be twice as frequent as $X^0 \to \rho\gamma$. If one takes 2.5% for the $X^0 \to 2\gamma$ branching ratio, one determines $\beta = -0.5$, which then gives $(X^0 \to \omega\gamma)/(X^0 \to \rho\gamma) = 0.6$.[13] There are no experimental values reported so far for this ratio.

We conclude by stressing that the measurement of the $X^0 \to 2\gamma$ and $X^0 \to \omega\gamma$ decay modes is highly desirable. This will hopefully allow us to differentiate between the η-X^0 mixing model and the model proposed here for the radiative decays of X^0. However, if none of these models succeed in accounting for the experimental data, one might have to use a combination of them or to give up vector dominance and possibly use a mixing angle from the linear mass formula.[14]

I should like to acknowledge a discussion with Professor L. M. Brown and Professor H. Munczek, and to thank the members of the theoretical group at DESY for their hospitality.

[13] In solving for β from (9) one gets two solutions; in both cases analyzed we used the β value which gives a lower $(X^0 \to \omega\gamma)/(X^0 \to \rho\gamma)$ ratio.

[14] F. Guerra, F. Vanoli, G. De Franceschi, and V. Silvestrini, Phys. Rev. **166**, 1587 (1968); R. S. Willey, *ibid.* **183**, 1397 (1969).

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Veneziano Model with Secondary Terms for Pion-Pion Scattering

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We show that the Veneziano-Lovelace one-term formula for pion-pion scattering, although a good first approximation, has some undesirable features. With the help of a few secondary terms, a model amplitude, which is consistent with the known properties of the pion-pion system up to the f^0 mass region and the high-energy phenomenology, is constructed.

I. INTRODUCTION

SOMETIME ago, Veneziano[1] proposed a very attractive model for scattering amplitudes of strongly interacting particles, incorporating analyticity, resonance-pole structure, crossing symmetry, and Regge behavior in a somewhat natural way. The original amplitude satisfies all the superconvergence relations and the finite-energy sum rules (FESR). Lovelace[2] took the next step by considering one-term Veneziano amplitude for the π-π scattering, and imposed Adler's self-consistency condition. This condition requires the π-π amplitude to vanish at $s=t=u=m_\pi^2$ (s, t, and u being the usual Mandelstam variables). This led to a condition on the ρ-trajectory parameter $\alpha_\rho(m_\pi^2)=\frac{1}{2}$ and the resulting π-π scattering lengths were in approximate agreement with those obtained from the current algebra. The Veneziano-Lovelace formula for the π-π scattering may be a good first approximation.[3] However, it has some undesirable features. First of all, $\alpha_\rho(m_\pi^2)$, from high-energy phenomenological fitting, is found to be somewhat larger $[\alpha_\rho(m_\pi^2)\approx 0.58]$ and the amplitude contains a term like $\Gamma(1-\alpha_\rho(t)-\alpha_\rho(u))$ which is extremely sensitive to the value of $\alpha_\rho(m_\pi^2)$ near $t=u=m_\pi^2$. If we take $\alpha_\rho(m_\pi^2)\approx 0.58$ in the one-term formula, the S-wave $I=0$ and $I=2$ π-π scattering lengths have the same sign and the $I=0$ scattering length comes out to be very large ($\sim 1/m_\pi$), in contrast to the current-algebra predictions.[4] Furthermore, the one-term formula gives the ratio of the width of the σ meson to that of the ρ meson to be about 5, which is unusually large. Since the Veneziano formula is based on a zero- (or very-narrow-) width resonance approximation, one feels uncomfortable about predictions of such large widths. Also, it is known that, in principle, one can have any number of secondary terms having essentially the same properties as the primary term.[5] In fact, it has been shown that for cases like π-ρ scattering, secondary terms are necessary to avoid parity doubling.[5] (In the case of the π-π system, Bose statistics rules out the parity doubling automatically.) Also, it appears that secondary terms may be necessary in the general case to assure factorization.[5]

In principle, one can add an infinite number of terms to the first term. However, in such a case the Regge behavior will not be necessarily obtained. Moreover, in practice one would like to have a limited number of terms to describe the physical amplitude. One can even think of determining the coefficients of the various terms by some bootstrap requirements.[6] However, in the

[1] G. Veneziano, Nuovo Cimento **57A**, 190 (1968).

