus, it seems reasonable to expand each $f_{j_1 \cdots j_m}(i_1, \ldots, i_m)$ as an asymptotic series in $c_{j_1 \cdots j_m}(i_1, \ldots, i_m)$, $m \geq 4$, including only the effect of $c_{j_1 j_2}(i_1, i_2)$ to all orders.

The calculation we will perform is greatly simplified if we smoothly extrapolate $P[Z(i)]$ in (3.4) to nonintegral values of $Z_3(i)$ then replace summations over $Z_3(i)$ in (3.4) and (3.8) with integrations with respect to $dZ_3(i)$ evaluated from $-\infty$ to ∞. Correspondingly the integrals with respect to $d\alpha_3(i)$ in (3.5), (3.6), (3.10), and (3.11) must be evaluated from $-\infty$ to ∞. For large values of m these changes will have some effect on the correlations $c_{j_1 \cdots j_m}(i_1, \ldots, i_m)$, which we now assume to be gotten from the extrapolated form of the probability distribution $P[Z(i)]$. But since the probability distribution on $Z(i)$ has been made smoother than before, we would expect higher $c_{j_1 \cdots j_m}(i_1, \ldots, i_m)$, if anything, to become somewhat smaller than before. The effect these changes will have on $a(q)$ can be evaluated in simple models and for charge exchange $(q_1 = q_2 = 0,\ q_3 = 1)$ $a(q)$ should be altered by much less than ± 0.01. For example, the results of Ref. 10 show that $Z_3(y)$ and $Z_3(y')$ have significant correlations only if $|y - y'|$ is not too much larger than 1. Thus, let us imagine that $Z_3(y)$ is always constant on each interval $I_m = \{y \mid 2m < y \leq 2m + 2\}$. Define $Z_3(m) = Z_3(y)$, $y \in I_m$. Assume that the probability distribution for $Z_3(m)$ extrapolated to nonintegral values is gotten by letting each $Z_3(m)$ be an independent Gaussian random variable uncorrelated with $Z_1(y)$ and $Z_2(y)$. Let $\langle [Z_3(m)]^2 \rangle$ be given by the observed value of $\langle [Z_3(0)]^2 \rangle$, which is ≈ 1.[10] The value of $a(q)$ can be obtained directly from (3.4), which becomes a product of many one-dimensional summations. Evaluating each sum numerically gives $a(q) \approx -0.0624$, while if each sum is approximated by an integral, we find $a(q) = -0.0625$.

The asymptotic series we want can be derived at last by recognizing the formal similarity between (3.6), (3.8), (3.11), and the Feynman path-integral expression for the Green's functions of a quantum field theory. Thus, if (3.6) is used to obtain $d_{j_1 \cdots j_m}(i_1, \ldots, i_m)$, $d_{j_1 \cdots j_m}(i_1, \ldots, i_m)$ may be thought of as the connected part of a Green's function with m external legs. Therefore, $d_{j_1 \cdots j_m}(i_1, \ldots, i_m)$ can be written as a sum including only connected Feynman diagrams. The vertices in each diagram are determined by $c_{j_1 \cdots j_m}(i_1, \ldots, i_{m'})$, $m' \geq 4$, and the propagators by $c_{j_1 j_2}(i_1, i_2)$. The full set of Feynman rules will not be given here, but can be derived without great difficulty following, for example, discussions in Ref. 18. For the leading terms we obtain

$$d_{j_1j_2}(i_1,i_2)=c_{j_1j_2}{}^{-1}(i_1,i_2)+\tfrac{1}{2}\sum_{\substack{i_1',\ldots,i_4'\\ j_1',\ldots,j_4'}} c_{j_1j_1'}{}^{-1}(i_1,i_1')c_{j_2j_2'}{}^{-1}(i_2,i_2')c_{j_3'j_4'}{}^{-1}(i_3',i_4')c_{j_1'j_2'j_3'j_4'}(i_1',i_2',i_3',i_4'),$$

$$d_{j_1j_2j_3j_4}(i_1,i_2,i_3,i_4,)=\sum_{\substack{i_1',\ldots,i_4'\\ j_1',\ldots,j_4'}} c_{j_1j_1'}{}^{-1}(i_1,i_1')c_{j_2j_2'}{}^{-1}(i_2,i_2')c_{j_3j_3'}{}^{-1}(i_3,i_3')\,c_{j_4j_4'}{}^{-1}(i_4,i_4')c_{j_1'j_2'j_3'j_4'}(i_1',i_2',i_3',i_4'), \tag{3.17}$$

where $c_{j_1j_2}{}^{-1}(i_1,i_2)$ is defined by

$$\sum_{i_2,j_2} c_{j_1j_2}{}^{-1}(i_1,i_2)c_{j_2j_3}(i_2,i_3)=\delta_{j_1j_3}\delta_{i_1i_3}. \tag{3.18}$$

If similar techniques are applied to (3.8) and (3.11), then the expression for $\{f_{j_1}\ldots{}_{j_m}(i_1,\ldots,i_m)\}$ to all orders in $c_{j_1j_2}(i_1,i_2)$ and including only the linear terms in $c_{j_1}\ldots{}_{j_4}(i_1,\ldots,i_4)$ becomes

$$f_{j_1j_2}(i_1,i_2)=\tfrac{1}{4}c_{j_1j_2}{}^{-1}(i_1,i_2),$$

$$f_{j_1j_2j_3j_4}(i_1,i_2,i_3,i_4)=\tfrac{1}{16}\sum_{\substack{i_1',\ldots,i_4'\\ j_1',\ldots,j_4'}} c_{j_1j_1'}{}^{-1}(i_1,i_1')c_{j_2j_2'}{}^{-1}(i_2,i_2')c_{j_3j_3'}{}^{-1}(i_3,i_3')c_{j_4j_4'}{}^{-1}(i_4,i_4')c_{j_1'j_2'j_3'j_4'}(i_1',i_2',i_3',i_4'). \tag{3.19}$$

