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Interaction of two black holes in the slow-motion limit*

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The interaction of two black holes is considered under the assumptions that each black hole moves in the far field of the other and that the relative velocity is small compared to the speed of light; the black holes may rotate and possess comparable masses. A method of matched asymptotic expansions developed previously by the author is used to relate the gravitational field far from the black holes, described by a slow-motion perturbation of flat space, to the gravitational field near the holes, described by perturbations of the Kerr geometry. This leads to equations of motion and spin propagation giving the main deviations from Newtonian behavior. The derivation uses only the vacuum Einstein equations, and the resulting equations are generalizations of well-known expressions for the interaction of massive or rotating bodies in general relativity.

I. INTRODUCTION

In this paper we shall describe a method of analyzing the interaction of two black holes in a wide variety of physical situations. The black holes may be rotating and are permitted in our approach to have comparable masses, subject to two types of restriction on the interaction. First, it is assumed that the spatial separation of the black holes is large compared to the radii of their event horizons, so that each black hole is in the far field of the other. Second, we suppose that the relative velocity of the black holes is small compared to the speed of light, so that their combined gravitational field can be described in terms of a slow-motion approximation scheme. Then we should expect that the dominant interaction between them is just that between a pair of Newtonian point particles; it is further assumed in a slow-motion scheme that the Newtonian potential energy of the interaction is comparable to the relative kinetic energy.

While a slow-motion perturbation of Minkowski space will represent the gravitational field throughout most of the space, it will break down in regions near the black holes, where the field strength becomes large. A related question was discussed in a previous paper by the author[1] which considered the interaction of a black hole with an external "background" universe. In that paper it was shown how the gravitational field can be described in terms of two approximation schemes, valid in rather different regions of spacetime, provided that a characteristic mass or length scale M of the black hole is much smaller than the length scales on which the background geometry varies. In one of the perturbation expansions, called the external scheme, the effect of the black hole is to produce a small perturbation of the background geometry; this expansion is valid in the far field of the black hole. In the present context, it corresponds to the slow-motion perturbation of flat space by the sum of the far fields of the two black holes, together with nonlinear effects of higher order, which is valid away from the strong-field regions of each black hole. The other perturbation expansion used in Ref. 1, called the internal scheme, is valid in the strong-field region of the black hole, and describes the structure of the field on length scales of order M. The background only produces small distortions of the basic Kerr black-hole geometry in this region, since the curvature of the background is much less than the curvature M^{-2} of the black hole on these length scales. In the present context there will be two such internal approximation schemes, one for each black hole. One hole feels only the far field of the other, which is weak since it can be described by a slow-motion perturbation of flat space; as a result the "background" curvature acting on one of the black holes is much less than the curvature in the near-field region, so that the interaction again produces only a small perturbation in the basic Kerr geometry near the black hole. The internal and external asymptotic expansions for the gravitational field must then be related by matching in some intermediate region near the relevant black hole where both schemes are valid; this matching procedure is discussed in some detail in Ref. 1.

Now that we have summarized our approach to the description of the gravitational field, let us turn our attention to the motion of the black holes. Over short periods of time, the motion will be dominated by the Newtonian interaction. But in order to follow the behavior of the black holes over longer periods, we must understand the higher-order relativistic corrections to the Newtonian motion. For example, if the black holes are rotating, there should be modifications due to spin-orbit and spin-spin forces. There will also be effects due to the emission of gravitational waves, and to the capture of gravitational radiation by the holes. Moreover, the presence of a second black hole will cause the direction of the spin of the first to precess slowly. It is the main aim of this work to calculate the most significant corrections to the equations of motion and the lowest-order equations of spin propagation. Before discussing our method further, we mention two previous approaches relevant to our problem, which will suggest the nature of the equations of motion and spin propagation which we seek.

Wald[2] has examined the interaction of spinning bodies in general relativity by studying the behavior of a spinning test particle in the far field of a rotating gravitational source. The test particle obeys the Papapetrou equations,[3] which are derived from the conservation equations[4]

$$T^{ab}{}_{;b}=0 \tag{1.1}$$

by making certain assumptions about the matter in the test particle, and describe the deviation of the particle's motion from a geodesic due to spin effects, together with the variation of the spin along the particle world line. Suppose that $M, \vec{J}$ are the mass and angular momentum of the source, which is at the origin of the spatial coordinates, and $\vec{r}$, $\vec{v}$, and $\vec{\Sigma}$ the position, velocity, and spin of the test body. We use units here such that Newton's constant G and the speed of light c are 1. Then, as shown in Ref. 2, the active gravitational effect of $\vec{J}$ gives the test body an acceleration

$$\vec{\alpha}=r^{-5}\vec{v}\times[2r^2\vec{J}-6\vec{r}(\vec{r}\cdot\vec{J})] . \tag{1.2}$$

The lowest-order spin-orbit force, due to deviation from geodesic motion in the dominant Schwarzschild part of the far field, is given as

$$\vec{F}=-3Mr^{-5}\{r^2\vec{v}\times\vec{\Sigma}+2\vec{r}[\vec{v}\cdot(\vec{r}\times\vec{\Sigma})] - (\vec{r}\cdot\vec{v})(\vec{r}\times\vec{\Sigma})\} . \tag{1.3}$$

The lowest-order spin-spin force, again due to deviation from geodesic motion in a far field which takes account of $\vec{J}$, is

$$\vec{F}=-\nabla\{r^{-5}[-r^2\vec{\Sigma}\cdot\vec{J}+3(\vec{r}\cdot\vec{J})(\vec{r}\cdot\vec{\Sigma})]\} . \tag{1.4}$$

The spin propagation equations show that the precession of the spin vector of the test body due to the Schwarzschild part of the far field is essentially

$$\frac{d\vec{\Sigma}}{dt}=\frac{3}{2}Mr^{-3}(\vec{r}\times\vec{v})\times\vec{\Sigma} . \tag{1.5}$$

The largest effect on the spin precession due to the active gravitational effect of $\vec{J}$ is

$$\frac{d\vec{\Sigma}}{dt}=r^{-5}\vec{\Sigma}\times[r^2\vec{J}-3\vec{r}(\vec{r}\cdot\vec{J})] . \tag{1.6}$$

These equations describe most of the dominant effects to be found in the distant interaction of two normal spinning bodies. If one believes that the energy-momentum tensor inside a small rotating black hole is such as to justify the use of the Papapetrou equations in describing the motion and spin propagation, then Eqs. (1.2)–(1.6) should apply to the situation where a small rotating black hole moves in the far field of a much larger black hole. Even if one admits ignorance about T_{ab} inside a black hole, Eqs. (1.2)–(1.6) are quite plausible candidates for the equations of motion and spin propagation in this situation. We might hope that any equations which we derive for the interaction of two black holes of comparable mass should enable us to recover the effects of Eqs. (1.2)–(1.6) by taking one of the masses to be zero, after making allowance (if necessary) for any differences in the choice of coordinates used to describe the gravitational field.

Another approach to the problem of equations of motion in general relativity is through the method of Einstein, Infeld, and Hoffmann[5-7] (which we refer to as the EIH method). EIH show that the empty-space field equations alone determine the motion of the type of particles which they consider. We encountered a similar phenomenon in Ref. 1, where we used only empty-space equations and found that a small black hole moves approximately on a geodesic of the background universe. The EIH method is a slow-motion approximation scheme, in which a finite number of low-mass, slowly moving particles interact by Newtonian forces together with small relativistic corrections. The gravitational field away from the particles is treated as a slow-motion perturbation of flat space. Just as we found for the metric terms $g^{(1)}_{ab}, g^{(2)}_{ab}, \ldots$ of Ref. 1, which describe the first-, second-, ... order perturbations induced on the background by the black hole, the EIH metric perturbations of flat space become singular at the world lines of the particles. The type of singularity occurring is restricted by EIH so that the particles should in some sense be as spherical as possible and represent nonrotating bodies. When the Einstein equations are solved order by order in the perturbation scheme, one finds that certain integrability conditions must be satisfied by the metric components up to a given perturbation order, before the field equations can be solved to find the next-order metric components. The EIH equations of motion are integrability conditions of this type, which state that certain integrals taken over two-surfaces surrounding the particle world lines must vanish. EIH showed by this means that the lowest-order equations of motion are indeed Newtonian, and also computed the lowest-order corrections to the equations of motion for the two-body problem. These corrections are of post-Newtonian order, i.e., of order GM/Rc^2 smaller than Newtonian terms, where R is a typical distance between the particles, M a typical mass, and we have written the G and c dependence explicitly. When applied to a bound two-body system, these equations show that the post-Newtonian perihelion precession is $6\pi(M_1+M_2)Gp^{-1}c^{-2}$ radians per revolution,[8] where M_1, M_2 are the particle masses and p is the semilatus rectum of the relative orbit.

Our approximation scheme should also allow us to recover EIH's post-Newtonian effects when we specialize to the case of two nonrotating black holes, after making allowance for differences in the choice of coordinates. This will provide an independent check on our analysis, quite apart from the comparison with Eqs. (1.2)–(1.6).

It would be possible to treat the interaction of two black holes by a combination of the matching technique of Ref. 1 with the EIH method. A small modification of EIH's surface integral method allows one to extract equations of spin propagation as well as equations of motion from the vacuum field equations. This approach was originally taken by the author; unfortunately, it leads to algebra much worse than that encountered by EIH, since it is hard to combine the matching technique with convenient coordinate conditions on the slow-motion perturbations of flat space. Recently an alternative approach has become available, since Chrzanowski[9] has shown how one may compute vacuum metric perturbations of the Kerr geometry explicitly from the Riemann tensor perturbations analyzed by Teukolsky.[10] In the new approach, to be described in this paper, we find equations of motion and spin propagation for the black holes by computing certain parts of the spacetime metric in two different ways. These parts of the metric are computed in one way in the course of the slow-motion perturbation of flat space; in this computation one imposes boundary conditions at infinity. They are also computed in terms of one of the internal schemes describing perturbations of a Kerr metric, which rely on the formulations of Teukolsky and Chrzanowski; in this computation one has imposed boundary conditions at the future horizon of the black hole. When one relates the internal scheme to the slow-motion scheme by matching, the equations of motion and spin propagation follow. This method is considerably simpler to apply than EIH's, and again uses only the empty-space field equations.

As in the EIH method, our metric perturbations of flat space which describe the combined far fields of the two black holes become singular near the black-hole world lines. Whereas EIH determine the singular behavior of the metric perturbations by solving the field equations and using rules about the addition of terms which represent extra multipoles in their particles, we find the singular terms near a world line by matching to the appropriate internal approximation scheme. This should allow us to express unambiguously the condition that the "particles" in our slow-motion scheme are really Kerr-like black holes. One expects that the entire physical system is defined by a small number of parameters describing the orbits, masses, and spins of the black holes, together with the boundary condition that no gravitational radiation is incoming from past null infinity. There should be no freely specifiable multipoles for the black holes beyond the basic Kerr parameters. For example, by the arguments of Ref. 1, the largest distortion of the geometry near one of the black holes is of a quadrupole character, determined by the strongest part of the "background" Riemann tensor, which will arise here from the Newtonian field of the other black hole. At a sufficiently high order in the approximation scheme, the quadrupole bulge will produce active gravitational effects via the matching. These effects are determined entirely by the "background" Riemann tensor, together with the boundary condition of regularity on the future event horizon, as we found in Ref. 1; higher-order internal distortions are similarly determined by the "external" field and horizon boundary conditions. This is in contrast to the EIH papers, where the particles could be given extra multipole structure, but high multipoles are suppressed in order to produce a unique approximation scheme.

We now explain the scaling which is used in ordering the various small parameters in the problem. The combined gravitational field of the two bodies, regarded as a slow-motion perturbation of flat space, is described in terms of time and space coordinates (u, w^i), where i runs from 1 to 3 and u is regarded as the zeroth coordinate. All metric perturbations depend on time through the variable (ϵu), where ϵ is a small parameter, and on space through w^i; hence spatial gradients are much larger than time derivatives. The black holes move approximately on orbits

$$\begin{aligned} w^i &= r_1^i(\epsilon u), \\ w^i &= r_2^i(\epsilon u) \end{aligned} \tag{1.7}$$

where r_1^i, r_2^i are certain position functions which we shall show later to obey Newtonian equations of motion for point particles. Thus as $\epsilon \to 0$, we fix the spatial positions of the orbits but lengthen the time needed to traverse them. The masses M_1, M_2 of the black holes scale as

$$M_1 = \epsilon^2 \mu_1,$$
$$M_2 = \epsilon^2 \mu_2. \tag{1.8}$$

The oriented Kerr spin parameters a_1^i, a_2^i (where $|\vec{a}_1|$ would be the Kerr parameter a_1 of the first black hole in the usual notation) scale as

$$\vec{a}_1 = \vec{\chi}_1 M_1,$$
$$\vec{a}_2 = \vec{\chi}_2 M_2, \tag{1.9}$$

where $|\vec{\chi}_1| < 1$, $|\vec{\chi}_2| < 1$. The indices $i, j, k, l, \ldots$ will always run from 1 to 3, and we treat quantities such as a_1^k, B^{kl}, labeled by these indices, as Cartesian vectors or tensors, to which a Cartesian summation convention applies. A dot over a quantity indicates differentiation with respect to (ϵu). For example, $\dot{r}_1^i = [d/d(\epsilon u)] r_1^i(\epsilon u)$.