[2] C. Lovelace, Phys. Letters **28B**, 264 (1968).

[3] K. Kawarabayashi, S. Kitakado, and H. Yabuki, Phys. Letters **28B**, 432 (1969). Also J. Yellin, J. Shapiro, and collaborators have considered extensively the implications of a one-term Veneziano model for π-π scattering: University of California Lawrence Radiation Laboratory Reports (unpublished).

[4] This is also noticed by K. Kang (report of work prior to publication).

[5] See, for example, Ref. 1; G. Altarelli and H. R. Rubinstein, Phys. Rev. **178**, 2165 (1969); J. E. Mandula (Cal-Tech report of work prior to publication); P. G. O. Freund and E. Schonberg, and also P. G. O. Freund (University of Chicago reports of work prior to publication).

[6] S. Mandelstam, Phys. Rev. Letters **21**, 1724 (1968).

present note we will adopt the following goal: We will attempt to construct an amplitude consisting of a superposition of a few Veneziano-type terms consistent with all the low-energy properties and high-energy phenomenology.

Out amplitude will be supposed to be a good representation of the π-π scattering amplitude up to the f^0 mass region. In particular, we require that the widths of the ρ, σ, and f^0 mesons come out to be 110, 240, and 145 MeV, respectively. Since the existence of the daughters of f (ρ' and σ') is in doubt, we will demand that they decouple from the π-π system. Adler's consistency condition and the current-algebra sum-rule condition will also be imposed. For all the terms, the value of $\alpha_\rho(0)$ will be taken to be about $0.56\sim0.58$. The S-wave scattering lengths obtained from the amplitude will turn out to be consistent with the current-algebra values.

Regarding the FESR, we note that Veneziano[1] has proved by induction that the one-term Γ-function formula satisfies the usual FESR for sufficiently large cutoff. It can be shown explicitly (by exactly summing the left-hand side) that this is true for the most general term.[6a] In this way one can find the exact value of the right-hand side of the FESR for arbitrary cutoff. This contains nonleading contributions also. Although the usual FESR are satisfied eventually for sufficiently large cutoff, we will require that they be satisfied for a somewhat low value of the cutoff also. The reason is that our amplitude will be valid only up to about the f^0-meson mass region. In the usual spirit of the FESR, it is natural to choose the cutoff near this mass region. Moreover, for the π-π system in particular, this cutoff is quite reasonable from the point of view of Regge dominance (see Ref. 14). Finally, it should be noted that this constraint is not severe. The one-term formula already satisfies the FESR within about 15%.

We discuss the general formalism in Sec. II. Various properties and constraints are discussed in Sec. III and the construction of the amplitude satisfying all the conditions is carried out. Comparison is made to the available S-wave phase shifts, the widths of the resonances, and the known sum rules. Finally, some remarks are made regarding the relevance of the present model to a realistic π-π amplitude.

II. GENERAL FORMALISM

The π-π scattering amplitude is given in a fairly standard notation by

$$T(s,t,u)=A(s,t,u)\delta_{\alpha\beta}\delta_{\gamma\delta}+B(s,t,u)\delta_{\alpha\gamma}\delta_{\beta\delta}+C(s,t,u)\delta_{\alpha\delta}\delta_{\beta\gamma}, \quad (1)$$

where α, β, γ, and δ are the isospin indices of the poins and s, t, and u are the usual Mandelstam variables. The amplitudes A, B, and C satisfy the crossing properties

$$A(s,t,u)=A(s,u,t)=B(t,s,u)=B(u,s,t)=C(u,t,s)=C(t,u,s). \quad (2)$$

Then, the $I=0$, 1, 2 isospin amplitudes in the s-channel are given by

$$\begin{aligned} T^0(s,t,u)&=3A+B+C,\\ T^1(s,t,u)&=B-C,\\ T^2(s,t,u)&=B+C. \end{aligned} \quad (3)$$

The corresponding partial-wave amplitudes are obtained by

$$T_l^I(s)=\tfrac{1}{2}\int_{-1}^{1} T^I(s,t,u)P_l(z)dz \quad (4)$$

and

$$T_l^I(s)=16\pi w\frac{e^{i\delta_l^I}\sin\delta_l^I}{q}, \quad (5)$$

w, q, and z being the c.m. energy, momentum, and the cosine of the scattering angle in the appropriate channel.