Now using condition C, Eqs. (3.1)–(3.4), (3.13)–(3.16), and choosing the spacing δ of the lattice $\{y_i\}$ sufficiently small that sums of $c_{j_1}\ldots{}_{j_m}(i_1,\ldots,i_m)$ over i_k, $1\le k\le m$, can be approximated by integrals of $c_{j_1}\ldots{}_{j_m}(y_1\cdots y_m)$ with respect to dy_k, $1\le k\le m$, we obtain

$$\left|\frac{\operatorname{Im}T(h_1+h_2\to h_1'+h_2')}{[\operatorname{Im}T(h_1+h_2\to h_1+h_2)\operatorname{Im}T(h_1'+h_2'\to h_1'+h_2')]^{1/2}}\right|\le c(q)b(q)s^{a(q)},$$

$$a(q)=-\frac{q_1{}^2+q_2{}^2}{8c_{20}}-\frac{q_3{}^2}{8c_{02}}+\frac{(q_1{}^2+q_2{}^2)^2c_{40}}{384c_{20}{}^4}+\frac{q_3{}^4c_{04}}{384c_{02}{}^4}+\frac{(q_1{}^2+q_2{}^2)q_3{}^2c_{22}}{64c_{20}{}^2c_{02}{}^2}\cdots, \tag{3.20}$$

where c_{mn} is related to $c_{j_1}\ldots{}_{j_{m+n}}(y_1,\ldots,y_{m+n})$ by (2.9).

Remarkably enough, the leading terms in the energy dependence of (3.20) are identical to the leading terms in (2.8). If we had included all terms in the s dependence of each expansion which are linear functions of any $c_{j_1}\ldots{}_{j_m}(y_1,\ldots,y_m)$, $m\ge4$, then the two results would still be identical. But as we argued in Sec. III, for charge exchange the contributions of the linear terms in $c_{j_1}\ldots{}_{j_m}(y_1,\ldots,y_m)$, $m\ge4$, are likely to be rather small in (2.8). Therefore, the same holds for (3.20). Thus, we would expect the effect of higher powers of $c_{j_1}\ldots{}_{j_m}(i_1,\ldots,i_m)$, $m\ge4$, to be most probably of no importance. These are the only terms which differ between the two expressions. Therefore, we can conclude that it is likely (2.8) provides very nearly the strongest bound on the energy dependence of charge exchange which can be obtained solely from many-particle production probabilities. By a simple extension of the preceding arguments this same conclusion can be gotten for the exchange of transverse momentum, or simultaneous exchange of transverse momentum and charge, if $q_1{}^2+q_2{}^2$ is not too large.

IV. AMPLITUDE FORMULATION OF LOCAL COMPENSATION OF QUANTUM NUMBERS

It follows from our results so far that if local compensation of charge does hold, as the data in Ref. 10 seem to suggest, and if, in addition, (2.5) is nearly correct, then most of the energy dependence of the amplitude for two-body charge exchange should be contributed by the phases of the inelastic amplitudes entering (2.5). Now it is possible that the energy dependence generated by phases arises in some very complicated manner. We will show, however, that an alternative to this can be gotten by reformulating the assumption of local compensation of quantum numbers as a restriction on many-particle amplitudes including their phases, rather than production probabilities alone which appear in A-D.

Precisely how to extend local compensation to production amplitudes can best be seen by returning to Eq. (2.5). If (2.5) is used to express the imaginary part of the two-body quantum-number-exchange amplitude using many-particle production amplitudes, we obtain

$$\mathrm{Im}\,T(h_1+h_2\to h_1'+h_2')=\tfrac{1}{2}\sum_n\sum_{p_{33},\ldots,p_{3n}}\int_{-Y/2}^{Y/2}dy_3\cdots\int_{-Y/2}^{Y/2}dy_n\iint dp_{13}dp_{23}\cdots\iint dp_{1n}\,dp_{2n}\,\theta(y_n-y_{n-1})\cdots\theta(y_4-y_3)$$
$$\times(2\pi)^4\delta\Big(\sum_{i=3}^n p_{1i}\Big)\delta\Big(\sum_{i=3}^n p_{2i}\Big)\delta\Big(\sum_{i=3}^n m_{\perp i}e^{y_i}-\sqrt{s}\Big)\delta\Big(\sum_{i=3}^n m_{\perp i}e^{-y_i}-\sqrt{s}\Big)$$
$$\times T(h_1+h_2\to h_3+\cdots)T^*(h_1'+h_2'\to h_3+\cdots),\tag{4.1}$$

where $m_{\perp i}{}^2=m^2+p_{1i}{}^2+p_{2i}{}^2$. All the kinematic variables in (4.1) are measured in the c.m. frame of h_1+h_2 and given by the definitions introduced at the beginning of Sec. II. Kinematic variables in the c.m. frame of $h_1'+h_2'$ will be distinguished by primes. Thus, y_3' is the rapidity of h_3 in the frame of $h_1'+h_2'$.

Now divide the space of rapidity and transverse momentum variables (y,p_1,p_2) into a collection of cells

$$I(i,j,k)=\{(y,p_1,p_2)|\,i\delta\le y\le(i+1)\delta,\ (j-\tfrac{1}{2})\epsilon\le p_1<(j+\tfrac{1}{2})\epsilon,\ (k-\tfrac{1}{2})\epsilon\le p_2<(k+\tfrac{1}{2})\epsilon\}\tag{4.2}$$