This scaling ensures that the Newtonian potential energy $M_1 M_2 |\vec{r}_1 - \vec{r}_2|^{-1}$ and kinetic energy $\frac{1}{2}\epsilon^2 M_1 |\dot{\vec{r}}_1|^2 + \frac{1}{2}\epsilon^2 M_2 |\dot{\vec{r}}_2|^2$ are both of order ϵ^4. It is applicable to a bound two-body system, but could also be used to describe a scattering orbit, provided that the velocities were small compared to c, and that the kinetic energy was not too much greater than the potential energy.

We may now classify the post-Newtonian corrections which we expect to find in the equations of motion and spin propagation, according to their order in the small parameter ϵ. We first outline our attitude toward the interpretation of these corrections.

The higher-order equations of motion or spin propagation will be written as time-evolution equations for certain correction terms to the lowest-order position or angular momentum. For example, the largest correction to the equation of motion for the first black hole will appear as an equation for $\ddot{r}_{12}^i(\epsilon u)$, where $\epsilon^2 r_{12}^i(\epsilon u)$ is a position displacement such that $r_1^i(\epsilon u) + \epsilon^2 r_{12}^i(\epsilon u)$ might be regarded as the position of the black hole, allowing for corrections up to $O(\epsilon^2)$. We prefer not to regard $r_{12}^i(\epsilon u)$ in this way, since the black hole is itself spread out over a length scale of $O(M_1) = O(\epsilon^2)$. Instead we remark that the relativistic corrections will gradually accumulate to produce effects of $O(1)$ over a sufficiently long time scale of $O(\epsilon^{-n})$, for some $n > 0$, and that the black-hole orbits should be well approximated by a sequence of Newtonian orbits in which the Newtonian orbit parameters change over a time scale $\gg O(\epsilon^{-1})$, as measured by the time u. Thus, although it will be convenient to refer to $\epsilon^2 r_{12}^i(\epsilon u)$ as the post-Newtonian position shift of the first hole, we prefer to treat it as describing part of a change in the orbit parameters which is taking place over a time scale of $O(\epsilon^{-3})$. Similarly, the spin-propagation equations will be regarded as describing fractional changes of $O(1)$ in the Kerr parameters $\vec{a}_1, \vec{a}_2$ over long time scales.

We shall say that an effect is n post-Newtonian if it produces $O(1)$ changes in the orbit or $O(1)$ fractional changes in $\vec{a}_1, \vec{a}_2$ over a time scale of $O(\epsilon^{-2n-1})$. Then the largest deviations from Newtonian motion are post-Newtonian, and should be equivalent to the post-Newtonian deviations found by EIH. In particular, it should be possible to recover the EIH perihelion shift for a bound orbit. The largest precession of the spin vectors will also be a post-Newtonian effect, given by some generalization of Eq. (1.5). Effects such as those given by Eqs. (1.2) and (1.3), in which a moving body is affected by the spin of another body, or in which a rotating moving body is affected by the Schwarzschild far field of another body, produce $1\text{-}\frac{1}{2}$ post-Newtonian changes in the orbit. The precession of one spin vector due to the rotation of the other body will also be a $1\text{-}\frac{1}{2}$ post-Newtonian effect, as in Eq. (1.6). The second post-Newtonian effects will be quite complicated, but among them one should find the largest spin-spin force, as in Eq. (1.4).

It has been convincingly demonstrated, at least in the context of slow-motion schemes for fluid bodies, that gravitational radiation effects first appear at the $2\text{-}\frac{1}{2}$ post-Newtonian order, and that quadrupole radiation is emitted in agreement with the linearized theory.[11,12] In our case the arguments of Ref. 11 can still be applied. This means that if the black holes are in a bound state, they will spiral towards each other and collide on a time scale of $O(\epsilon^{-6})$; some details of this process are given by Peters.[13]

Secular changes in the masses and scalar angular momenta of the black holes only appear at the $4\text{-}\frac{1}{2}$ post-Newtonian order and higher. For, by the arguments of Ref. 1, the largest internal perturbations of (say) black hole 1 will be due to the dominant part of the "external" Riemann tensor; as mentioned already, this dominant part arises from the Newtonian field of black hole 2, and will be of order $M_2 |\vec{r}_1 - \vec{r}_2|^{-3}$, i.e., of $O(\epsilon^2)$. Now in Ref. 1 we found it convenient to describe the gravitational field near a black hole of small mass M_1 in terms of "internal" time and space coordinates scaled by a factor M_1, so that the black hole has structure on a coordinate scale of $O(1)$, which corresponds to distances in spacetime of $O(M_1)$. When a conformal factor $(M_1)^2$ is removed from the metric, we are left with perturbations of a unit-mass Kerr metric (with some angular momentum χ_1). In our present case, the "external" Riemann tensor which perturbs black hole 1 is some function of (ϵu). A rescaled time coordinate appropriate for describing the structure of black hole 1 (provided we are not too close to the event horizon) is $U = u/M_1 = u/(\epsilon^2 \mu_1)$. Hence, when we view the internal scheme in terms of perturbations of a unit-mass Kerr metric, the perturbations will depend on time through functions of $(\epsilon^3 U)$. Viewed in this way, the internal scheme will be called the QS (quasistationary) scheme, since the perturbations change so slowly with internal time U; an extended discussion of the various viewpoints on the geometry of a black hole in an external field is given in Sec. II of Ref. 1. We further showed in Ref. 1 that the largest QS perturbations will be of order $(M_1)^2$ times the external Riemann tensor, as measured in some orthonormal frame. Hence in our case the QS perturbations will be of order $(M_1)^2\epsilon^2 = O(\epsilon^6)$. Since the area change of the event horizon is quadratic in the perturbations (Ref. 1, Sec. V), $[d(\text{area})/dU]/\text{area} = O(\epsilon^{12})$. Hence $[d(\text{area})/du]/\text{area} = O(\epsilon^{10})$, so that secular changes in the black hole's Kerr parameters are 4-$\frac{1}{2}$ post-Newtonian, as claimed above.

We have used the scaling of small parameters described by Eqs. (1.7)–(1.9) since it would apply to a large class of interactions between two black holes, and in particular to the interesting case where the black holes form a bound system and will eventually collide. But it is not necessary to combine our matching technique with a slow-motion expansion about flat space. One could instead use a fast-motion scheme in which the masses and spins scale as before, but now the lowest-order orbits are $r_1^i(u)$, $r_2^i(u)$, and all metric components in the perturbation of flat space depend on time through u rather than (ϵu). This would correspond to a distant scattering in which the two bodies are only slightly deflected from straight-line motion. This scaling would have the merit of exhibiting gravitational radiation at much lower orders than in the slow-motion scheme; see, for example, the work of Thorne and Kovács.[14] Yet another regime is described by letting the relative velocity of the holes tend to the speed of light; in this case the black holes behave almost like two colliding impulsive plane-fronted gravitational waves.[15]

In Sec. II we shall discuss the basic assumptions of our method, and illustrate the matching at low orders in the slow-motion scheme. In Secs. III and IV, certain 1st and 1-$\frac{1}{2}$ post-Newtonian metric terms, which are needed for finding equations of motion and spin propagation, are computed by using the post-linearized Einstein equations together with the matching conditions. Section V describes the computation of certain parts of the largest internal perturbations of the Kerr metric for either black hole, where we find boundary conditions on the perturbations by matching in towards the black holes and then apply Chrzanowski's work. The matching arguments which lead to the equations of motion and spin propagation up to 1-$\frac{1}{2}$ post-Newtonian order are given in Sec. VI, together with the calculation of the largest spin-spin force. Section VII comments on the results.

II. BASIC METHOD, LOWEST ORDERS

We begin the exposition of our method by making some basic assumptions about the approximation scheme. In the coordinate system (u, w^i) we expand the metric as

$$\begin{aligned} g_{00} &= -1 + \epsilon Q^1_{00}(\epsilon u, w^k) + \epsilon^2 Q^2_{00}(\epsilon u, w^k) + \cdots, \\ g_{0i} &= \epsilon Q^1_{0i}(\epsilon u, w^k) + \epsilon^2 Q^2_{0i}(\epsilon u, w^k) + \cdots, \\ g_{ij} &= \delta_{ij} + \epsilon Q^1_{ij}(\epsilon u, w^k) + \epsilon^2 Q^2_{ij}(\epsilon u, w^k) + \cdots. \end{aligned} \tag{2.1}$$

Traditionally, a metric term Q^n_{00} is known as $(n/2-1)$ post-Newtonian, Q^n_{0i} as $(n/2-\frac{1}{2})$ post-Newtonian, and Q^n_{ij} as $n/2$ post-Newtonian. These names are explained by the facts that a particle moving on a geodesic with a velocity of $O(\epsilon)$ is given a coordinate acceleration of $O(\epsilon^{2n+2})$ by an nth post-Newtonian term, that in the Newtonian limit particle accelerations are $O(\epsilon^2)$, and that the largest deviations from Newtonian dynamics ("post-Newtonian effects") are always $O(\epsilon^2)$ smaller than Newtonian effects.

We assume that

$$\begin{aligned} Q^1_{00} &= Q^3_{00} = 0\,, \\ Q^2_{00} &= 2\mu_1|\vec{w}-\vec{r}_1|^{-1} + 2\mu_2|\vec{w}-\vec{r}_2|^{-1}, \\ Q^1_{0i} &= Q^2_{0i} = 0\,, \\ Q^3_{0i} &= -2\mu_1\dot{r}^i_1|\vec{w}-\vec{r}_1|^{-1} \\ &\quad - 2\mu_1\dot{r}^k_1(w^k - r^k_1)(w^i - r^i_1)|\vec{w}-\vec{r}_1|^{-3} \\ &\quad - 2\mu_2\dot{r}^i_2|\vec{w}-\vec{r}_2|^{-1} \\ &\quad - 2\mu_2\dot{r}^k_2(w^k - r^k_2)(w^i - r^i_2)|\vec{w}-\vec{r}_2|^{-3}, \\ Q^1_{ij} &= Q^3_{ij} = 0\,, \\ Q^2_{ij} &= 2\mu_1(w^i - r^i_1)(w^j - r^j_1)|\vec{w}-\vec{r}_1|^{-3} \\ &\quad + 2\mu_2(w^i - r^i_2)(w^j - r^j_2)|\vec{w}-\vec{r}_2|^{-3}. \end{aligned} \tag{2.2}$$

These terms should describe, at the lowest orders, the conditions that the black holes have "masses" $\epsilon^2\mu_1$, $\epsilon^2\mu_2$ and move approximately along world lines $\vec{w}=\vec{r}_1(\epsilon u)$, $\vec{r}_2(\epsilon u)$. For if we put $\mu_2=0$ (say), then the metric at low orders is that of a Kerr black hole with mass $\epsilon^2\mu_1$, and oriented Kerr parameter $\vec{a}_1$, which has been given a translation and a slow Lorentz boost by velocity $\epsilon\dot{\vec{r}}_1$, so that it moves along a straight line $\vec{w}=\vec{r}_1(\epsilon u)$. Thus the low-order metric for $\mu_2=0$ is obtained from a low-mass Kerr solution, expressed in "Cartesian" coordinates based on Boyer-Lindquist coordinates as

$$ds^2=-dt^2+dr^k\,dr^k+2\epsilon^2\mu_1 r^{-1}dt^2 + 2\epsilon^2\mu_1 r^k r^l r^{-3}dr^k\,dr^l+O(\epsilon^4), \quad (2.3)$$

by a Poincaré transformation

$$t=(1-\epsilon^2|\dot{\vec{r}}_1|^2)^{-1/2}(u-\epsilon\dot{r}_1^k w^k),$$
$$r^i=w^i-\epsilon(1-\epsilon^2|\dot{\vec{r}}_1|^2)^{-1/2}\dot{r}_1^i u-r_1^i(0) + |\dot{\vec{r}}_1|^{-2}[(1-\epsilon^2|\dot{\vec{r}}_1|^2)^{-1/2}-1]\dot{r}_1^k w^k\dot{r}_1^i\,. \quad (2.4)$$

The terms representing the two moving black holes may be superposed at low orders to give Eq. (2.2), since nonlinear terms are not present in the lowest-order Einstein equations

$$Q^2_{00,kk}=0\,,$$
$$Q^2_{ij,kk}+Q^2_{kk,ij}-Q^2_{ik,jk}-Q^2_{jk,ik}=Q^2_{00,ij}\,, \quad (2.5)$$
$$Q^3_{0i,kk}-Q^3_{0k,ik}=\epsilon^{-1}Q^2_{ik,0k}-\epsilon^{-1}Q^2_{kk,0i}\,.$$