The $I=0$ and $I=2$ S-wave scattering lengths are given by

$$a_0^0=\frac{T_0^0(4m_\pi^2)}{32\pi m_\pi} \quad\text{and}\quad a_0^2=\frac{T_0^2(4m_\pi^2)}{32\pi m_\pi}. \quad (6)$$

Now define a function $F(s,t)$ of the Veneziano form[7]

$$F(s,t)=g^2\sum_{n,l=1}^{\infty}R_{nl}\frac{\Gamma(n-\alpha_s(s))\Gamma(n-\alpha_t(t))}{\Gamma(l-\alpha_s(s)-\alpha_t(t))}, \quad (7)$$

where $\alpha_s(s)$ and $\alpha_t(t)$ are the s- and the t-channel trajectory functions. n and l are arbitrary positive integers with the restriction

$$n\leqslant l\leqslant 2n.$$

The first condition, $n\leqslant l$, is necessary to assure the maximal allowed Regge behavior. The second condition, $l\leq 2n$, guarantees the required polynomial character of the resonance-pole residues and absence of the ancestors. g^2 is the over-all normalizing factor. R_{11} is taken as 1. In the Veneziano-Lovelace model for pion-pion scattering, only R_{11} is taken to be nonzero.

The amplitude $A(s,t,u)$ is then given by

$$A(s,t,u)=F(t,u)-F(s,t)-F(s,u). \quad (8)$$

Similar expressions hold good for B and C.

The s-channel isospin amplitudes are now given by

$$\begin{aligned} T^0&=F(t,u)-3F(s,t)-3F(s,u),\\ T^1&=2F(s,u)-2F(s,t),\\ T^2&=-2F(t,u). \end{aligned} \quad (9)$$

[6a] See, for example, K. V. Vasavada, Can. J. Phys. (to be published).

[7] A term of the type $\Gamma(n-\alpha(s))\Gamma(m-\alpha(t))$ $(n\neq m)$ can also occur. However, in the present case, crossing symmetry requires that there should be another term with n and m interchanged and with identical coefficient. Hence these terms can always be combined to give series of terms of the above form where only terms with $n=m$ occur. It should be noted, however, that the proof of the FESR goes through completely even for the general case $n\neq m$.

If we demand absence of resonances in the $I=2$ channels, we are forced to have $\alpha_s=\alpha_t=\alpha_u$ and, hence, we have equivalence of odd- (ρ) and even- (f) signature trajectories.[8] This also follows as a consistency condition by analyzing the amplitudes near $\alpha_\rho(t)=1$. For the exchange-degenerate trajectory, let $x=\alpha(s)$, $y=\alpha(t)$, and $w=\alpha(u)$. For linear trajectories, we have

$$x=\alpha_0+\alpha' s$$

and

$$x+y+w=3\alpha_0+4m_\pi^2\alpha'=D. \tag{10}$$

We will take α_0 as an independent parameter; α' will be determined by the condition that $\alpha(m_\rho^2)=1$.

The s-channel resonance poles can be exhibited by using the expansion[9]

$$\frac{\Gamma(n-x)\Gamma(m-y)}{\Gamma(l-x-y)}=(-1)^{n+m-l}\sum_{j=0}^{\infty}\frac{\Gamma(j+1+n-l+y)}{\Gamma(j+1)\Gamma(1+y-m)}\times\frac{1}{j+n-x}, \tag{11}$$

$$\max(m,n)\leq l\leq m+n.$$

We consider only the case where $n=m$. The residue at the pole located at $x=j+n$ is a polynomial in y of the order $j+n$ and, hence, gives rise to a set of $j+n+1$ resonances alternating in odd and even spins and consequently odd and even isospins. Thus the ρ resonance ($J=1$, $I=1$) has a degenerate S-wave partner $\sigma(J=0, I=0)$. The f^0 resonance ($J=2$, $I=1$) has two degenerate partners: $\rho'(J=1, I=1)$ and $\sigma'(J=0, I=0)$. By comparing the isospin amplitudes with the Breit-Wigner form, we find the following formula for the widths of the resonances:

$$\Gamma_R=\frac{q_R X}{32\pi\alpha' m_R^2}\int_{-1}^{1}(\mathrm{Res})_R P_l(z)dz, \tag{12}$$

where $(\mathrm{Res})_R$ is the residue of the pole at $x=j+n$. X takes the values -4 or -6 according as $I=1$ or 0. The value of y at a particular resonance is obtained by

$$y=\tfrac{1}{2}[2\alpha_0+\alpha(4m_\pi^2)-j]+\tfrac{1}{2}[j-\alpha(4m_\pi^2)]z. \tag{13}$$

In the next section we consider different properties of the π-π system and construct an amplitude satisfying them.