where ϵ is a small positive real number, $\delta=Y/2n$, and i, j, k, and n are integers with $-n\le i<n$, and $2n$ is the largest even integer less than Y/δ' for an energy-independent constant δ'. Divide the space of variables (y',p_1',p_2') into cells in an identical way and call these $I'(i,j,k)$. A Lorentz transform does not convert a collection of cells $I(i,j,k)$ precisely into a set of $I'(i,j,k)$, but for large regions covered by $I(i,j,k)$ a corresponding set of $I'(i,j,k)$ can be found which almost covers the Lorentz image exactly. Since the measure $dydp_1dp_2$ is equal to $dy'\,dp_1'\,dp_2'$, and the volume of each $I(i,j,k)$, $\delta\epsilon^2$, is the same as that of each $I'(i,j,k)$, the number of cells needed to cover a large region is almost unchanged under Lorentz transformation. If ϵ and δ are sufficiently small, the sum in (4.1) will include negligible contributions from states $h_3+\cdots$ with more than one particle in any cell $I(i,j,k)$ or $I'(i,j,k)$ or in any union of cells of the form

$$J(i)=\bigcup_{j,k}I(i,j,k)\ \text{or}\ J'(i)=\bigcup_{j,k}I'(i,j,k).$$

Moreover, the total contribution to (4.1) given by $h_3+\cdots$ with a particle in any cell $I(i,0,0)$ or $I'(i,0,0)$ will also be negligible. Now restrict the possible values of the pairs of variables p_1,p_2, and p_1',p_2' to the centers of $I(i,j,k)$ and $I'(i,j,k)$, respectively. Then each intermediate state which contributes to (4.1) can be specified by $p(i)=[p_1(i),p_2(i),p_3(i)]$ giving the transverse momenta $p_1(i),p_2(i)$, and charge $p_3(i)$ contained in the volume $J(i)$, where we adopt the convention that if $p_1(i)=p_2(i)=0$ the volume $J(i)$ is assumed empty. An equivalent description using $p'(i)=[p_1'(i),p_2'(i),p_3'(i)]$ can be made in the frame of $h_1'+h_2'$. Let ψ be the set of values of i such that $J(i)$ contains no particles and define ψ' similarly for $J'(i)$. Assume $N(\psi)$ and $N(\psi')$ are the number of elements of ψ and ψ', respectively. We have $N(\psi)=N(\psi')$.

If we replace the δ functions in (4.1) with smooth approximations which change slowly over any cell $I(i,j,k)$ or $I'(i,j,k)$, Eq. (4.1) can be rewritten

$$\mathrm{Im}\,T(h_1+h_2\to h_1'+h_2')=\tfrac{1}{2}\prod_i\Big[\sum_{p(i)}\delta\epsilon^2\Big]\delta\Big[\sum_i p_1(i)\Big]\delta\Big[\sum_i p_2(i)\Big]\delta\Big[\sum_{i\notin\psi}m_\perp(i)e^{y(i)}-\sqrt{s}\Big]\delta\Big[\sum_{i\notin\psi}m_\perp(i)e^{-y(i)}-\sqrt{s}\Big]$$
$$\times T[p(i)]\,T'^*[p'(i)]\,(\delta\epsilon^2)^{-N(\psi)}\prod_{i\in\psi}\delta[p_3(i),0],\tag{4.3}$$

where

$$T[p(i)]=T(h_1+h_2\to h_3+\cdots),\qquad T'[p'(i)]=T(h_1'+h_2'\to h_3+\cdots),$$

$$m_\perp{}^2(i)=m^2+p_1{}^2(i)+p_2{}^2(i),\qquad y(i)=(i+\tfrac{1}{2})\delta.$$

The quantity $\delta[p,r]$ in (4.3) is 1 for $p=r$ and 0 otherwise, and the factor $(\delta\epsilon^2)^{-N(\psi)}$ appears to adjust the phase volume for the missing degrees of freedom corresponding to each element of ψ. The Lorentz-transform properties of $I(i,j,k)$, $I'(i,j,k)$, and the energy-momentum δ functions of (4.1) imply that (4.3) is essentially unchanged if we replace all unprimed quantities in (4.3) with primed quantities, but leave the scattering amplitudes unchanged.

Define a zone graph $Z(i)=[Z_1(i),Z_2(i),Z_3(i)]$ from $p(i)=[p_1(i),p_2(i),p_3(i)]$ by the obvious extension of (2.2) and let $Z'(i)$ be obtained from $p'(i)$ in the same way. As before $p(i)$ and $p'(i)$ are both completely determined by either $Z(i)$ or $Z'(i)$. Thus, if we introduce the definitions

$$\mathcal{T}[Z(i)]=T[p(i)]\left\{\delta\Big[\sum_i p_1(i)\Big]\delta\Big[\sum_i p_2(i)\Big]\delta\Big[\sum_{i\in\psi} m_\perp(i)e^{y(i)}-\sqrt{s}\Big]\delta\Big[\sum_{i\notin\psi} m_\perp(i)e^{-y(i)}-\sqrt{s}\Big]\right\}^{1/2}$$
$$\times\,\delta^{n/2}(\delta\epsilon^2)^{-N(\psi)/2}\prod_{i\in\psi}\delta[p_3(i),0],$$
$$\mathcal{T}'[Z'(i)]=T'[p'(i)]\left\{\delta\Big[\sum_i p_1'(i)\Big]\delta\Big[\sum_i p_2'(i)\Big]\delta\Big[\sum_{i\notin\psi} m_\perp'(i)e^{y(i)}-\sqrt{s}\Big]\delta\Big[\sum_{i\notin\psi} m_\perp'(i)e^{-y(i)}-\sqrt{s}\Big]\right\}^{1/2}$$
$$\times\,\delta^{n/2}(\delta\epsilon^2)^{-N(\psi')/2}\sum_{i\in\psi'}\delta[p_3'(i),0], \tag{4.4}$$

Eq. (4.3) becomes simply

$$\mathrm{Im}\,T(h_1+h_2\to h_1'+h_2')=\tfrac{1}{2}\prod_i\Big[\sum_{Z(i)}\epsilon^2\Big]\mathcal{T}[Z(i)]\,\mathcal{T}'^*[Z'(i)]. \tag{4.5}$$