When matching the gravitational field far outside the black holes to the field near (say) black hole 1, we shall work in terms of matching coordinates (t,r^i), which will shortly be related to (u,w^i). In the internal approximation scheme for black hole 1, the metric is that of a Kerr solution with mass $\epsilon^2\mu_1$ and spin parameter $\vec{a}_1$, perturbed by the interaction with the second black hole. We have already pointed out in the Introduction that when we remove a conformal factor $(\epsilon^2\mu_1)^2$ from the internal metric to produce a QS scheme, the perturbations will be of $O(\epsilon^6)$ and higher; we suppose that "internal" coordinates have been chosen so that no gauge terms of order less than ϵ^6 have been left in the QS scheme. The matching coordinates (t,r^i) are related to the Boyer-Lindquist coordinates (T,R,θ,ϕ) used to describe the internal scheme just as in Sec. II of Ref. 1 for the matching coordinates (τ,x,y,z). For example, if the spin of the black hole is in the r^3 direction, then

$$t=t_0+M_1T\,,$$
$$r^1=M_1R\sin\theta\cos\phi\,,$$
$$r^2=M_1R\sin\theta\sin\phi\,, \quad (2.6)$$
$$r^3=M_1R\cos\theta$$

provided that (T,R,θ,ϕ) is outside the ergosurface of the unperturbed Kerr metric, where t_0 is some "external time." If the spin of the black hole is in some other direction, then Eq. (2.6) should be modified by the obvious rotation of spatial coordinates. Note the rescaling of time and distance by the factor $M_1=\epsilon_1\mu^2$ which we mentioned in the Introduction. As a result, when we match out a QS perturbation of order ϵ^I, it produces metric terms in the matching coordinates which behave as $\epsilon^I(r\epsilon^{-2})^J$ near $r=0$, apart from time and angular dependence; here J depends on the radial behavior of the QS perturbation, and $r=|\vec{r}|$. When we match out the dominant Kerr part of the internal metric, we obtain a postlinearized Kerr metric.

Combining all these terms, and making one extra assumption, we take the metric in (t,r^i) coordinates to be

$$g_{00}=-1+\epsilon P^1_{00}(\epsilon t,r^k)+\epsilon^2P^2_{00}(\epsilon t,r^k)+\cdots\,,$$
$$g_{0i}=\epsilon P^1_{0i}(\epsilon t,r^k)+\epsilon^2P^2_{0i}(\epsilon t,r^k)+\cdots\,, \quad (2.7)$$
$$g_{ij}=\delta_{ij}+\epsilon P^1_{ij}(\epsilon t,r^k)+\epsilon^2P^2_{ij}(\epsilon t,r^k)+\cdots\,,$$

where

$$P^1_{00}=0,\quad P^2_{00}=2\mu_1 r^{-1}+O(r^2),\quad P^3_{00}=O(r^2),$$
$$P^4_{00}=O(r),\quad P^5_{00}=O(r),\quad \ldots,$$
$$P^1_{0i}=0,\quad P^2_{0i}=O(r^2),\quad P^3_{0i}=O(r^2),$$
$$P^4_{0i}=-2(\mu_1)^2\epsilon_{ijk}\chi^j_1 r^k r^{-3}+O(r),\quad P^5_{0i}=O(r),\quad \ldots, \quad (2.8)$$
$$P^1_{ij}=0,\quad P^2_{ij}=2\mu_1 r^i r^j r^{-3}+O(r^2),\quad P^3_{ij}=O(r^2),$$
$$P^4_{ij}=(\mu_1)^2[(4-2|\vec{\chi}_1|^2)r^{-4}r^ir^j-r^{-2}\chi^i_1\chi^j_1 + r^{-4}\chi^k_1 r^k(r^i\chi^j_1+\chi^i_1 r^j)+|\vec{\chi}_1|^2r^{-2}\delta_{ij}]+O(r),$$
$$P^5_{ij}=O(r),\quad \cdots$$

as $r\to 0$. Here the terms given as $O(r^n)$ arise from matching out the internal perturbations, while the remaining terms are the lowest-order parts of the linearized Kerr metric. The extra assumption is that the terms due to internal perturbations appear only in P^2_{00}, P^2_{0i}, P^2_{ij} and higher-order components, so that $I-2J\geq 2$ in the expression in the previous paragraph. This assumption is plausible since the internal perturbations are caused by "background" curvature, which is here $O(\epsilon^2)$, and so should admit a description in (t,r^i) coordinates by metric perturbations of

$O(\epsilon^2)$ or higher; no inconsistencies are apparent between the assumption and the rest of the approximation scheme. The $O(r^n)$ bounds on radial behavior follow from the matching. For example, the largest $O(\epsilon^6)$ QS perturbations produce terms behaving as r^2 in P^2_{ab}, as r^1 in P^4_{ab}, as r^0 in P^6_{ab}, etc. The $O(\epsilon^7)$ QS perturbations produce terms behaving as r^2 in P^3_{ab}, as r^1 in P^5_{ab}, as r^0 in P^7_{ab}, etc., and the $O(\epsilon^8)$ QS perturbations produce terms behaving as r^3 in P^2_{ab}, as r^2 in P^4_{ab}, as r^1 in P^6_{ab}, etc.

The transformation between the coordinates (t, r^i) and (u, w^i) is assumed to be of the form

$$
\begin{aligned}
t &= u + \epsilon f_1(\epsilon u, w^k) + \epsilon^2 f_2(\epsilon u, w^k) + \cdots, \\
r^i &= w^i - r^i_1(\epsilon u) + \epsilon^2 h^i_2(\epsilon u, w^k) + \epsilon^3 h^i_3(\epsilon u, w^k) + \cdots,
\end{aligned}
\tag{2.9}
$$

where $f_1, f_2, \ldots,\ h^i_2, h^i_3, \ldots$ are regular functions of $(\epsilon u, w^k)$. We only look for one such transformation that is consistent with our assumptions about the P^n_{ab} and Q^n_{ab}. In fact, there is a large variety of these transformations, and we choose the one which apparently corresponds to the simplest perturbation schemes; any alternative transformation should merely lead to a description of the same physical system in different coordinates.

The transformation chosen has the form

$$
\begin{aligned}
t = {} & u - \epsilon \dot r^k_1 w^k + \epsilon A + \tfrac{1}{2}\epsilon(\epsilon u)\dot r^k_1 \dot r^k_1 - \tfrac{1}{2}\epsilon^3 \dot r^k_1 \dot r^k_1 \dot r^l_1 w^l + \tfrac{3}{8}\epsilon^3(\epsilon u)(\dot r^k_1 \dot r^k_1)^2 - \epsilon^3 \dot r^k_{12} w^k + \epsilon^3 B \\
& + \epsilon^3 B^k(w^k - r^k_1) + \epsilon^3 B^{kl}(w^k - r^k_1)(w^l - r^l_1) + \epsilon^3 B^{klm}(w^k - r^k_1)(w^l - r^l_1)(w^m - r^m_1) \\
& - \epsilon^4 \dot r^k_{13} w^k + \epsilon^4 G + \epsilon^4 G^k(w^k - r^k_1) + \epsilon^4 G^{kl}(w^k - r^k_1)(w^l - r^l_1) + O(\epsilon^5), \\
r^i = {} & w^i - r^i_1 + \tfrac{1}{2}\epsilon^2 \dot r^k_1 w^k \dot r^i_1 - \tfrac{1}{2}\epsilon^2 \dot r^k_1 \dot r^k_1 r^i_1 - \epsilon^2 r^i_{12} + \epsilon^2 F^{ik}_{12}(w^k - r^k_1) + \epsilon^2 D^{ik}(w^k - r^k_1) \\
& + \epsilon^2 D^{ikl}(w^k - r^k_1)(w^l - r^l_1) + \epsilon^2 D^{iklm}(w^k - r^k_1)(w^l - r^l_1)(w^m - r^m_1) - \epsilon^3 r^i_{13} + \epsilon^3 F^{ik}_{13}(w^k - r^k_1) + O(\epsilon^4).
\end{aligned}
\tag{2.10}
$$

Here we can separate out the terms which would effect the transformation if black hole 2 were absent. These are

$$
\begin{aligned}
t &= u - \epsilon \dot r^k_1 w^k + \tfrac{1}{2}\epsilon(\epsilon u)\dot r^k_1 \dot r^k_1 - \tfrac{1}{2}\epsilon^3 \dot r^k_1 \dot r^k_1 \dot r^l_1 w^l + \tfrac{3}{8}\epsilon^3(\epsilon u)(\dot r^k_1 \dot r^k_1)^2 + O(\epsilon^5), \\
r^i &= w^i - r^i_1 + \tfrac{1}{2}\epsilon^2 \dot r^k_1 w^k \dot r^i_1 - \tfrac{1}{2}\epsilon^2 \dot r^k_1 \dot r^k_1 r^i_1 + O(\epsilon^4)
\end{aligned}
\tag{2.11}
$$

and arise from a slow-motion Poincaré transformation. The terms $A, B, B^k, B^{kl}, B^{klm}, D^{ik}, D^{ikl}, D^{iklm}, G, G^k, G^{kl}$ are functions of (ϵu), where

$$
\begin{aligned}
& B^{kl} = B^{(kl)}, \quad B^{klm} = B^{(klm)}, \\
& D^{ik} = D^{(ik)}, \quad D^{ikl} = D^{i(kl)}, \quad D^{iklm} = D^{i(klm)},
\end{aligned}
\tag{2.12}
$$

and are needed in the matching at low orders. For in the QS scheme we have already chosen that no gauge terms of order less than ϵ^6 should be present in the metric, and we shall also impose coordinate restrictions on the QS perturbations at orders ϵ^6 and ϵ^7. These coordinate restrictions dictate part of the form of the $P^n_{ab}(\epsilon t, r^i)$ in Eq. (2.8). But it will also be convenient to impose certain coordinate conditions on the $Q^n_{ab}(\epsilon u, w^i)$ found in the course of a slow-motion perturbation of flat space. The terms $A, B, \ldots$ are required to make the two types of coordinate condition compatible.

The terms $r^i_{12}, r^i_{13}, F^{ik}_{12}, F^{ik}_{13}$ are also functions of (ϵu), where

$$
F^{ik}_{12} = F^{[ik]}_{12}, \quad F^{ik}_{13} = F^{[ik]}_{13}. \tag{2.13}
$$

They are again determined by the matching conditions at low orders, but have a more direct physical significance than $A, B, \ldots$. We shall later be able to interpret r^i_{12} by regarding $\epsilon^2 r^i_{12}$ as the post-Newtonian position shift of the first black hole, under the interpretation given in the Introduction. Similarly, $\epsilon^3 r^i_{13}$ will be a 1-$\frac{1}{2}$ post-Newtonian position shift. Were we to write down the transformation equation (2.10) to higher orders, we should include $n/2$ post-Newtonian terms r^i_{1n} for $n \geq 4$ in the same way. At a lower order, r^i_1 is effectively r^i_{10}; the Newtonian interpretation for r^i_1, r^i_2 will shortly be justified when we show that r^i_1, r^i_2 obey the Newtonian equations for point particles. Note that we have not included a $\frac{1}{2}$ post-Newtonian term r^i_{11} in the transformation. Roughly speaking, this is because there are no $\frac{1}{2}$ post-Newtonian metric terms Q^n_{ab} to produce $\frac{1}{2}$ post-Newtonian deflections. Had we included terms r^i_{11} and r^i_{21} in the transformations to matching coordinates (where r^i_{2n} are introduced for the 2nd black hole just as r^i_{1n} for the 1st), we should have found that r^i_{11}, r^i_{21} obey the Newtonian equations describing small displacements $\epsilon r^i_{11}, \epsilon r^i_{21}$ from the lowest-order orbits r^i_1, r^i_2.

We shall also be able to interpret F^{ij}_{12} and F^{ij}_{13} as describing the precession of the black-hole spin axis on post-Newtonian and 1-$\frac{1}{2}$ post-Newtonian time scales: $\delta_{ij} + \epsilon^2 F^{ij}_{12} + \epsilon^3 F^{ij}_{13}$ is essentially a rotation matrix close to the identity. At higher orders in the transformation of Eq. (2.10), we would include $n/2$ post-Newtonian terms F^{ij}_{1n} for

$n \geq 4$ in the same way. Lower-order rotational terms can be shown to be excluded from the transformation by the matching conditions. Corresponding terms F^{ij}_{2n} will arise in the transformation from (u, w^i) to matching coordinates for the 2nd black hole.