III. PROPERTIES OF π-π AMPLITUDES

Now we consider the π-π scattering amplitude of the general form (7). The over-all normalization constant g^2 can be found by evaluating the width of the ρ meson in the model:

$$g^2=6\pi m_\rho^2\Gamma_\rho/q_\rho^3=g_{\rho\pi\pi}^2, \tag{14}$$

where $g_{\rho\pi\pi}$ is the usual ρ-π-π coupling constant. The coefficients R_{nl} will be determined in the following.

The Adler consistency condition has already been discussed. There is a current-algebra sum rule called the Adler sum rule, which requires that the derivative of the $I_t=1$ t-channel amplitude for the scattering of a zero-mass pion by a physical pion be given by

$$\left.\frac{dT^1}{ds}\right|_{s=m_\pi^2,\,t=0}=\frac{4}{f_\pi^2}. \tag{15}$$

Here, f_π is the pion decay constant. Experimentally, it is found to be 135 MeV. This makes the right-hand side of Eq. (15) to be 4.28 (in units of the pion mass).

If this condition is imposed on the one-term formula with $\alpha(m_\pi^2)=\frac{1}{2}$, the width of the ρ meson turns out to be about 90 MeV. This value is rather low.

We have already mentioned that the high-energy phonemonological fits to the ρ-trajectory parameter require considerably larger α [$\alpha(0)\approx 0.56-0.58$]. Since $F(m_\pi^2,m_\pi^2)$ contains a factor $\Gamma(1-\alpha(m_\pi^2)-\alpha(m_\pi^2))$, the results of the one-term formula are extremely sensitive to the value of $\alpha(m_\pi^2)$.

With $\alpha(m_\pi^2)=\frac{1}{2}$, one finds the S-wave scattering lengths to be $a_0^0=0.20/m_\pi$, $a_0^2=-0.054/m_\pi$, and $a_0^0/a_0^2\approx -\frac{7}{2}$. These results agree roughly with the current-algebra predictions $a_0^0=0.15/m_\pi$, $a_0^2=-0.043/m_\pi$, and $a_0^0/a_0^2\approx -\frac{7}{2}$. However, if we take $\alpha_0=0.56$, the Adler condition is not satisfied in the one-term model and the scattering lengths become $a_0^0=0.81/m_\pi$ and $a_0^2=0.18/m_\pi$. Clearly, these are in complete disagreement with the current-algebra values.

Now consider the implications for the widths of the resonances. These can be obtained from Eq. (12). The σ enhancement in the ρ-mass region seems to have been established recently.[10] This appears to have a mass $m_\sigma\approx 730$ MeV and a width $\Gamma_\sigma\approx 240$ MeV. The other solution, $m_\sigma\approx 930$ MeV and $\Gamma_\sigma\approx 650$ MeV, seems to be much less likely.[11] We will return to this point later while considering the S-wave phase shifts. Here we

[8] We have left out the Pomeranchuk trajectory as usual. According to a conjecture by Freund and Harari, its contribution may be related to the nonresonant background [see P. G. O. Freund, Phys. Rev. Letters **20**, 235 (1968); H. Harari, *ibid.* **20**, 1395 (1968)]. More recently, however, some attempts have been made to include the Pomeranchuk and the cut contribution in the Veneziano model. See, for example, D. Wong, Phys. Rev. **181**, 1900 (1969); M. Cassandro and M. Greco (to be published).