The amplitude $\mathcal{T}[Z(i)]$ is the quantity on which we will impose local compensation of quantum numbers. Before doing so, however, it is useful to reconstruct (4.5) by a rather formal derivation which perhaps makes the significance of $\mathcal{T}[Z(i)]$ somewhat clearer. Let Ω be a quantity which specifies the number of particles, charges and momenta appearing in any state $h_3+\cdots$. We assume the kinematic variables in Ω are given in the frame of h_1+h_2. Let Ω' contain the same information as Ω, but with its kinematic quantities given in the frame of $h_1'+h_2'$. Define $T(\Omega)$ and $T'(\Omega')$ by

$$T(\Omega)=T(h_1+h_2\to h_3+\cdots),\quad T'(\Omega')=T(h_1'+h_2'\to h_3+\cdots).$$

Equation (4.1) becomes

$$\mathrm{Im}\,T(h_1+h_2\to h_1'+h_2')=\tfrac{1}{2}\int d\mu(\Omega)T(\Omega)T'^*(\Omega'), \tag{4.6}$$

for an appropriate measure $d\mu(\Omega)$ on Ω. Since Ω and Ω' can also be specified by the zone graphs $Z(y)$ or $Z'(y)$, it is possible to change the variable of integration in (4.6) and obtain

$$\mathrm{Im}\,T(h_1+h_2\to h_1'+h_2')=\tfrac{1}{2}\prod_y\Big[\int\!\!\int dZ_1(y)dZ_2(y)\sum_{Z_3(y)}\Big]T(\Omega)T'^*(\Omega')\frac{d\mu(\Omega)}{\prod_y dZ_1(y)dZ_2(y)}. \tag{4.7}$$

With the definitions

$$\mathcal{T}[Z(y)]=T(\Omega)\left(\frac{d\mu(\Omega)}{\prod_y dZ_1(y)dZ_2(y)}\right)^{1/2},\quad \mathcal{T}'[Z'(y)]=T'(\Omega')\left(\frac{d\mu(\Omega')}{\prod_y dZ_1'(y)dZ_2'(y)}\right)^{1/2}, \tag{4.8}$$

Eq. (4.7) is equivalent to

$$\mathrm{Im}\,T(h_1+h_2\to h_1'+h_2')=\tfrac{1}{2}\prod_y\Big[\int\!\!\int dZ_1(y)dZ_2(y)\sum_{Z_3(y)}\Big]\mathcal{T}[Z(y)]\,\mathcal{T}'[Z'(y)]\left(\frac{\prod_y dZ_1'(y)dZ_2'(y)}{\prod_y dZ_1(y)dZ_2(y)}\right)^{1/2}. \tag{4.9}$$

The quantities $\mathcal{T}[Z(y)]$ and $\mathcal{T}'[Z'(y)]$ in (4.8) and (4.9) may be considered the continuum limits of $\mathcal{T}[Z(i)]$ and $\mathcal{T}'[Z(i)]$, respectively. Equations (4.8) and (4.9) are the continuum limits of (4.4) and (4.5), respectively. In addition, our derivation of (4.5) implies that the Jacobian which appears as the argument of the square root in (4.9) is equal to 1.

Let us extract from $\mathcal{T}[Z(i)]$ the set of moments $\{G_{j_1\cdots j_m}(i_1,\ldots,i_m)\}$

$$G=\prod_i\Big[\sum_{Z(i)}\epsilon^2\Big]\mathcal{T}[Z(i)],\quad G_{j_1\cdots j_m}(i_1,\ldots,i_m)=G^{-1}\prod_i\Big[\sum_{Z(i)}\epsilon^2\Big]Z_{j_1}(i_1)\cdots Z_{j_m}(i_m)\mathcal{T}[Z(i)], \tag{4.10}$$

and take a cluster decomposition of these to obtain the correlations $\{g_{j_1\cdots j_m}(i_1,\ldots,i_m)\}$. The assumption of local compensation of quantum numbers can be extended to multiparticle amplitudes by requiring that the set $\{g_{j_1\cdots j_m}(i_1,\ldots,i_m)\}$ obeys A–D.

In addition, however, we will need a restriction governing the normalization G. The motivation for this rule can be gotten by examining a sum of $\mathcal{T}[Z(i)]$ over a subset of the $Z(i)$. Assume s is sufficiently large that Y is much greater than the correlation length λ which appears in A–D. Divide rapidity space into a collection of regions

$$R(k)=\{y\,|\,kY/(2N+1)-Y/2\le y<(k+1)Y/(2N+1)-Y/2\},$$

where k and N are integers, $0\le k<2N+1$, $2N+1$ is the largest odd integer less than Y/ρ, and ρ is an energy-independent constant much greater than λ. Let $z^k(i)$ represent the function obtained by restricting $Z(i)$ to lattice points within the interval $R(k)$. Define $\overline{\mathcal{T}}[z^0(i),z^2(i),\ldots,z^{2N}(i)]$ by summing over odd $z^k(i)$

$$\overline{\mathcal{T}}[z^0(i),z^2(i),\ldots,z^{2N}(i)]=\prod_i\Big[\sum_{z^1(i)}\epsilon^2\Big]\prod_i\Big[\sum_{z^3(i)}\epsilon^2\Big]\cdots\prod_i\Big[\sum_{z^{2N-1}(i)}\epsilon^2\Big]\mathcal{T}[Z(i)].$$

Since each of the remaining variables $z^k(i)$ in $\overline{\mathcal{T}}[z^0(i),z^2(i),\ldots,z^{2N}(i)]$ is separated from the others by a region many correlation lengths long, it can be shown that A–D imply $\overline{\mathcal{T}}[z^0(i),z^2(i),\ldots,z^{2N}(i)]$ is nearly given by a product of the form

$$\overline{\mathcal{T}}[z^0(i),z^2(i),\ldots,z^{2N}(i)]=t^0[z^0(i)]\,t^2[z^2(i)]\cdots t^{2N}[z^{2N}(i)]. \tag{4.11}$$