We now show how some of the functions A, B, etc., are determined by the matching. For example, by transforming the (t, r^i) form of the metric [Eq. (2.8)] to (u, w^i) coordinates, we find that near the 1st black hole

$$\begin{aligned} g_{uu} &= -1 + 2\epsilon^2\mu_1 r^{-1} + 2\epsilon^2 \ddot{r}^k_1 w^k - 2\epsilon^2 \dot{A} - 2\epsilon^2(\epsilon u)\ddot{r}^k_1 \dot{r}^k_1 + \epsilon^2 O(r^2) + O(\epsilon^3) \\ &= -1 + 2\epsilon^2\mu_1 |\vec{w} - \vec{r}_1|^{-1} + 2\epsilon^2 \ddot{r}^k_1 w^k - 2\epsilon^2 \dot{A} - 2\epsilon^2(\epsilon u)\ddot{r}^k_1 \dot{r}^k_1 + \epsilon^2 O(|\vec{w} - \vec{r}_1|^2) + O(\epsilon^3). \end{aligned} \tag{2.14}$$

But from our assumptions [Eq. (2.2)] about the low-order Q^n_{ab},

$$g_{uu} = -1 + 2\epsilon^2\mu_1 |\vec{w} - \vec{r}_1|^{-1} + 2\epsilon^2\mu_2 |\vec{r}_1 - \vec{r}_2|^{-1} - 2\epsilon^2\mu_2 (w^k - r^k_1)(r^k_1 - r^k_2)|\vec{r}_1 - \vec{r}_2|^{-3} + \epsilon^2 O(|\vec{w} - \vec{r}_1|^2) + O(\epsilon^3). \tag{2.15}$$

Hence,

$$\dot{A} = \dot{r}^k_1 \dot{r}^k_1 - \mu_2 |\vec{r}_1 - \vec{r}_2|^{-1} - (\epsilon u)\ddot{r}^k_1 \dot{r}^k_1, \tag{2.16}$$

$$\ddot{r}^i_1 = -\mu_2 (r^i_1 - r^i_2)|\vec{r}_1 - \vec{r}_2|^{-3}. \tag{2.17}$$

Similarly, for the second black hole

$$\ddot{r}^i_2 = \mu_1 (r^i_1 - r^i_2)|\vec{r}_1 - \vec{r}_2|^{-3}. \tag{2.18}$$

Consistently with Eqs. (2.17) and (2.18) we may work in the center-of-mass frame, where

$$\mu_1 r^i_1 + \mu_2 r^i_2 = 0. \tag{2.19}$$

Thus, the two black holes do indeed move approximately on Newtonian orbits in each other's far field. This is related to the property established in Ref. 1, that a small black hole moves (at lowest order) on a geodesic of the background universe which surrounds it. In the present case the "background" for one black hole is essentially the Newtonian field of the other, contained in Q^2_{00}, and Eqs. (2.17) and (2.18) are just the lowest-order geodesic equations in these backgrounds. Our derivation of the Newtonian equations has used only the vacuum Einstein equations in conjunction with the matching approach; the higher-order equations of motion will again follow from matching Q^n_{00}, for $n \geq 4$.

Continuing with our determination of A, B, etc., we next compute D^{ik}, D^{ikl} by examining Q^2_{ij}, for by transforming to (u, w^i) coordinates, given the (t, r^i) metric form of Eq. (2.8), we find

$$Q^2_{ij} = 2\mu_1 (w^i - r^i_1)(w^j - r^j_1)|\vec{w} - \vec{r}_1|^{-3} + 2D^{ij} + (2D^{ijk} + 2D^{jik})(w^k - r^k_1) + O(|\vec{w} - \vec{r}_1|^2) \tag{2.20}$$

as $|\vec{w} - \vec{r}_1| \to 0$. But, by Eq. (2.2),

$$\begin{aligned} Q^2_{ij} = {} & 2\mu_1 (w^i - r^i_1)(w^j - r^j_1)|\vec{w} - \vec{r}_1|^{-3} + 2\mu_2 (r^i_1 - r^i_2)(r^j_1 - r^j_2)|\vec{r}_1 - \vec{r}_2|^{-3} \\ & + 2\mu_2 [\delta_{ik}(r^j_1 - r^j_2)|\vec{r}_1 - \vec{r}_2|^{-3} + (r^i_1 - r^i_2)\delta_{jk}|\vec{r}_1 - \vec{r}_2|^{-3} \\ & - 3(r^i_1 - r^i_2)(r^j_1 - r^j_2)(r^k_1 - r^k_2)|\vec{r}_1 - \vec{r}_2|^{-5}](w^k - r^k_1) + O(|\vec{w} - \vec{r}_1|^2). \end{aligned} \tag{2.21}$$

Hence

$$D^{ij} = \mu_2 (r^i_1 - r^i_2)(r^j_1 - r^j_2)|\vec{r}_1 - \vec{r}_2|^{-3}, \tag{2.22}$$

$$D^{ijk} = \mu_2 (r^i_1 - r^i_2)\delta_{jk}|\vec{r}_1 - \vec{r}_2|^{-3} - \tfrac{3}{2}\mu_2 (r^i_1 - r^i_2)(r^j_1 - r^j_2)(r^k_1 - r^k_2)|\vec{r}_1 - \vec{r}_2|^{-5}. \tag{2.23}$$

Then, by examining the matching for Q^3_{0i} in a similar way, we find

$$B^i = -\mu_2(\mu_1 + \mu_2)^{-2}[2(\mu_1)^2 + 3\mu_1\mu_2 + \tfrac{3}{2}(\mu_2)^2]\dot{r}^i_1 |\vec{r}_1|^{-1} - \mu_2(\mu_1 + \mu_2)^{-2}[2(\mu_1)^2 + 4\mu_1\mu_2 + \tfrac{3}{2}(\mu_2)^2]\dot{r}^k_1 r^k_1 r^i_1 |\vec{r}_1|^{-3}, \tag{2.24}$$

$$B^{ij} = \tfrac{1}{2}\mu_1(\mu_2)^2(\mu_1 + \mu_2)^{-2}(\dot{r}^i_1 r^j_1 + r^i_1 \dot{r}^j_1)|\vec{r}_1|^{-3} + (\mu_2)^2(\mu_1 + \mu_2)^{-1}\dot{r}^k_1 r^k_1 (\tfrac{3}{2} r^i_1 r^j_1 |\vec{r}_1|^{-5} - \delta_{ij}|\vec{r}_1|^{-3}), \tag{2.25}$$

$$\dot{F}^{ij}_{12} = (\mu_2)^2(\mu_1 + \mu_2)^{-2}(2\mu_1 + \tfrac{3}{2}\mu_2)(r^i_1 \dot{r}^j_1 - \dot{r}^i_1 r^j_1)|\vec{r}_1|^{-3}. \tag{2.26}$$

If we take the metric given by Eqs. (2.1) and (2.2) and transform it to the form of Eq. (2.7) using the transformation Eq. (2.10), then the resulting P^n_{ab} now have the form of Eq. (2.8) for $n \leq 3$. We must await more detailed information about the internal perturbations and about the higher-order Q^n_{ab} before we can determine the remaining functions B, B^{klm}, D^{iklm}, G, G^k, G^{kl} or their time derivatives.

III. COMPUTATION OF Q^4_{00}

We are now in a position to calculate some higher-order metric terms which will later be needed in finding equations of motion and spin propagation, and begin with the remaining post-Newtonian term Q^4_{00}. This satisfies the field equation

$$Q^4_{00,kk}=2\epsilon^{-1}Q^3_{0k,0k}-\epsilon^{-2}Q^2_{kk,00}+Q^2_{kl}\,Q^2_{00,kl}+Q^2_{kl,l}\,Q^2_{00,k}-\tfrac{1}{2}Q^2_{00,k}\,Q^2_{ll,k}-\tfrac{1}{2}Q^2_{00,k}\,Q^2_{00,k}\,, \tag{3.1}$$

which follows from postlinearizing the Einstein equations. If we write

$$R^i_1=w^i-r^i_1,\quad R^i_2=w^i-r^i_2\,, \tag{3.2}$$

then, using Eq. (2.2),

$$Q^4_{00,kk}=\sum_{\alpha=1,2}\left[-2\mu_\alpha\ddot r^k_\alpha R^k_\alpha|\vec R_\alpha|^{-3}+2\mu_\alpha\dot r^k_\alpha\dot r^k_\alpha|\vec R_\alpha|^{-3}-6\mu_\alpha(\dot r^k_\alpha R^k_\alpha)^2|\vec R_\alpha|^{-5}\right]-4\mu_1\mu_2(|\vec R_1|^{-1}|\vec R_2|^{-3}+|\vec R_1|^{-3}|\vec R_2|^{-1})$$
$$+12\mu_1\mu_2(R^k_1R^k_2)^2(|\vec R_1|^{-3}|\vec R_2|^{-5}+|\vec R_1|^{-5}|\vec R_2|^{-3})-16\mu_1\mu_2R^k_1R^k_2|\vec R_1|^{-3}|\vec R_2|^{-3}\,. \tag{3.3}$$

By transforming the (t,r^i) form of the metric given in Eq. (2.8) to (u,w^i) coordinates, we find

$$Q^4_{00}=\mu_1(\dot r^k_1\dot r^k_1 r^l_1-\dot r^k_1 r^k_1\dot r^l_1+2r^l_{12})R^l_1|\vec R_1|^{-3}+\mu_1(\dot r^k_1R^k_1)^2|\vec R_1|^{-3}+2\mu_1(\dot r^k_1\dot r^k_1-2\mu_2|\vec r_1-\vec r_2|^{-1})|\vec R_1|^{-1}+O(1) \tag{3.4}$$

as $|\vec R_1|\to 0$. Solving Eq. (3.3) subject to Eq. (3.4), a similar condition as $|\vec R_2|\to 0$, and the condition that $Q^4_{00}\to 0$ as $|\vec w|\to\infty$, we find

$$Q^4_{00}=\sum_{\alpha=1,2}\Big[\mu_\alpha\ddot r^k_\alpha R^k_\alpha|\vec R_\alpha|^{-1}+2\mu_\alpha\dot r^k_\alpha\dot r^k_\alpha|\vec R_\alpha|^{-1}-2\mu_1\mu_2|\vec r_1-\vec r_2|^{-1}|\vec R_\alpha|^{-1}+\mu_\alpha(\dot r^k_\alpha R^k_\alpha)^2|\vec R_\alpha|^{-3}$$
$$-\mu_\alpha\dot r^k_\alpha r^k_\alpha\dot r^l_\alpha R^l_\alpha|\vec R_\alpha|^{-3}+\mu_\alpha\dot r^k_\alpha\dot r^k_\alpha r^l_\alpha R^l_\alpha|\vec R_\alpha|^{-3}+2\mu_\alpha r^k_{\alpha 2}R^k_\alpha|\vec R_\alpha|^{-3}-2\mu_1\mu_2r^k_\alpha|\vec r_\alpha|^{-1}R^k_\alpha|\vec R_\alpha|^{-3}\Big]$$
$$+2\mu_1\mu_2R^k_1R^k_2(|\vec R_1|^{-1}|\vec R_2|^{-3}+|\vec R_1|^{-3}|\vec R_2|^{-1})-4\mu_1\mu_2|\vec R_1|^{-1}|\vec R_2|^{-1}\,. \tag{3.5}$$

We remark on the usefulness of the matching method in this calculation, since it has easily given us the information in Eq. (3.4) on the singular parts of Q^4_{00} near the black-hole world line. Without the use of matching, we might have had some difficulty in establishing which solution of the Poisson equation (3.3) to take.

One can then calculate $\dot B$, where $B(\epsilon u)$ is defined in Eq. (2.10), by examining the condition that the terms appearing in P^4_{00} due to internal perturbations are $O(r)$ as $r\to 0$. We omit the calculation, since a knowledge of $\dot B$ will not be of use in the rest of this work.

Our expression for Q^4_{00} now allows us to interpret $\epsilon^2 r^i_{12}$ and $\epsilon^2 r^i_{22}$ as the post-Newtonian position shifts of the black holes, according to the interpretation given in the Introduction, for slow changes in the Newtonian orbit parameters over a time scale of $O(\epsilon^{-3})$ will be expressed in the post-Newtonian metric components by terms which may grow to the size of Newtonian metric components over this time scale. The only such terms are $2\mu_1 r^l_{12}R^l_1|\vec R_1|^{-3}+2\mu_2 r^l_{22}R^l_2|\vec R_2|^{-3}$ in Q^4_{00}, and these would be produced by replacing r^i_1, r^i_2 in Q^2_{00} by $r^i_1+\epsilon^2 r^i_{12}, r^i_2+\epsilon^2 r^i_{22}$. Thus r^i_{12}, r^i_{22} determine the changes in Newtonian orbit parameters over a post-Newtonian time scale.

IV. COMPUTATION OF 1-$\frac{1}{2}$ POST-NEWTONIAN METRIC TERMS

Usually one does not encounter any metric terms at 1-$\frac{1}{2}$ post-Newtonian order in a slow-motion perturbation of flat space, but in the present context we are forced to include them. This occurs because we are scaling the angular momentum of each black hole as mass^2, whereas in an ordinary slow-motion scheme for fluid bodies the angular momentum scales as mass$^{3/2}$.