[9] This expansion can be derived from the series expansion of the hypergeometric function $F(a,b,c,1)=\Gamma(c)\Gamma(c-a-b)/\Gamma(c-a)\Gamma(c-b)$ [see E. T. Whittaker and G. N. Watson, *A Course on Modern Analysis* (Cambridge University Press, New York, 1962), p. 282]. The convergence conditions are not satisfied in the case where $n=l$. However, the expansion can be considered for $l=n+1$, and then the pole residues for the case $n=l$ can be obtained by suitable multiplication. It turns out that the residues are given by exactly the same expression. The factor $(-1)^{n+m-l}$ is important. It seems to have been left out of some of the recent works.

[10] W. D. Walker, J. Carroll, A. Garfinkel, and B. Y. Oh, Phys. Rev. Letters **18**, 630 (1967); E. Malamud and P. E. Schlein, *ibid.* **19**, 1056 (1967); L. J. Gutay, D. D. Carmony, P. L. Csonka, F. J. Loeffler, and F. T. Meiere, Nucl. Phys. **B12**, 31 (1969).

[11] S. Marateck, V. Hagopian, W. Selove, L. Jacobs, F. Oppenheimer, W. Schultz, L. J. Gutay, D. H. Miller, J. Prentice, E. West, and W. D. Walker, Phys. Rev. Letters **21**, 1613 (1968); **22**, 219(E) (1969).

TABLE I. Comparison between our results and those of the one-term model.

No.[a]	α_0	Γ_ρ (MeV)	Γ_σ (MeV)	Γ_f (MeV)	Γ_g (MeV)	Γ_ρ' (MeV)	Γ_σ' (MeV)	a_0^0 (m_π^{-1})	a_0^2 (m_π^{-1})	Adler[b] sum rule	FESR[c]	$\delta_0^0(m_K)-\delta_0^2(m_K)$ (deg)
1	0.58	110	240	145	95	~ 0	~ 0	0.16	-0.043	$+4.26$	1.03	34
2	0.56	110	240	145	98	~ 0	~ 0	0.16	-0.043	4.26	1.00	34
3	0.48	90	458	74	29	86	-10	0.14	-0.048	4.18	1.15	36
4	0.48	110	559	90	35	105	-13	0.17	-0.059	5.11	1.15	43

[a] 1 and 2 are our results; 3 and 4 are obtained from the one-term formula. 1 and 2 differ in the values of the coefficients which are given in the text.
[b] These numbers represent the values of the left-hand side of Eq. (15). The right-hand side is 4.28.
[c] These numbers represent the values of the left-hand side of the FESR, divided by the right-hand side when only the leading trajectory contribution is taken into account. Exact satisfaction of the FESR will thus give 1.

note that if we take $\alpha(m_\pi^2)=\frac{1}{2}$ in the one-term formula, we find Γ_σ to be too large ($\Gamma_\sigma/\Gamma_\rho\approx 5$). $\alpha_0=0.56$ gives even larger Γ_σ ($\Gamma_\sigma/\Gamma_\rho\approx 8.7$). Finally, the f^0-meson width comes out to be too small ($\Gamma_f/\Gamma_\rho=9\sqrt{3}/20$).

In view of these difficulties, it is interesting to see whether some improvement can be made by adding a few additional terms. As mentioned above, there is no reason why the secondary terms cannot be present.

One can try to have just one extra (satellite) term (R_{12}). Then the Adler condition determines R_{12} by

$$R_{12}=2\alpha(m_\pi^2)-1. \tag{16}$$

For $\alpha(m_\pi^2)=\frac{1}{2}$, this gives $R_{12}=0$, which is consistent. Using the expressions for the widths [Eq. (12)], it can be shown that

$$\begin{aligned}\frac{\Gamma_\sigma}{\Gamma_\rho}&=\frac{9}{2}\left(\frac{2\alpha_0+\alpha(4m_\pi^2)-1-2R_{12}}{1-\alpha(4m_\pi^2)}\right)\\&=\frac{9}{2}\left(\frac{1}{1-4m_\pi^2/m_\rho^2}\right).\end{aligned} \tag{17}$$

Unfortunately, this ratio is also very large. The scattering lengths are found to be roughly consistent with the current-algebra values. However, this is not surprising. The π-π threshold is quite close to the $s=t=u=m_\pi^2$ point. Thus, when the Adler consistency condition and the sum-rule constraint are satisfied, the scattering lengths cannot be too much different from the current-algebra values.