In addition, translational invariance, condition C, implies that if s is sufficiently large a single energy-independent $t[z^k(i)]$ exists such that

$$t^k[z^k(i)]=t[z^k(i)],\quad 2\le k\le 2N-2. \tag{4.12}$$

But since $z^0(i)$ is many correlation lengths from h_2 and $z^{2N}(i)$ is many correlation lengths from h_1, it seems reasonable to suppose that an energy-independent $u[z^k(i),h_l]$ exists for which

$$t^0[z^0(i)]=u[z^0(i),h_1],\quad t^{2N}[z^{2N}(i)]=u[z^{2N}(i),h_2]. \tag{4.13}$$

Summing $\overline{\mathcal{T}}[z^0(i),z^2(i),\ldots,z^{2N}(i)]$ over its remaining arguments we obtain

$$G=\Big\{\prod_i\Big[\sum_{z(i)}\epsilon^2\Big]u[z(i),h_1]\Big\}\Big\{\prod_i\Big[\sum_{z(i)}\epsilon^2\Big]u[z(i),h_2]\Big\}\Big\{\prod_i\Big[\sum_{z(i)}\epsilon^2\Big]t[z(i)]\Big\}^{N-1},$$

so that as s becomes large G assumes the form

$$G=g(h_1)g(h_2)s^\gamma, \tag{4.14}$$

where

$$\gamma=\rho^{-1}\ln\Big\{\prod_i\Big[\sum_{z(i)}\epsilon^2\Big]t[z(i)]\Big\},\quad g(h_k)=m^{-\gamma}\prod_i\Big[\sum_{z(i)}\epsilon^2\Big]u[z(i),h_k].$$

We will therefore require that when s becomes large G has the form given by (4.14) for some energy-independent function $g(h_k)$ and a constant γ. Equation (4.14) will be referred to as condition E. If A–E are satisfied by G and $\{g_{j_1\cdots j_m}(i_1,\ldots,i_m)\}$, we will say that $\mathcal{T}[Z(i)]$ obeys local compensation of quantum numbers. It can be shown that if $\mathcal{T}[Z(i)]$ does fulfill local compensation of quantum numbers, then Eq. (4.13) which we guessed for $t^0[z^0(i)]$ and $t^{2N}[z^{2N}(i)]$ will also be satisfied.

The content of A–E is perhaps made somewhat clearer by recognizing that if these conditions are fulfilled and higher-order correlations are sufficiently well behaved, then it can be shown that $\mathcal{T}[Z(i)]$ must be nearly given by an expression of the form

$$\mathcal{T}[Z(i)]=U[z^0(i),z^1(i),h_1]\,V[z^1(i),z^2(i)]\cdots V[z^{2N-2}(i),z^{2N-1}(i)]\,U[z^{2N-1}(i),z^{2N}(i),h_2],$$

where $U[z(i),z'(i),h_k]$ and $V[z(i),z'(i)]$ are energy-independent functions and $z^0(i),z^1(i),\ldots$ are the functions obtained before by restricting $Z(i)$ to the intervals $R(0),R(1),\ldots$.

If the lattice $\{y_i\}$ in Sec. III is chosen correctly and each $Z_1(i)$ and $Z_2(i)$ restricted to appropriate discrete points, then the probability distribution $P[Z(i)]$ is related to $\mathcal{T}[Z(i)]$ by

$$P[Z(i)]=\frac{\mathcal{T}^*[Z(i)]\,\mathcal{T}[Z(i)]}{\prod_i\Big[\sum_{Z(i)}\epsilon^2\Big]\mathcal{T}^*[Z(i)]\,\mathcal{T}[Z(i)]}. \tag{4.15}$$

If $\mathcal{T}[Z(i)]$ fulfills A–E, it can be shown that the correlations gotten from $P[Z(i)]$ fulfill the form of local compensation of quantum numbers discussed in Secs. II and III. Conversely, if $P[Z(i)]$ fulfills the version of local compensation considered in Secs. II and III, then one can show that the correlations gotten from $\mathcal{T}[Z(i)]$, after removing the phase of this amplitude, fulfill A–D. The complete set of conditions A–E is obeyed by the amplitude $\mathcal{T}[Z(i)]$ given by most versions of the multiperipheral model, such as the model of Ref. 12. Thus, the formulation of local compensation of quantum numbers we have imposed on $\mathcal{T}[Z(i)]$ would appear to be a fairly reasonable extension of the weaker version of this condition considered in Secs. II and III for $P[Z(i)]$.

Combining (2.5) and A–E, we can calculate the amplitude for two-body quantum-number exchange. The amplitude $\mathcal{T}[Z(i)]$ can be written

$$\mathcal{T}[Z(i)] = g(h_1)g(h_2)s^\gamma \prod_i \left[\int \frac{d^3\alpha(i)}{(2\pi)^3} \right] \times \exp\left\{ -i \sum_{i,j} \alpha_j(i) Z_j(i) + \mathcal{G}[\alpha(i)] \right\}, \tag{4.16}$$

$$\mathcal{G}[\alpha(i)] = \sum_{m=1}^{\infty} \sum_{\substack{i_1, \ldots, i_m \\ j_1, \ldots, j_m}} [i\alpha_{j_1}(i_1)] \cdots [i\alpha_{j_m}(i_m)] \times \frac{g_{j_1 \cdots j_m}(i_1, \ldots, i_m)}{m!}. \tag{4.17}$$