We have already taken the 1-$\frac{1}{2}$ post-Newtonian term Q^3_{ij} to be zero. Then Q^4_{0i} must satisfy the field equation

$$Q^4_{0i,kk}-Q^4_{0k,ik}=0. \tag{4.1}$$

By transforming from the (t,r^i) form of the metric in Eq. (2.8), we find

$$Q^4_{0i}=-2(\mu_1)^2\epsilon_{ikl}\chi^k_1R^l_1|\vec R_1|^{-3}+O(1) \tag{4.2}$$

as $|\vec R_1|\to 0$, with a similar condition as $|\vec R_2|\to 0$. We take the solution

$$Q^4_{0i} = -2(\mu_1)^2\epsilon_{ikl}\chi^k_1 R^l_1|\vec{R}_1|^{-3} -2(\mu_2)^2\epsilon_{ikl}\chi^k_2 R^l_2|\vec{R}_2|^{-3} \tag{4.3}$$

of Eq. (4.1), which satisfies the boundary conditions at the holes and tends to zero as $|\vec{w}|\to\infty$.

The field equation for Q^5_{00} gives

$$Q^5_{00,kk} = 2\epsilon^{-1}Q^4_{0k,0k} = 0. \tag{4.4}$$

When we transform from Eq. (2.8), we find

$$Q^5_{00} = 2\mu_1 r^k_{13} R^k_1|\vec{R}_1|^{-3} +4(\mu_1)^2\epsilon_{klm}\dot{r}^k_1\chi^l_1 R^m_1|\vec{R}_1|^{-3} + O(1) \tag{4.5}$$

as $|\vec{R}_1|\to 0$, with a similar condition as $|\vec{R}_2|\to 0$. If we solve Eq. (4.4), subject to these boundary conditions and the requirement that $Q^5_{00}\to 0$ as $|\vec{w}|\to\infty$, we obtain simply

$$Q^5_{00} = 2\mu_1 r^k_{13} R^k_1|\vec{R}_1|^{-3} + 4(\mu_1)^2\epsilon_{klm}\dot{r}^k_1\chi^l_1 R^m_1|\vec{R}_1|^{-3} + 2\mu_2 r^k_{23} R^k_2|\vec{R}_2|^{-3} + 4(\mu_2)^2\epsilon_{klm}\dot{r}^k_2\chi^l_2 R^m_2|\vec{R}_2|^{-3}. \tag{4.6}$$

Given these expressions for Q^4_{0i} and Q^5_{00}, one can now calculate the functions G^k and $\dot{G}$, where G^k and G appear in Eq. (2.10). We find G^k by transforming the Q^n_{ab} to (t, r^i) coordinates and requiring that the terms in P^4_{0i} due to internal perturbations be $O(r)$ as $r\to 0$, as in Eq. (2.8). This gives

$$G^k = 2(\mu_2)^2\epsilon_{klm}\chi^l_2(r^m_1 - r^m_2)|\vec{r}_1 - \vec{r}_2|^{-3}, \tag{4.7}$$

which is needed in calculating the $1\text{-}\frac{1}{2}$ post-Newtonian equation of motion. Similarly, one finds $\dot{G}$ by transforming coordinates and requiring that $P^5_{00} = O(r)$ as $r\to 0$; the expression for G is not used later and we do not give it.

The appearance of r^i_{13} and r^i_{23} in Q^5_{00} shows that they may be interpreted as describing slow changes in the Newtonian orbit parameters over a $1\text{-}\frac{1}{2}$ post-Newtonian time scale, just as r^i_{12} and r^i_{22} determine post-Newtonian orbital effects.

V. EXPLICIT INTERNAL PERTURBATIONS

Before we can proceed to find equations of motion and spin propagation at higher orders by the matching method, we need some more explicit information about the internal perturbations at the lowest orders in the QS scheme. As we pointed out in the Introduction, the lowest-order QS perturbations of black hole 1 will be at $O(\epsilon^6)$, because of the Newtonian field of the other black hole. Since the time dependence of the slow-motion perturbations is slowed by a factor ϵ, and since another factor ϵ^2 is gained in the course of matching to the internal scheme [Eq. (2.6)], the QS perturbations of a unit-mass Kerr metric depend on "internal time" T through functions of $(\epsilon^3 T)$. This implies that the QS perturbations at orders ϵ^6, ϵ^7, and ϵ^8 separately obey the field equations describing stationary linearized vacuum gravitational perturbations of a unit-mass Kerr metric. Only at order ϵ^9 do time derivatives start to appear in the field equations for the QS scheme, while nonlinear effects only become important at order ϵ^{12}.

In Teukolsky's analysis[10] of vacuum gravitational perturbations of the Kerr metric, the perturbed geometry may be described in terms of the Newman-Penrose quantity

$$\Psi_0 = C_{abcd}\, l^a m^b l^c m^d. \tag{5.1}$$

Here C_{abcd} is the Weyl tensor of the perturbed spacetime, and $l^a, n^a, m^a, \overline{m}^a$ is the Kinnersley null tetrad,[16] such that $l^a n_a = -1$, $m^a\overline{m}_a = 1$, and all other inner products are zero. In Boyer-Lindquist coordinates (T, R, θ, ϕ) for a unit-mass Kerr solution with angular momentum χ,

$$\begin{aligned} l^a &\equiv [(R^2+\chi^2)\Delta_K^{-1}, 1, 0, \chi\Delta_K^{-1}],\\ n^a &\equiv [R^2+\chi^2, -\Delta_K, 0, \chi]/2\Sigma,\\ m^a &\equiv [i\chi\sin\theta, 0, 1, i/\sin\theta]/2^{1/2}(R+i\chi\cos\theta), \end{aligned} \tag{5.2}$$

where $\Sigma(R,\theta) = R^2+\chi^2\cos\theta$, $\Delta_K(R) = R^2 - 2R + \chi^2$. The unperturbed value of Ψ_0 is zero, and the part of Ψ_0 corresponding to linearized perturbations is independent of infinitesimal coordinate transformations (gauge transformations) and of infinitesimal tetrad rotations. Teukolsky derived an equation for Ψ_0 alone, not involving other perturbed field quantities, from the field equations linearized about the Kerr metric. He further showed that the equation for Ψ_0 is completely separable in Boyer-Lindquist coordinates, with solutions of the form $W(R)S(\theta)e^{i\omega T}e^{im\phi}$. It has been shown by Wald[17] that under reasonable conditions, Ψ_0 uniquely determines the geometry for linearized perturbations, apart from the addition of gauge terms and of perturbations which merely change the mass and angular momentum of the black hole. Chrzanowski[9] has since given constructive expression to Wald's theorem by providing explicit formulas for the metric perturbations in terms of the perturbed Ψ_0.

The matching conditions reveal that the $O(\epsilon^6)$ QS metric perturbations are $O(R^2)$ as $R\to\infty$, where the fastest growing R^2 parts correspond to the r^2 parts of P^2_{ab} in Eq. (2.8). Similarly, the $O(\epsilon^7)$ QS metric perturbations are again $O(R^2)$ as $R\to\infty$, with the R^2 parts corresponding to the r^2 parts of P^3_{ab}. The $O(\epsilon^8)$ QS metric perturbations are $O(R^3)$ as $R\to\infty$, where the R^3 parts arise from the r^3 parts of P^2_{ab}. It follows from arguments given in Ref. 1, where we use the Teukolsky equation for Ψ_0 together with boundary conditions of regularity at the future event horizon of the black hole, that the $O(\epsilon^6)$ and $O(\epsilon^7)$ QS perturbations each consist only of modes with angular quantum number $L=2$. By a slight extension of these arguments, it follows that the $O(\epsilon^8)$ QS perturbations consist of a sum of modes which only have $L=3$ or 2. It will be particularly important to have explicit information about the $L=2$ stationary modes for these lowest-order QS perturbations. We found in Ref. 1 that the largest distortions of the Kerr geometry for a black hole in a background universe are also $L=2$ stationary perturbations, and we now summarize (somewhat heuristically) relevant details of the treatment of these dominant modes in Ref. 1.

When a small black hole moves in a background universe under the assumptions of Ref. 1, it is possible to choose coordinates (τ, x, y, z) for space-time such that the black hole stays approximately at the spatial origin $l_0=\{x=y=z=0\}$ as time τ varies. The coordinates may further be chosen such that the background metric $g^{(0)}_{ab}$ is $\mathrm{diag}(-1,1,1,1)+O(x^2+y^2+z^2)$ as $x^2+y^2+z^2\to 0$. Thus the 4-velocity u^a of the "black-hole world line" l_0 is $(1,0,0,0)$ in (τ,x,y,z) coordinates. The largest internal perturbations are caused by the background Weyl tensor $C^{(0)}_{abcd}$ on l_0, and it is convenient to decompose $C^{(0)}_{abcd}$ into its electric and magnetic parts[18] with respect to u^a. We define the electric tensor

$$E_{ac}=C^{(0)}_{abcd}u^b u^d=E_{ca} \tag{5.3}$$

and magnetic tensor

$$H_{ac}=\tfrac{1}{2}\eta_{ab}{}^{gh}C^{(0)}_{ghcd}u^b u^d=H_{ca}\ , \tag{5.4}$$

where η^{abcd} is the totally antisymmetric tensor such that $\eta^{0123}=[-\det(g^{(0)}_{ab})]^{-1/2}$. Then

$$\begin{aligned} &E_{ac}u^c=H_{ac}u^c=0,\\ &g^{(0)ac}E_{ac}=g^{(0)ac}H_{ac}=0 \end{aligned} \tag{5.5}$$

so that E_{ac}, H_{ac} define 3-dimensional Cartesian symmetric tracefree tensors E_{ij}, H_{ij}, where the indices i,j refer to spatial coordinates x,y,z. These two spatial tensors describe the background Riemann tensor on l_0 completely in vacuo.

Now suppose, making a spatial rotation if necessary, that the black-hole rotation axis is in the z direction. Then we decompose

$$\begin{aligned} E_{ij}=\alpha_0\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&2\end{bmatrix}+\alpha_1\sqrt{\tfrac{3}{2}}\begin{bmatrix}0&0&1\\0&0&i\\1&i&0\end{bmatrix}+\alpha_2\sqrt{\tfrac{3}{2}}\begin{bmatrix}1&i&0\\i&-1&0\\0&0&0\end{bmatrix}\\ +\alpha_{-1}\sqrt{\tfrac{3}{2}}\begin{bmatrix}0&0&1\\0&0&-i\\1&-i&0\end{bmatrix}+\alpha_{-2}\sqrt{\tfrac{3}{2}}\begin{bmatrix}1&-i&0\\-i&-1&0\\0&0&0\end{bmatrix}\ , \end{aligned} \tag{5.6}$$

where α_0 is real, $\alpha_{-1}=\bar{\alpha}_1$, $\alpha_{-2}=\bar{\alpha}_2$, and the α_m, in general, depend on the time τ along the world line. We decompose H_{ij} similarly in terms of coefficients β_m $(-2\leq m\leq 2)$. In this way we have analyzed E_{ij} and H_{ij} into parts with convenient rotational properties.

We showed in Ref. 1 that the resulting internal perturbations produce in the QS scheme

$$\begin{aligned}\Psi_0=&\tfrac{1}{2}M^2(\alpha_0+i\beta_0)\,{}_2Y_2^0(\theta,\phi)\\ &+M^2\sum_{m=-2,-1,1,2}\frac{(\alpha_m+i\beta_m)\Gamma(3+2\gamma_m)(R_+-R_-)^5}{48\Gamma(-1+2\gamma_m)}\,{}_2Y_2^m(\theta,\phi)(R-R_-)^{-3-\gamma_m}\\ &\times(R-R_+)^{-2+\gamma_m}F\left[1,3+2\gamma_m;-1+2\gamma_m;\left(\frac{R-R_+}{R-R_-}\right)\right]+O(M^3)\,. \end{aligned} \tag{5.7}$$

Here M is the (small) mass of the black hole, which has angular momentum χM^2. We have defined $R_+=1+(1-\chi^2)^{1/2}$, $R_-=1-(1-\chi^2)^{1/2}$, $\gamma_m=im\chi/(R_+-R_-)$. Also $F[\,,\,;\,;\,]$ is the hypergeometric function,[19] and ${}_2Y_2^m(\theta,\phi)$ are spin-weight-2 spherical harmonics,[20] which we normalize as

$$\begin{aligned} &{}_2Y_2^0(\theta,\phi)=6\sin^2\theta\,,\\ &{}_2Y_2^{\pm1}(\theta,\phi)=-2\sqrt{6}\,\sin\theta(\cos\theta\mp 1)e^{\pm i\phi}\,,\\ &{}_2Y_2^{\pm2}(\theta,\phi)=2\sqrt{6}\,(\cos\theta\mp 1)^2e^{\pm 2i\phi}\,. \end{aligned} \tag{5.8}$$

In the context of the two-black-hole problem, we see that the lowest-order QS metric perturbations can be found explicitly provided that we have a means of computing metric perturbations from Ψ_0 and that we can identify the tensors E_{ij}, H_{ij} which act as "background" curvature, producing these low-order $L=2$

modes.