Clearly, one extra term is not enough. Next, we introduce a number of terms and demand that $\Gamma_\rho=110$ MeV, $\Gamma_\sigma=240$ MeV, and $\Gamma_f=145$ MeV be obtained. Adjustment of values of g^2, R_{12}, and R_{22} can do this, but the Adler condition and the sum rule are not satisfied. Hence it is necessary to include at least R_{23} and R_{24} terms. An additional problem arises as we add the secondary terms. The widths of the resonances obtained from the R_{11} term are mostly positive up to large values of j. Negative-width ghosts appear, as secondary terms are introduced. This problem seems to have been unresolved at present. However, it is possible to get rid of the ghosts up to a certain energy by adjusting the coefficients. Beyond this energy, however, ghosts do appear.[12] In the present case the daughter pole of ρ has been already identified with the σ meson. The degenerate daughter poles of f^0 (ρ' and σ') have not been observed so far. Presumably, even if they exist, they will have very small elastic widths. Hence, we will require our amplitude to give $\Gamma_{\rho'}$ and $\Gamma_{\sigma'}$ to be nearly zero (or at least small and positive).

Now we consider the FESR.[13] If a crossing-odd or even amplitude $G(\nu,t)$ corresponding to a definite isospin eigenstate in the t channel is such that $\operatorname{Im} G(\nu,t)\to\beta(t)\nu^{\alpha(t)}$ as $\nu\to\infty$, then the FESR reads

$$\int_0^{\bar\nu}\nu^p\operatorname{Im}G(\nu,t)d\nu=\frac{\beta(t)\,\bar\nu^{\alpha(t)+p+1}}{\alpha(t)+p+1}. \tag{18}$$

Here we have $\nu=\frac{1}{4}(s-u)$, and only the leading trajectory contribution has been retained on the right-hand side of Eq. (18). p is an even or odd integer according as $G(\nu,t)$ is odd or even under $s\leftrightarrow u$ crossing. $\bar\nu$ is some cutoff energy above which the Regge representation can be assumed to be good. For the $I=1$ state in the t channel of the π-π system, we have

$$G(\nu,t)=2F(t,u)-2F(s,t). \tag{19}$$

It can be shown that the FESR are satisfied by the Γ-function formula for sufficiently large $\bar\nu$. As mentioned in the Introduction, we also demand satisfaction of the zero-moment FESR for $I_t=1$ in the t channel with a low cutoff. Since our amplitude will be supposed to represent the physical amplitude up to the f^0 mass region and will be only partly valid in the g-meson ($J^P=3^-$ recurrence of the ρ meson) mass region, we will choose the cutoff at the midpoint between m_f^2 and m_g^2.[14] We have checked that with this cutoff the FESR is satisfied by the leading trajectory contribution for the values of the coefficients mentioned below. We have also checked that it is *not* satisfied for widely different values of the coefficients. Of course, in the latter case it is satisfied eventually for sufficiently large cutoff.

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[12] A similar problem arises in saturation of the current-algebra equations by single-particle states. There it has been suggested that the ghosts may represent multiparticle and continuum effects. This may be true here also.

[13] K. Igi, Phys. Rev. Letters **9**, 76 (1962); K. Igi and S. Matsuda, *ibid.* **18**, 625 (1967); A. A. Logunov, L. D. Soloviev, and A. N. Tavkhelidze, Phys. Letters **24B**, 181 (1967); R. Dolen, D. Horn, and C. Schmid, Phys. Rev. **166**, 1768 (1968).

[14] Setting of Regge behavior should be a good assumption at this cutoff for the π-π system. The cosine of the angle of scattering in the t channel is already about 65 at this point.

We have assumed that there is no $I_t=2$ Regge trajectory. Hence the right-hand side of the corresponding FESR should be zero. From the expression for the residues at the s-channel poles, it can be seen that the successive terms contribute with opposite signs. As a result of these large cancellations, the FESR is satisfied on the average.

In order to maintain all the above conditions, at least seven terms are necessary. These are R_{11}, R_{12}, R_{22}, R_{23}, R_{24}, R_{33}, and R_{34}. To have "good" results at the f^0-meson level, we cannot stop at the R_{24} term; we run into problems of unacceptable widths of ρ' and σ' mesons. Again it is not possible to adjust the width of the g meson by adjusting R_{33} and R_{34} without ruining the rest of the conditions. With these coefficients, the daughters of the g meson already contain ghosts. Additional terms are necessary to eliminate them.