The integrations with respect to $d\alpha_1(i)$ and $d\alpha_2(i)$ in (4.16) are from $-\pi/\epsilon$ to π/ϵ, and the integration with respect to $d\alpha_3(i)$ is from $-\pi$ to π. These limits will also be used on all other integrations with respect to $d^3\alpha(i)$ in the present section. We will assume higher $g_{j_1 \cdots j_m}(i_1, \ldots, i_m)$ grow sufficiently slowly that the summation of (4.16) converges within the range of integration of (4.15). Since the size of higher $g_{j_1 \cdots j_m}(i_1, \ldots, i_m)$ may be considered a measure of the degree of randomness of the fluctuations of $Z(i)$ in $\mathcal{T}[Z(i)]$, we assume, in effect, that $Z(i)$ is "sufficiently random." This assumption would seem to be a natural extension of the arguments given earlier suggesting that higher correlations gotten from $P[Z(i)]$ should not be too large. If we let the correlations obtained from $\mathcal{T}'[Z'(i)]$ be $\{g'_{j_1 \cdots j_m}(i_1, \ldots, i_m)\}$, then $\mathcal{T}'[Z'(i)]$ is given by (4.16) and (4.17) with primes on appropriate quantities. As we mentioned before, however, it is shown in Refs. 1 and 3–5 that for values of i outside finite, energy-independent regions at both ends of rapidity space we have

$$Z'_j(i) = Z_j(i) - q_j, \quad j = 1, 2, 3. \tag{4.18}$$

In addition, for values of $i_1, \ldots, i_m$ many correlation lengths from the ends of rapidity space, condition D implies

$$g'_{j_1 \cdots j_m}(i_1, \ldots, i_m) = g_{j_1 \cdots j_m}(i_1, \ldots, i_m), \tag{4.19}$$

while near the ends of rapidity space $g_{j_1 \cdots j_m}(i_1, \ldots, i_m)$ and $g'_{j_1 \cdots j_m}(i_1, \ldots, i_m)$ are determined only by whichever of the hadrons h_k or h'_k, respectively, are located nearby. Suppose $\alpha(i) = [\alpha_1(i), \alpha_2(i), \alpha_3(i)]$ is an arbitrary three-component function and let $\alpha^k(i) = [\alpha_1^k(i), \alpha_2^k(i), \alpha_3^k(i)]$ be obtained by restricting $\alpha(i)$ to the region hear h_k in which (4.18) and (4.19) are violated. Let $g''_{j_1 \cdots j_m}(i_1, \ldots, i_m)$ be given by $g_{j_1 \cdots j_m}(i_1, \ldots, i_m)$ in the region in which (4.18) and (4.19) are fulfilled and by 0 elsewhere. Then combining Eq. (2.5) with the arguments we have just given it follows that a function $f[\alpha^k(i), h_k, h'_k, q]$ exists such that the imaginary part of the amplitude for two-body quantum-number exchange is given by

$$\operatorname{Im} T(h_1 + h_2 \to h'_1 + h'_2) = \tfrac{1}{2} g(h_1) g^*(h'_1) g(h_2) g^*(h'_2) s^{\gamma + \gamma^*} \prod_i \left[\int \frac{d^3\alpha(i)}{(2\pi)^3} \right] f[\alpha^1(i), h_1, h'_1, q] f[\alpha^2(i), h_2, h'_2, q] \times \exp\left\{ -i \sum_{i,j} \alpha_j(i) q_j + \mathcal{G}''[\alpha(i)] + \mathcal{G}''^*[\alpha(i)] \right\}, \tag{4.20}$$

where $\mathcal{G}''[\alpha(i)]$ is gotten from $\{g''_{j_1 \cdots j_m}(i_1, \ldots, i_m)\}$ by an equation of the same form as (4.17), and none of the quantities in (4.20) depend on h_1, h_2, h'_1, h'_2 or q except as explicitly indicated.

Consider the expression obtained by replacing q with 0 in the argument of the exponential of (4.20)

$$B = \prod_i \left[\int \frac{d^3\alpha(i)}{(2\pi)^3} \right] f[\alpha^1(i), h_1, h'_1, q] f[\alpha^2, h_2, h'_2, q] \exp\{\mathcal{G}''[\alpha(i)] + \mathcal{G}''^*[\alpha(i)]\}. \tag{4.21}$$

Since by B and C the correlations from which $\mathcal{G}''[\alpha(i)]$ is constructed are short range and translationally invariant at points removed from the boundaries of rapidity space, B has the form of a partition function for a one-dimensional system with short-range interactions. Again by a slight extension of results by Araki,[17] it follows that there exists an energy-independent function $b(h_k, h'_k, q)$ and a real number ϕ such that as s becomes large, B assumes the form

$$B = b(h_1, h'_1, q) b(h_2, h'_2, q) s^\phi. \tag{4.22}$$

Now define another set of moments

$$B_{j_1 \cdots j_m}(i_1, \ldots, i_m) = B^{-1}\prod_i\left[\int \frac{d^3\alpha(i)}{(2\pi)^3}\right]\alpha_{j_1}(i_1)\cdots\alpha_{j_m}(i_m) f[\alpha^1(i), h_1, h_1', q] f[\alpha^2(i), h_2, h_2', q] \times \exp\{\mathcal{G}''[\alpha(i)] + \mathcal{G}''^*[\alpha(i)]\}. \tag{4.23}$$

The correlations obtained from a cluster decomposition of $\{B_{j_1 \cdots j_m}(i_1, \ldots, i_m)\}$ will be written $\{b_{j_1 \cdots j_m}(i_1, \ldots, i_m)\}$. Using (4.20)–(4.23), $\operatorname{Im} T(h_1 + h_2 \to h_1' + h_2')$ can now be written

$$\operatorname{Im} T(h_1 + h_2 \to h_1' + h_2') = \tfrac{1}{2} s^{\gamma+\gamma^*+\phi} g(h_1) g^*(h_1') g(h_2) g^*(h_2') \times b(h_1, h_1', q) b(h_2, h_2', q) \exp\left[\sum_{m=1}^{\infty} \sum_{j_1 \cdots j_m} (iq_{j_1}) \cdots (iq_{j_m}) \frac{b_{j_1 \cdots j_m}}{m!}\right], \tag{4.24}$$

$$b_{j_1 \cdots j_m} = \sum_{i_1, \ldots, i_m} b_{j_1 \cdots j_m}(i_1, \ldots, i_m).$$