Chrzanowski's expressions for vacuum metric perturbations of the Kerr geometry in terms of Riemann tensor perturbations have not to date been completely checked.[9] However, the evidence for their validity is very strong and we accept them here. Essentially Chrzanowski shows that if Ψ_0 satisfies Teukolsky's equation for vacuum perturbations about the Kerr metric, then there is a complex metric perturbation h_{ab} constructed from Ψ_0 such that both $\mathrm{Re}(h_{ab})$ and $i\,\mathrm{Im}(h_{ab})$ satisfy the linearized Einstein equations and lead back to (complex) multiples of the original Ψ_0, where we use the defining equation (5.1). Explicitly,

$$h_{ab}=\rho^{-4}\{n_a n_b(2\pi+2\bar\beta+\bar\delta)(3\pi-4\bar\beta-\bar\delta)+\bar m_a\bar m_b(\bar\mu+4\mu-2\gamma+\Delta)(4\gamma-\mu-\Delta)$$
$$-n_{(a}\bar m_{b)}[(2\bar\gamma+2\gamma-4\mu-\Delta)(4\bar\beta-3\pi+\bar\delta)+(2\pi+4\bar\beta+\bar\tau+\bar\delta)(4\gamma-\mu-\Delta)]\}\Psi_0 \tag{5.9}$$

Here the Newman-Penrose operators Δ, $\bar\delta$ act on functions f according to $\Delta f=n^a f_{,a}$ and $\bar\delta f=\bar m^a f_{,a}$. The quantities $\rho,\beta,\pi,\tau,\mu,\gamma$ are Newman-Penrose spin coefficients relating to the Kinnersley tetrad. In Boyer-Lindquist coordinates for a unit-mass Kerr metric with angular momentum χ, they take the values

$$\rho=-1/(R-i\chi\cos\theta),\quad \beta=-\bar\rho(\cot\theta)/2\sqrt2,$$
$$\pi=i\chi\rho^2(\sin\theta)/\sqrt2,\quad \tau=-i\chi\rho\bar\rho(\sin\theta)/\sqrt2, \tag{5.10}$$
$$\mu=\rho^2\bar\rho\Delta_K/2,\quad \gamma=\mu+\rho\bar\rho(R-1)/2.$$

We can now use Eq. (5.9) to find explicitly the Kerr $L=2$ stationary metric perturbations corresponding to the Ψ_0 of Eq. (5.7). Since we shall only need the first two terms in an asymptotic expansion of the metric perturbations about $R=\infty$, we expand Eq. (5.7) as

$$\Psi_0=\sum_{m=-2}^{2}\tfrac12(\alpha_m+i\beta_m)\,{}_2Y_2^m(\theta,\phi)[1+im\chi/R+O(R^{-2})] \tag{5.11}$$

near $R=\infty$; at present it is convenient to ignore scaling factors such as M^2 in Eq. (5.7), and to regard all perturbations as linear fields on the Kerr spacetime. One proceeds by taking in turn the only nonzero member of $\{\alpha_m,\beta : 0\le m\le 2\}$ to be $\alpha_0,\alpha_1,\alpha_2$ with α_1,α_2 real, and computing h_{ab} in Eq. (5.9); the expressions corresponding to α_1 or α_2 imaginary can be found by performing rotations about the axis $\theta=0$. Since only the largest terms of h_{ab} near $R=\infty$ are required, one may also use expansions of the Kinnersley tetrad components [Eq. (5.2)] and spin coefficients [Eq. (5.10)] about $R=\infty$ to simplify the calculation.

In the case of $L=2$ stationary modes caused only by an electric tensor E_{ij}, so that all $\beta_m=0$, we find the metric perturbation

$${}'j_{ab}=-\tfrac23\mathrm{Re}(h_{ab}). \tag{5.12}$$

For, if we accept Chrzanowski's conclusions, both $\mathrm{Re}(h_{ab})$ and $i\,\mathrm{Im}(h_{ab})$ are vacuum metric perturbations regular at the future event horizon which lead back to complex multiples of the original Ψ_0 via Eq. (5.1). By explicit computation we find that the ratio of Ψ_0 for $\mathrm{Re}(h_{ab})$ and Ψ_0 for $i\,\mathrm{Im}(h_{ab})$ is purely imaginary. Hence there is a unique linear combination of $\mathrm{Re}(h_{ab})$ and $i\,\mathrm{Im}(h_{ab})$ which leads back to the original Ψ_0. By Wald's uniqueness theorem, this is precisely the metric perturbation corresponding to the given Ψ_0, modulo gauge transformations, and the addition of angular momentum and mass to the black hole. The correct linear combination of Eq. (5.12) is found by computing $\lim_{R\to\infty}\Psi_0$ and comparing with Eq. (5.11). We remark that it follows from Ref. 9 that the perturbations caused by an electric tensor are even under the parity transformation $(T,R,\theta,\phi)\to(T,R,\pi-\theta,\pi+\phi)$.

We write down the $L=2$ perturbations in terms of coordinates (T,R^1,R^2,R^3), where $R^1=R\sin\theta\cos\phi$, $R^2=R\sin\theta\sin\phi$, $R^3=R\cos\theta$ and the labels $i,j,\ldots$ refer to $R^i,R^j,\ldots$ ($i=1,2,3$, etc.). In the gauge of Eq. (5.9) we find

$$\begin{aligned}
{}'j_{TT}&=E_{kl}R^kR^l-4R^{-1}E_{kl}R^kR^l+O(1),\\
{}'j_{Ti}&=\tfrac13R^{-1}R^iE_{jk}R^jR^k+\tfrac23RE_{ij}R^j-\tfrac23R^{-2}R^iE_{jk}R^jR^k-\tfrac43E_{ij}R^j-\tfrac23E_{ij}\epsilon_{jkl}\chi^kR^l+\tfrac43\epsilon_{ijk}E_{jl}\chi^kR^l+O(1),\\
{}'j_{ij}&=\tfrac23E_{ij}R^kR^k+\tfrac13\delta_{ij}E_{kl}R^kR^l+(\tfrac23R^{-1}E_{km}\chi^lR^m+\tfrac13R^{-3}E_{mn}\chi^lR^kR^mR^n)(R^i\epsilon_{jkl}+R^j\epsilon_{ikl})\\
&\quad+\tfrac23R^{-3}R^iR^j\epsilon_{kmn}E_{kl}\chi^mR^lR^n-\tfrac23R^{-1}\delta_{ij}\epsilon_{kmn}E_{kl}\chi^mR^lR^n+\tfrac23R^{-1}R^l\chi^mR^m(E_{ki}\epsilon_{jkl}+E_{kj}\epsilon_{ikl})+O(1)
\end{aligned} \tag{5.13}$$

as $R\to\infty$. By a gauge transformation, this perturbation can be put in the simpler form

$$\begin{aligned}
j_{TT}&=E_{kl}R^kR^l-4R^{-1}E_{kl}R^kR^l+O(1),\\
j_{Ti}&=-\tfrac23E_{ij}\epsilon_{jkl}\chi^kR^l+\tfrac43\epsilon_{ijk}E_{jl}\chi^kR^l+O(1),\\
j_{ij}&=\tfrac23E_{ij}R^kR^k+\tfrac13\delta_{ij}E_{kl}R^kR^l+O(1).
\end{aligned} \tag{5.14}$$

Similarly, for $L=2$ stationary modes caused only by a magnetic tensor H_{ij}, we find the metric perturbation

$$'k_{ab}=\tfrac{2}{3}\,\mathrm{Re}(h_{ab}), \tag{5.15}$$

where now all coefficients α_m are zero. These perturbations are odd under the parity transformation. In the gauge of Eq. (5.9),

$$\begin{aligned}
'k_{TT}&=-2H_{kl}R^k\chi^l+O(1),\\
'k_{Ti}&=\tfrac{2}{3}\epsilon_{ijk}H_{jl}R^kR^l-\tfrac{4}{3}R^{-1}\epsilon_{ijk}H_{jl}R^kR^l+\tfrac{1}{3}R^{-3}R^iH_{jk}\chi^lR^jR^kR^l\\
&\quad-\tfrac{2}{3}R^{-1}H_{ij}\chi^kR^jR^k-\tfrac{1}{3}R^{-1}\chi^iH_{jk}R^jR^k-\tfrac{2}{3}R^{-1}R^iH_{jk}\chi^jR^k-\tfrac{2}{3}RH_{ij}\chi^j+O(1),\\
'k_{ij}&=\tfrac{1}{3}R^{-1}H_{km}R^lR^m(\epsilon_{ikl}R^j+\epsilon_{jkl}R^i)+\tfrac{1}{3}RR^l(H_{ik}\epsilon_{jkl}+H_{jk}\epsilon_{ikl})-\tfrac{4}{3}H_{ij}\chi^kR^k-\tfrac{2}{3}\delta_{ij}H_{kl}R^k\chi^l+O(1)
\end{aligned} \tag{5.16}$$

as $R\to\infty$. This perturbation can be simplified by a gauge transformation to give

$$\begin{aligned}
k_{TT}&=-2H_{kl}R^k\chi^l+O(1),\\
k_{Ti}&=\tfrac{2}{3}\epsilon_{ijk}H_{jl}R^kR^l-\tfrac{4}{3}R^{-1}\epsilon_{ijk}H_{jl}R^kR^l+O(1),\\
k_{ij}&=-\tfrac{4}{3}H_{ij}\chi^kR^k-\tfrac{2}{3}\delta_{ij}H_{kl}R^k\chi^l+O(1).
\end{aligned} \tag{5.17}$$

Armed with Eqs. (5.14) and (5.17) we can now find the parts of the QS metric perturbations which are relevant for computing equations of motion and spin propagation. Since we know that the $O(\epsilon^6)$ and $O(\epsilon^7)$ QS perturbations for each black hole are separately $L=2$ stationary, we may assume that they are in the gauges of Eqs. (5.14) and (5.17). For example, the $O(\epsilon^6)$ QS perturbations of black hole 1 are taken to be $\epsilon^6(\mu_1)^2(j_{ab}+k_{ab})$, where we have reinstated the perturbation factors $\epsilon^6(\mu_1)^2$ which involve mass², and where χ^i, E_{kl}, H_{kl} appearing in Eqs. (5.14) and (5.17) are replaced by $\chi_1^i, E_{kl}^{16}, H_{kl}^{16}$; the label "16" appearing in E_{kl}^{16}, H_{kl}^{16} refers to the $O(\epsilon^6)$ perturbations of black hole 1.

We find E_{kl}^{16} and H_{kl}^{16} by examining the matching conditions for the metric perturbations Q_{ab}^2 of flat space. Recall that we have already dealt with the matching for Q_{ab}^2 up to terms behaving as $|\vec{w}-\vec{r}_1|$ near the first black hole. Our assumption of a specific form based on Eqs. (5.14) and (5.17) for the largest $O(\epsilon^6)$ QS perturbations now allows us to deal with all terms in Q_{ab}^2 behaving as $|\vec{w}-\vec{r}_1|^2$ near the first black hole. First, by examining the matching for Q_{00}^2 at $O(|\vec{w}-\vec{r}_1|^2)$, we find

$$E_{kl}^{16}=3\mu_2(r_1^k-r_2^k)(r_1^l-r_2^l)|\vec{r}_1-\vec{r}_2|^{-5}-\mu_2\delta_{kl}|\vec{r}_1-\vec{r}_2|^{-3}. \tag{5.18}$$

This expression just arises from making a Taylor expansion of the part $2\mu_2|\vec{w}-\vec{r}_2|^{-1}$ of Q_{00}^2 due to the second black hole. From the matching for Q_{0i}^2 at $O(|\vec{w}-\vec{r}_1|^2)$ we find

$$H_{kl}^{16}=0. \tag{5.19}$$

Thus the dominant internal perturbations of black hole 1 are purely even, due to the electric tensor E_{kl}^{16} which represents the Newtonian curvature caused by black hole 2. Matching the $|\vec{w}-\vec{r}_1|^2$ part of Q_{ij}^2 gives us the quantity D^{ijkl} appearing in the transformation Eq. (2.10); a knowledge of D^{ijkl} is necessary in computing the post-Newtonian equation of motion. We have

$$\begin{aligned}
D^{ijkl}&=\tfrac{5}{2}\mu_2S^iS^jS^kS^lS^{-7}+\tfrac{1}{6}\mu_2S^{-3}(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})\\
&\quad-\tfrac{1}{6}\mu_2S^{-5}(\delta_{ij}S^kS^l+\delta_{ik}S^jS^l+\delta_{il}S^jS^k)\\
&\quad-\tfrac{5}{6}\mu_2S^iS^{-5}(\delta_{jk}S^l+\delta_{jl}S^k+\delta_{kl}S^j),
\end{aligned} \tag{5.20}$$

where

$$S^i=r_1^i-r_2^i,\quad S=|\vec{S}|. \tag{5.21}$$

The $O(\epsilon^7)$ QS perturbations of black hole 1 are taken to be of the form $\epsilon^7(\mu_1)^2(j_{ab}+k_{ab})$, where χ^i, E_{kl}, H_{kl} appearing in Eqs. (5.14) and (5.17) for j_{ab}, k_{ab} are replaced by $\chi_1^i, E_{kl}^{17}, H_{kl}^{17}$. By analogy with our calculation of E_{kl}^{16} and H_{kl}^{16}, we compute E_{kl}^{17} and H_{kl}^{17} from matching the Q_{ab}^3. Again, we have already dealt with matching for Q_{ab}^3 up to $O(|\vec{w}-\vec{r}_1|)$ near black hole 1, and our assumption about the gauge for the $O(\epsilon^7)$ QS perturbations now leads to complete information about the matching for Q_{ab}^3 up to $O(|\vec{w}-\vec{r}_1|^2)$. By matching the $|\vec{w}-\vec{r}_1|^2$ part of Q_{00}^3 we obtain

$$E_{kl}^{17}=0. \tag{5.22}$$

From the $|\vec{w}-\vec{r}_1|^2$ part of Q_{0i}^3 we find

$$H_{kl}^{17}=-3(\mu_1+\mu_2)S^{-5}\dot{r}_1^nS^m(\epsilon_{knm}S^l+\epsilon_{lnm}S^k). \tag{5.23}$$

We see that the second-largest QS perturbations are odd under parity change, and are due to a magnetic tensor H_{kl}^{17} which represents mass current effects. Examination of the $|\vec{w}-\vec{r}_1|^2$ part of Q_{0i}^3 also gives us the matching term B^{klm} of Eq. (2.10); this term is not needed later and we do not write it down. Matching for Q_{ij}^3 at $O(|\vec{w}-\vec{r}_1|^2)$ does not give any new information, but merely acts as a check on the consistency of our treatment so far.