Since the range of integration with respect to each $d^3\alpha(i)$ is finite in (4.23), all the integrals in (4.23) converge and the summation in (4.24) must converge at least in some region near $q_1 = q_2 = q_3 = 0$. But conditions A–D for the set $\{g_{j_1 \cdots j_m}(i_1, \ldots, i_m)\}$ combined with the arguments of Sec. III imply the set $\{b_{j_1 \cdots j_m}(i_1, \ldots, i_m)\}$ fulfills A–D. Therefore, as s becomes large each $b_{j_1 \cdots j_m}$ assumes the form

$$b_{j_1 \cdots j_m} = \ln(s)\alpha_{j_1 \cdots j_m} + \beta_{j_1 \cdots j_m}(h_1, h_1', q) + \beta_{j_1 \cdots j_m}(h_2, h_2', q),$$
$$\alpha_{j_1 \cdots j_m} = \delta^{-1} \sum_{i_2, \ldots, i_m} b_{j_1 \cdots j_m}(0, i_2, \ldots, i_m), \tag{4.25}$$

where each $\alpha_{j_1 \cdots j_m}$ is independent of s and q, each $\beta_{j_1 \cdots j_m}$ is independent of s, and δ is the quantity which appears in Eq. (4.2). Equations (4.24) and (4.25) finally yield the result that when s becomes large the imaginary part of the amplitude for two-body quantum-number exchanges becomes

$$\operatorname{Im} T(h_1 + h_2 \to h_1' + h_2') = \beta(h_1, h_1', q)\beta(h_2, h_2', q) s^{\alpha(q)},$$
$$\alpha(q) = \alpha + \sum_{\substack{m=2 \\ m\,\text{even}}}^{\infty} \sum_{j_1 \cdots j_m} (-1)^{m/2} q_{j_1} \cdots q_{j_m} \frac{\alpha_{j_1 \cdots j_m}}{m!}, \tag{4.26}$$

where $\alpha = \gamma + \gamma^* + \phi$ is a real number and $\beta(h_k, h_k', q)$ is independent of s. Condition D, rotational invariance and charge-conjugation invariance yield $\alpha_{j_1 \cdots j_m} = 0$ if an odd number of the subscripts $j_1 \cdots j_m$ assume a single value. Thus, we have shown that (2.5) combined with local compensation of quantum numbers by many-particle production amplitudes implies that the imaginary part of the amplitude for two-body quantum-number exchange asymptotically approaches the form given by the exchange of a Regge pole with factorizable residues. As we mentioned in Sec. II, however, if the phase of $T(h_1 + h_2 \to h_1' + h_2')$ is now restored by applying a dispersion relation to (4.26), the entire amplitude will have the same s dependence as (4.26) and, therefore, the entire amplitude will assume the form given by the exchange of a Regge pole with factorizable residues.

V. APPLICATIONS

We will consider two different applications of (4.26). First, let us examine a simplified model which illustrates the way in which the phases of multiparticle amplitudes suppress (4.26) below the bound given by (3.1). Assume that all $g_{j_1 \cdots j_m}(i_1, \ldots, i_m)$ are negligible except $g_{j_1 j_2}(i_1, i_2)$ and replace the sums in (4.5) and (3.1) with integrals. The trajectory $a(q)$ which appears in (3.15) can be obtained by evaluating multidimensional Gaussian integrals. We find

$$a(q) = -(q_1^2 + q_2^2)\frac{\cos\theta_{11}}{4g_{11}} - q_3^2 \frac{\cos\theta_{33}}{4g_{33}},$$
$$g_{ij}\exp(i\theta_{ij}) = \delta^{-1}\sum_k g_{ij}(0, k), \tag{5.1}$$

where δ is the lattice parameter from Eq. (4.2), and we have used rotational invariance, charge-conjugation invariance, and assumption D to conclude that g_{ij} and θ_{ij} are diagonal with $g_{11} = g_{22}$, $\theta_{11} = \theta_{22}$. The trajectory $\alpha(q)$ in (4.26), on the other hand becomes

$$\alpha(q) = \alpha - \frac{q_1^2 + q_2^2}{4g_{11}\cos\theta_{11}} - \frac{q_3^2}{4g_{33}\cos\theta_{33}}. \tag{5.2}$$

As either θ_{11} or θ_{33} approach $\pi/2$ the trajectory given by (5.2) falls progressively below the bound given by (5.1).

Equation (4.26) can also be used to obtain a number of qualitative conclusions concerning exotic mesons. Now the power series for $\alpha(q)$ is absolutely convergent in some region including