By using Chrzanowski's work together with our matching method in this section, we have found enough explicit information on the $O(\epsilon^6)$ and $O(\epsilon^7)$ QS perturbations to enable us to derive all equations of motion and spin propagation up to and including $1\text{-}\frac{1}{2}$ post-Newtonian order; the derivation of these equations will be described in the next section. We also leave to the next section any discussion of the explicit metric perturbations at $O(\epsilon^8)$ in the QS scheme, which are needed in computing the dominant 2nd post-Newtonian part of the spin-spin force.

VI. EQUATIONS OF MOTION AND SPIN PROPAGATION

The post-Newtonian equation of motion for black hole 1 now follows from matching Q^4_{00} at $O(|\vec{w}-\vec{r}_1|)$ near the black-hole world line. This computation uses Eq. (5.14) together with Eq. (5.18) to find the contribution of internal perturbations to the $|\vec{w}-\vec{r}_1|$ part of Q^4_{00}. It also uses the information which we derived previously on the matching quantities A, B^k, B^{kl}, D^{ik}, D^{ikl}, D^{iklm}, F^{ik}_{12} of Eq. (2.10). We do not give the details, but merely the final result

$$\begin{aligned}\ddot{r}^i_{12} &= \mu_2 r^i_{22} S^{-3} - 3\mu_2 r^k_{22} S^k S^i S^{-5} - \mu_2 r^i_{12} S^{-3} + 3\mu_2 r^k_{12} S^k S^i S^{-5}\\ &+ [3\mu_1\mu_2 + 2(\mu_2)^2] S^i S^{-4} - [2(\mu_1)^2 + 7\mu_1\mu_2 + \tfrac{1}{2}(\mu_2)^2]\,\mu_2(\mu_1)^{-2}\dot{r}^k_2 \dot{r}^k_2 S^i S^{-3}\\ &+ \tfrac{1}{2}[3(\mu_1)^2 + 5\mu_1\mu_2 + 7(\mu_2)^2]\,\mu_2(\mu_1)^{-2}\dot{r}^k_2 S^k \dot{r}^i_2 S^{-3} + [3(\mu_1)^2 + \tfrac{15}{2}\mu_1\mu_2]\,\mu_2(\mu_1)^{-2}(\dot{r}^k_2 S^k)^2 S^i S^{-5}\,. \end{aligned} \tag{6.1}$$

The first four terms in Eq. (6.1) are just the Newtonian terms which we would expect for small perturbations $\epsilon^2 r^i_{12}$, $\epsilon^2 r^i_{22}$ in the orbits. The remaining terms are relativistic in origin. Of course, the post-Newtonian equation of motion for the second black hole is found by interchanging the particle labels 1 and 2 in Eq. (6.1) (so that $S^i \to -S^i$ also).

Next we turn to the post-Newtonian equation of spin propagation. We first justify our interpretation suggested in Sec. II of $F^{ij}_{12}(\epsilon u)$ as describing the lowest-order precession of the spin of the first black hole, where $F^{ij}_{12}(\epsilon u)$ is the antisymmetric matrix appearing in the transformation Eq. (2.10). The Kerr spin parameters first appear in the slow-motion scheme at the $1\text{-}\frac{1}{2}$ post-Newtonian order, in the terms Q^4_{0i} and Q^5_{00}. Metric terms which represent the variation of the Kerr parameters over long time scales first appear at the $2\text{-}\frac{1}{2}$ post-Newtonian order, in Q^6_{0i} and Q^7_{00}. These describe changes taking place over a post-Newtonian time scale of $O(\epsilon^{-3})$. We pointed out in the Introduction that secular fractional changes of $O(1)$ in the masses and scalar angular momenta of the black holes only take place over a much longer time scale of $O(\epsilon^{-10})$. Thus, the only significant effect on the Kerr spin parameters over a post-Newtonian time scale will be a slow precession. This will appear in Q^6_{0i} through the slow rotation of the Kerr parameters in

$$\begin{aligned} Q^4_{0i} &= -2(\mu_1)^2 \epsilon_{ikl}\,\chi^k_1 (w^l - r^l_1)|\vec{w}-\vec{r}_1|^{-3}\\ &\quad - 2(\mu_2)^2 \epsilon_{ikl}\,\chi^k_2 (w^l - r^l_2)|\vec{w}-\vec{r}_2|^{-3}\,. \end{aligned} \tag{6.2}$$

By examining the matching conditions for the P^n_{ab} and the transformation to (u, w^i) coordinates, one finds that Q^6_{0i} contains the term $2(\mu_1)^2 F^{ik}_{12} \times \epsilon_{klm}\chi^l_1 (w^m - r^m_1)|\vec{w}-\vec{r}_1|^{-3}$ together with a similar term for the second black hole. Other terms involving the spin and the transformation terms D^{ik}, D^{ikl}, D^{iklm} are also present in Q^6_{0i}, but these will remain bounded over a post-Newtonian time scale and are not of the form of precession terms. The expression $2(\mu_1)^2 F^{ik}_{12}\epsilon_{klm}\chi^l_1 (w^m - r^m_1)|\vec{w}-\vec{r}_1|^{-3}$ is the only term in Q^6_{0i} which describes $O(1)$ post-Newtonian variations in the normalized Kerr parameter of the first black hole, and is just as if χ^i_1 were replaced by the rotated parameter $(\delta_{ik} + \epsilon^2 F^{ki}_{12})\chi^k_1$ in Q^4_{0i}; a similar argument holds for the second black hole.

Equation (2.26) determining $\dot{F}^{ij}_{12}$ from the matching conditions now gives the post-Newtonian spin precession of the first black hole. Written in vectorial form, the equation for the propagation of the spin $\vec{\Sigma}$ of the first black hole is

$$\begin{aligned}\frac{d\vec{\Sigma}}{du} &= \epsilon^3(\mu_2)^2(\mu_1+\mu_2)^{-2}(2\mu_1 + \tfrac{3}{2}\mu_2)|\vec{r}_1|^{-3}\\ &\quad \times(\vec{r}_1 \times \dot{\vec{r}}_1)\times\vec{\Sigma} + O(\epsilon^4|\vec{\Sigma}|)\,. \end{aligned} \tag{6.3}$$

It might appear that we have calculated this spin precession without really using the field equations. However, our expression for $\dot{F}^{ij}_{12}$ arose from imposing the condition that the terms in P^3_{0i} of Eq. (2.8) due to internal perturbations are $O(r^2)$ as $r \to 0$. This follows from the condition that the QS internal perturbations are $O(\epsilon^6)$, which in turn depends on arguments of the type used in Ref. 1, involving the Teukolsky equation, to show that any lower-order internal perturbations are trivial. Thus the expression for $\dot{F}^{ij}_{12}$ does ultimately rely on the field equations near the black hole, together with a boundary condition of regularity at the future event horizon. The same comment can be made of all our other derivations of equations of motion and spin propagation.

We now proceed to find the $1\text{-}\frac{1}{2}$ post-Newtonian equation of spin propagation by matching for Q^4_{0i}. We have already dealt with the matching up to $O(|\vec{w}-\vec{r}_1|^0)$ in Q^0_{0i}; this led to the expression for G^k in Eq. (4.7). Equations (5.14) and (5.17) describing the dominant internal perturbations, together with Eqs. (5.18) and (5.19), allow us to treat the matching for Q^4_{0i} up to $O(|\vec{w}-\vec{r}_1|)$. By examining the $|\vec{w}-\vec{r}_1|$ part of Q^4_{0i} and using the previous expression for G^k, we find

$$\dot{F}^{ik}_{13} = -2(\mu_2)^2\epsilon_{imk}\chi_2^m S^{-3} - 2\mu_1\mu_2\epsilon_{imk}\chi_1^m S^{-3} + 3(\mu_2)^2\epsilon_{iml}\chi_2^m S^k S^l S^{-5} + 3\mu_1\mu_2\epsilon_{iml}\chi_1^m S^k S^l S^{-5} - 3(\mu_2)^2\epsilon_{kml}\chi_2^m S^i S^l S^{-5} - 3\mu_1\mu_2\epsilon_{kml}\chi_1^m S^i S^l S^{-5}, \tag{6.4}$$

$$G^{kl} = -\tfrac{3}{2}(\mu_2)^2\epsilon_{kmn}\chi_2^m S^n S^l S^{-5} - \tfrac{3}{2}(\mu_2)^2\epsilon_{lmn}\chi_2^m S^n S^k S^{-5} + \tfrac{1}{2}\mu_1\mu_2\epsilon_{lmn}\chi_1^n S^k S^m S^{-5} + \tfrac{1}{2}\mu_1\mu_2\epsilon_{kmn}\chi_1^n S^l S^m S^{-5}, \tag{6.5}$$

where G^{kl} is a matching term appearing in Eq. (2.10).

The interpretation of F^{ik}_{13} as describing $1\text{-}\frac{1}{2}$ post-Newtonian spin precession is analogous to the post-Newtonian interpretation of F^{ik}_{12}; the evolution of the spin of the first black hole over $1\text{-}\frac{1}{2}$ post-Newtonian time scales is determined by the rotated parameter $(\delta_{ik}+\epsilon^2 F^{ki}_{12}+\epsilon^3 F^{ki}_{13})\chi_1^k$. In vectorial form, the spin propagation equation for black hole 1, allowing for the $1\text{-}\frac{1}{2}$ post-Newtonian effects of Eq. (6.4), is

$$\begin{aligned}\frac{d\vec{\Sigma}}{du} = \vec{\Sigma}\times\{&-\epsilon^3(\mu_2)^2(\mu_1+\mu_2)^{-2}(2\mu_1+\tfrac{3}{2}\mu_2)(\vec{r}_1\times\dot{\vec{r}}_1)|\vec{r}_1|^{-3} \\ &+\epsilon^4(\mu_2)^2\vec{\chi}_2|\vec{r}_1-\vec{r}_2|^{-3} - 3\epsilon^4(\mu_2)^2(\vec{r}_1-\vec{r}_2)[\vec{\chi}_2\cdot(\vec{r}_1-\vec{r}_2)]\,|\vec{r}_1-\vec{r}_2|^{-5} \\ &+\epsilon^4\mu_1\mu_2\vec{\chi}_1|\vec{r}_1-\vec{r}_2|^{-3} - 3\epsilon^4\mu_1\mu_2(\vec{r}_1-\vec{r}_2)[\vec{\chi}_1\cdot(\vec{r}_1-\vec{r}_2)]\,|\vec{r}_1-\vec{r}_2|^{-5}\} + O(\epsilon^5|\vec{\Sigma}|).\end{aligned} \tag{6.6}$$

Next we find the $1\text{-}\frac{1}{2}$ post-Newtonian equation of motion by matching the $|\vec{w}-\vec{r}_1|$ parts of Q^5_{00}. Here we use Eqs. (5.14) and (5.17) together with Eqs. (5.22) and (5.23) to find the explicit $O(\epsilon^7)$ QS metric perturbations which contribute to the $|\vec{w}-\vec{r}_1|$ part of Q^5_{00} through the matching. We also need the expression for Q^5_{00} in Sec. IV and the matching quantities G^k, G^{kl}, $\dot{F}^{ik}_{13}$. The resulting equation of motion is

$$\begin{aligned}\ddot{r}^i_{13} = {}&\mu_2 r^i_{23}S^{-3} - 3\mu_2 r^k_{23}S^k S^i S^{-5} - \mu_2 r^i_{13}S^{-3} + 3\mu_2 r^k_{13}S^k S^i S^{-5} - 4\mu_2(\mu_1+\mu_2)\epsilon_{ikl}\dot{r}^k_1\chi_2^l S^{-3} \\ &+ 3\mu_1(\mu_1+\mu_2)S^i\epsilon_{klm}\dot{r}^k_1\chi_1^l S^m S^{-5} + 6\mu_2(\mu_1+\mu_2)S^i\epsilon_{klm}\dot{r}^k_1\chi_2^l S^m S^{-5} - 3\mu_1(\mu_1+\mu_2)\epsilon_{ikl}\dot{r}^k_1 S^l\chi_1^m S^m S^{-5} \\ &- 6\mu_2(\mu_1+\mu_2)\epsilon_{ikl}\chi_2^k S^l\dot{r}^m_1 S^m S^{-5}.\end{aligned} \tag{6.7}$$

The first four terms arise as we might expect from the Newtonian force. The remaining terms describe various types of spin-orbit interaction.