$q_1 = q_2 = q_3 = 0$. But if, as we argued in Sec. IV, higher $g_{j_1 \cdots j_m}(i_1, \ldots, i_m)$ do not grow too violently as m becomes large, then we would expect the set $\{b_{j_1 \cdots j_m}(i_1, \ldots, i_m)\}$ gotten from (4.23), and, therefore, the set $\{\alpha_{j_1 \cdots j_m}\}$ in (4.26), to behave smoothly for large m and not to grow too rapidly. Evidence that higher $\alpha_{11 \cdots 1}$ are small is provided by the approximate linearity of experimentally determined meson trajectories.[4] Thus, it would appear reasonable to assume the radius of convergence of the power series in (4.26) is fairly large and that, in addition, $\alpha(q)$ can be analytically continued to the entire complex plane with at most cuts, poles, and essential singularities, but without natural boundaries. If this is the case, (4.26) yields exotic meson trajectories with arbitrarily high charge. In each such multiplet we expect only the trajectory with highest charge to enter (4.26) directly. The trajectories carrying lower charge, however, appear in the corrections to (4.26) which are suppressed by some power of s below the leading asymptotic behavior for each q. These trajectories could be extracted from (4.20) but, of course, by isospin invariance, are degenerate with the trajectories of highest charge in each multiplet. In any case, if it were possible to obtain the set $\{g_{j_1 \cdots j_m}(i_1, \ldots, i_m)\}$ in some way, we could predict the trajectories of an infinite family of exotic mesons (and a number of normal meson trajectories in addition). This cannot be done at present, but we might guess that if higher $\alpha_{j_1 \cdots j_m}(i_1, \ldots, i_m)$ in the power series of (4.26) do not grow rapidly, then to obtain the intercept of the leading charge 2 trajectory (presumably composed of $qq\overline{q}\overline{q}$) we will not need more than the first few terms of the series. If we use only the constant term and the term quadratic in q_3, then since the Pomeron intercept is close to 1 and the intercept of the ρ is close to 0.6, we predict a leading exotic intercept close to -0.6. This result is consistent with a recent analysis of $B\overline{B}$ scattering yielding an intercept of -0.5 ± 0.3 (see Ref. 19). In addition, since the slope of the ρ differs significantly from the slope of the Pomeron, at least some higher $\alpha_{j_1 \cdots j_m}$ coupling q_1 and q_2 to q_3 must be fairly large. Therefore, we expect the slope of exotic trajectories to differ significantly from the ρ slope and the Pomeron slope and to depend on the maximum charge available in the isospin multiplet including the trajectory. Equivalently, since the leading trajectory with each charge would be expected to contain the minimum number of quark-antiquark pairs required to produce that charge, the slopes of meson trajectories should vary with the number of $q\overline{q}$ pairs each contains.

VI. CONCLUSION

It was shown in Refs. 1–7 that if two-body scattering processes occur primarily as the shadow of a class of multiparticle production events which obey the probability formulation of local compensation of quantum numbers, then data obtained from many particle production can be used to place an upper bound on the energy dependence of two-body quantum-number exchange. In the present paper we showed that the strongest bound of this sort, which involves an infinite-dimensional integration, can be expanded as an asymptotic series in the set of zone correlations. But we argued that the relation between zone correlations and density correlations combined with the result that higher density correlations appear to be small[13] suggests that higher zone correlations will also be small. Thus, the asymptotic series we obtained should be nearly given by its first few terms, which are the same as the leading terms in the bound given in Refs. 3, 5. Recent data[10] imply that this limit is significantly weaker than the observed energy dependence of two-body charge exchange. We concluded that if two-body scattering is expressed as the shadow of many-particle events, then the main contribution to the energy dependence of charge exchange is probably generated by the phase of many-particle amplitudes. This result is not completely surprising since it was shown some time ago[20,21] that the momentum-transfer dependence of two-body elastic scattering (at fixed energy) is also determined primarily by the phase of many-particle production. It may be useful to add, however, that in the multiperipheral overlap calculation of Ref. 12 phases do not make any contribution to the energy dependence of forward charge-exchange scattering. Thus, our results suggest this particular aspect of the model in Ref. 12 is not realistic.

Then we reformulated local compensation of quantum numbers as a condition on many-particle production amplitudes including their phases. From this assumption we derived an exact expression for the asymptotic behavior of two-body quantum-number exchange. The expression we obtained splits into a pair of terms which we interpreted as a factorized Regge residue and another term interpreted as a Regge propagator. Each of the terms of the residue is generated by contributions to the overlap integral near a corresponding end of rapidity space, while the propagator arises from contributions in the central region. Factorization of the two-body amplitude is a direct consequence of the absence of correlations in $\mathcal{T}[Z(y)]$ coupling these three regions, which in turn follows from the amplitude formulation of local compensation of quantum numbers.

The Regge trajectory we derived is an analytic function both of the squares of the two components of transverse momentum transfer, q_1^2 and q_2^2, and of the square of the charge transfer, q_3^2. Since only the highest charge in each isospin multiplet is expected to appear in this trajectory, and since, in addition, the trajectories of all charges in each multiplet are degenerate, q_3^2 is simply the square of the isospin for a particular isospin multiplet. Thus, we have found a natural means of organizing a collection of exotic Regge trajectories of increasing isospin into a single family. Perhaps this idea may have other applications.

If one wishes, the amplitude formulation of local compensation of quantum numbers may be thought of as a condition abstracted from versions of the multiperipheral model. From this point of view, the main result of Sec. IV is that the property we have abstracted by itself is sufficient to prove Regge behavior for the shadow of many-particle production; the numerical details of a multiperipheral amplitude and the unrealistic assumptions used in multiperipheral overlap calculations (for example, the absence of various "interference terms," or of "spill-over graphs" in multiperipheral cluster models) are not needed. We would like to believe that since the amplitude formulation of local compensation of quantum numbers is a weaker requirement than the more detailed assumptions of multiperipheral models, it stands a better chance of actually being fulfilled.

Although we assumed for convenience that all particles are pions, we could just as well have included any meson resonances which decay into pions through strong interactions. The zone graph for a final state including such resonances could be calculated simply by replacing each one with the pions produced when it decays. Our results could also be extended to include kaons by adding a fourth zone graph determined by the distribution of strangeness. However, since the rate for kaon production at high energy is of the order of $\frac{1}{10}$ the rate for pions, including the possibility of kaon production would not significantly alter our results for charge exchange.

A question which remains unanswered is how best to extend our calculation to include production of SU(2) or SU(3) singlets, or the production of baryons and other particles with spin. Another problem which remains is how to make a somewhat realistic calculation of $\mathcal{T}[Z(i)]$ or $\{g_{j_1 \cdots j_m}(i_1, \ldots, i_m)\}$. Either of these objects, of course, could be extracted from any of the familiar models of particle production. It seems that a more interesting possibility would be to find some way to calculate $\mathcal{T}[Z(i)]$ or $\{g_{j_1 \cdots j_m}(i_1, \ldots, i_m)\}$ directly, for example, from a field theory. Since higher $g_{j_1 \cdots j_m}(i_1, \ldots, i_m)$ are expected to be small, $\mathcal{T}[Z(i)]$ may actually be a simpler object that it seems.

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