To find the dominant 2nd post-Newtonian part of the spin-spin force, we examine the matching for Q^6_{00}. Since $\ddot{r}^i_{14}$ appears in Q^6_{00} in the form $2\ddot{r}^k_{14}(w^k-r^k_1)$ when we match out the internal scheme, it is sufficient to look for terms in Q^6_{00} which are proportional to both black-hole spins and have radial dependence $|\vec{w}-\vec{r}_1|$. Now when one computes Q^6_{00} in the course of the slow-motion perturbation of flat space, no such terms depending on $\chi_1^k\chi_2^l$ appear. To see this, we first remark that there are no such terms in Q^4_{ij}, for Q^4_{ij} satisfies the field equation

$$Q^4_{ij,kk}+Q^4_{kk,ij}-Q^4_{ik,jk}-Q^4_{jk,ik}=S^4_{ij}, \tag{6.8}$$

where S^4_{ij} is complicated expression involving only Q^n_{ab} for $n<4$, and hence contains no spin terms. Since there are no products $\chi_1^k\chi_2^l$ in the boundary conditions for Q^4_{ij} near the two black holes found from the matching, it is possible to choose Q^4_{ij} such that it involves no products $\chi_1^k\chi_2^l$. In fact the author has computed an explicit form for Q^4_{ij}, which satisfies Eq. (6.8) and the matching conditions near the holes; the only spin terms in this Q^4_{ij} involve $\chi_1^k\chi_1^l$ or $\chi_2^k\chi_2^l$ through Kerr far-field terms of the type given in P^4_{ij} of Eq. (2.8). Next consider Q^5_{0i}, which satisfies a field equation of the form

$$Q^5_{0i,kk}-Q^5_{0k,ik}=\epsilon^{-1}Q^4_{ik,0k}-\epsilon^{-1}Q^4_{kk,0i}+S^5_{0i}. \tag{6.9}$$

Again S^5_{0i} is a complicated quantity depending only on the Q^n_{ab} for $n<4$, so that the right-hand side of Eq. (6.9) contains no terms involving $\chi_1^k\chi_2^l$. It can be shown that the boundary conditions on Q^5_{0i} near the black holes arising from the matching involve no $\chi_1^k\chi_2^l$ terms, and hence that Q^5_{0i} may be chosen to involve no such product terms. Now Q^6_{00} satisfies the field equation

$$Q^6_{00,kk}=2\epsilon^{-1}Q^5_{0k,0k}+S^6_{00}, \tag{6.10}$$

where S^6_{00} depends only on the Q^n_{ab} for $n<5$ and contains no spin product terms $\chi_1^k\chi_2^l$. The most singular parts of Q^6_{00} near the black holes, found from the matching, involve no spin product terms. Hence the full expression for Q^6_{00} involves no spin product terms $\chi_1^k\chi_2^l$.

We see that spin product terms involving $\chi_1^k \chi_2^l$ in Q_{00}^6 to conpensate for $2\ddot{r}_{14}^k(w^k - r_1^k)$ can only arise themselves from matching out the internal scheme for black hole 1. To find such terms produced in Q_{00}^6 by the matching we need to understand the $O(\epsilon^8)$ QS metric perturbations, since these produce terms in P_{00}^6 which behave as r and hence terms in Q_{00}^6 which behave as $|\vec{w} - \vec{r}_1|$. We recall that the $O(\epsilon^8)$ QS metric perturbations are $O(R^3)$ as $R \to \infty$, consisting of $L = 2$ and $L = 3$ stationary modes. The $L = 3$ modes are determined by the r^3 parts of P_{ab}^2, and arise ultimately from derivatives of the curvature due to the Schwarzschild far field of black hole 2; they can only produce $|\vec{w} - \vec{r}_1|$ terms in Q_{00}^6 which are proportional to products $\chi_1^k \chi_1^l$. The $L = 2$ $O(\epsilon^8)$ QS modes are determined by the r^3 parts of P_{ab}^2 and the r^2 parts of P_{ab}^4. These $L = 2$ modes can only produce $\chi_1^k \chi_2^l$ terms in Q_{00}^6 with $|\vec{w} - \vec{r}_1|$ radial dependence if the corresponding electric and magnetic tensors E_{ij}^{18}, H_{ij}^{18} contain parts ${}^{\rm spin}E_{ij}^{18}$, ${}^{\rm spin}H_{ij}^{18}$ which involve χ_2^k; such spin-dependent tensors can only arise from the r^2 terms in P_{ab}^4 which are proportional to χ_2^k. We find that the only contribution to ${}^{\rm spin}E_{ij}^{18}$, ${}^{\rm spin}H_{ij}^{18}$ is (through the matching) from the term $-2(\mu_2)^2 \epsilon_{ikl}\chi_2^k(w^l - r_2^l)|\vec{w} - \vec{r}_2|^{-3}$ in Q_{0i}^4. This leads to

$$ {}^{\rm spin}E_{ij}^{18} = 0 , \tag{6.11} $$

$$ {}^{\rm spin}H_{ij}^{18} = (\mu_2)^2(-3S^i\chi_2^j S^{-5} - 3S^j\chi_2^i S^{-5} - 3\delta_{ij}S^k\chi_2^k S^{-5} + 15 S^i S^j S^k \chi_2^k S^{-7}) . \tag{6.12} $$

When we use Eq. (5.17) to find the explicit QS metric perturbations for a magnetic $L = 2$ stationary mode, and then match out again, we find that ${}^{\rm spin}H_{ij}^{18}$ contributes $-2\mu_1\, {}^{\rm spin}H_{kl}^{18}\, \chi_1^k(w^l - r_1^l)$ to the $|\vec{w} - \vec{r}_1|$ part of Q_{00}^6. It may be checked that no other terms involving products $\chi_1^k \chi_2^l$ are contributed to the $|\vec{w} - \vec{r}_1|$ part of Q_{00}^6 when we match out the internal scheme for black hole 1. Hence the dominant 2nd post-Newtonian part of the spin-spin force corresponds to the acceleration

$$ \text{spin-spin } \ddot{r}_{14}^i = \mu_1\, {}^{\rm spin}H_{ik}^{18}\chi_1^k . \tag{6.13} $$

VII. COMMENTS AND CONCLUSIONS

We have now fulfilled the aims set out in the Introduction by finding the dominant equations of motion and spin propagation describing the dynamics of two rotating black holes of comparable mass, which move at nonrelativistic speeds in each other's far fields. As a by-product of our calculation, we have given descriptions of the gravitational field which together are accurate both in the combined far field of the black holes and also in the regions near the black holes where the curvature is large.

The equations of motion and spin propagation found in Sec. VI agree with the expressions discussed in the Introduction which were found by previous authors and should apply to various special cases of the two-black-hole interaction considered here. To verify this, let us first compare our work with that of EIH. The EIH equations of motion refer to the interaction of nonrotating bodies, so that we only need to check our post-Newtonian equation of motion Eq. (6.1) against EIH's. Written in our language, the EIH post-Newtonian equations of motion describe the post-Newtonian shifts $\epsilon^2\, {}'r_{12}^i$ and $\epsilon^2\, {}'r_{22}^i$ of two bodies from their Newtonian positions r_1^i, r_2^i. The equation for the first body is

$$ {}'\ddot{r}_{12}^i = \mu_2\, {}'r_{22}^i S^{-3} - 3\mu_2\, {}'r_{22}^k S^k S^i S^{-5} - \mu_2\, {}'r_{12}^i S^{-3} + 3\mu_2\, {}'r_{12}^k S^k S^i S^{-5} + [5\mu_1\mu_2 + 4(\mu_2)^2] S^i S^{-4} - [2(\mu_1)^2 + 4\mu_1\mu_2 + (\mu_2)^2]\mu_2(\mu_1)^{-2}\dot{r}_2^k \dot{r}_2^k S^i S^{-3} + [(3\mu_1)^2 + 7\mu_1\mu_2 + 4(\mu_2)^2]\mu_2(\mu_1)^{-2}\dot{r}_2^k S^k \dot{r}_2^i S^{-3} + \tfrac{3}{2}\mu_2(\dot{r}_2^k S^k)^2 S^i S^{-5} . \tag{7.1} $$

But this is equivalent to our Eq. (6.1) if we define

$$ {}'r_{12}^i = r_{12}^i - \tfrac{1}{2}\dot{r}_1^k r_1^k \dot{r}_1^i + \tfrac{1}{2}\dot{r}_1^k \dot{r}_1^k r_1^i - \mu_2 r_1^i |\vec{r}_1|^{-1} , $$
$$ {}'r_{22}^i = r_{22}^i - \tfrac{1}{2}\dot{r}_2^k r_2^k \dot{r}_2^i + \tfrac{1}{2}\dot{r}_2^k \dot{r}_2^k r_2^i - \mu_1 r_2^i |\vec{r}_2|^{-1} . \tag{7.2} $$

Now the functions $({}'r_{12}^i - r_{12}^i)$ and $({}'r_{22}^i - r_{22}^i)$ remain bounded over post-Newtonian time scales, whether or not the Newtonian orbit is bound. Hence we agree with EIH on all changes of $O(1)$ in the Newtonian orbit parameters which take place over a post-Newtonian time scale. The only difference between our equations and EIH's lies in the terms $\epsilon^2({}'r_{12}^i - r_{12}^i)$ and $\epsilon^2({}'r_{22}^i - r_{22}^i)$ which remain $O(\epsilon^2)$. This difference corresponds to the freedom of choosing reference points for the black holes anywhere inside regions of the dimension $O(\epsilon^2)$ on which the holes are smeared out, and is unimportant.

We can further compare our equations with Eqs. (1.2)–(1.6), which refer to the interaction of two rotating bodies with very disparate masses. We let $\mu_1/\mu_2 \to 0$ in our combined post-Newtonian and 1-$\frac{1}{2}$ post-Newtonian spin-propagation equation (6.6), and immediately recover the effects of Eqs. (1.5) and (1.6). When we take this limit for our

1-$\frac{1}{2}$ post-Newtonian equation of motion Eq. (6.7), we recover the two types of spin-orbit force given by Eqs. (1.2) and (1.3), on using the vector identity

$$\vec{r}[\vec{v}\cdot(\vec{\Sigma}\times\vec{r})]-(\vec{v}\cdot\vec{r})(\vec{\Sigma}\times\vec{r})=\vec{v}\times[r^2\vec{\Sigma}-(\vec{r}\cdot\vec{\Sigma})\vec{r}]\,. \tag{7.3}$$

Here the effects of Eq. (1.2) arise from the terms in Eq. (6.7) proportional to $(\mu_2)^2\chi_2^k$, those of Eq. (1.3) arise from the terms proportional to $\mu_1\mu_2\chi_1^k$. Finally, our 2nd post-Newtonian spin-spin equation (6.13) agrees with Eq. (1.4).

Thus, by our use of the matching method, we have found a generalization of EIH's equations which allows for the rotation of the black holes. Similarly, our equations are more general versions of well-known expressions for the interaction of rotating bodies, since we have allowed the black holes to have comparable masses in addition to spin. In common with the EIH method, our treatment uses only the vacuum Einstein field equations; we have avoided the need for making dubious assumptions about the matter inside the black holes.

The assumptions which underlie our approximation scheme would be expected to hold in a wide class of interactions between pairs of black holes. They should only break down in two situations. One possibility is that the black holes approach within a few Schwarzschild radii of each other. In this case it seems very unlikely that a perturbation method can be made to work; instead one is forced back to the problem of solving the full nonlinear Einstein equations. The second possibility is that the black holes move in each other's far fields, but that their relative velocity is not small compared to the speed of light. In this case perturbation theory is still useful, and a combination of our matching method with a fast-motion scheme might well succeed in describing the system adequately.

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