Some of the results are shown in Table I for $\alpha_0=0.58$ and 0.56. Also shown are the results of the one-term Veneziano-Lovelace model for comparison. The values of the coefficients are as follows: For $\alpha_0=0.58$, we find

$$R_{11}=1,\quad R_{12}=0.31,\quad R_{22}=-0.58,\quad R_{23}=0.21,$$
$$R_{24}=0.015,\quad R_{33}=0.25,\quad R_{34}=-0.77. \tag{20}$$

For $\alpha_0=0.56$, these become

$$R_{11}=1,\quad R_{12}=0.28,\quad R_{22}=-0.59,\quad R_{23}=0.28,$$
$$R_{24}=-0.055,\quad R_{33}=0.30,\quad R_{34}=-0.87. \tag{21}$$

The value of g^2 in both the cases is 25. $\Gamma_{\rho'}$ and $\Gamma_{\sigma'}$ are very sensitive to the values of R_{23} and R_{24}. So by small variations in the values of these coefficients, small positive values of $\Gamma_{\rho'}$ and $\Gamma_{\sigma'}$ can be easily obtained without disturbing the other conditions.

As we have already discussed, the Adler condition and the sum rule have been required to be satisfied. In connection with the contributions to the Adler sum rule, we note an interesting point. In our case, the contribution from the low-energy region up to the f^0-meson mass is only about 67% of the total contribution. 33% contribution comes from the energy region above this. A similar conclusion was arrived at in a recent work by us.[15] In that work, the high-energy contributions to the current-algebra sum rules were evaluated by combining these sum rules with the FESR. Since our model amplitude satisfies both the Adler sum rule and the FESR, it is not surprising that we are led to identical conclusions. However, in the present case, we have an explicit model which clearly exhibits the relative contributions. We should note that even in the one-term model (with very large Γ_σ), the high-energy contribution is still about 20% of the total contribution.

Now, there is also an $I_t=2$ current-algebra sum rule. Let $\partial_\mu A_\mu{}^i$ be the divergence of the axial-vector current and Q_i be the corresponding charge. Then the assumption that the commutator $[\partial_\mu A_\mu{}^i, Q^j]$ does not contain any $I=2$ piece leads to a condition

$$T_t^2|_{s=m_\pi^2,u=m_\pi^2,t=0}=0. \tag{22}$$

The assumption is consistent with the absence of mesons having $I=2$. The last condition leads to a sum rule. In the present model, $T_t^2=-2F(t,u)$ automatically satisfies the above condition, since the Adler condition has been already imposed.

Now we consider the S-wave phase shifts in the present model. The amplitudes considered above are essentially real. Imaginary parts are introduced only through δ functions corresponding to zero- (or very-narrow-) width resonances. Unitarizing the amplitude without destroying the crossing symmetry remains a very important problem. One faces a similar problem in the case of the current-algebra calculations. A possible way to arrive at approximately valid amplitude in such cases is to interpret the real amplitude as a K-matrix element instead of the T-matrix element. Then, in terms of the phase shifts, one takes

$$\tan\delta_l=(q/16\pi w)T_l, \tag{23}$$

where T_l is the matrix element obtained above. This is certainly not a rigorous procedure and undoubtedly destroys some crossing symmetry. However, it may be a valid approximation at low energies. This procedure leads to reasonable results in the case of the hard-pion current-algebra calculation of the π-π scattering.[16] Also, while unitarizing the Veneziano amplitude by effective-range theory, Kang has noted that the unitarity corrections in the ρ-mass region are small.[4] Thus one can hope that our amplitude is a realistic π-π amplitude in the low-energy region.

Using Eq. (23), we have evaluated the $I=0$ and $I=2$ S-wave phase shifts in the present model. In Fig. 1 we compare the $I=0$ S-wave phase shifts with the results of analysis by Gutay *et al.*[10] It can be seen that our phase shifts are in good agreement with the set-I solution of these authors. A further analysis by Marateck *et al.*,[11] in the region 500–900 MeV, shows that the set-I solution is favored. The slight disagreement at the low-energy end may be due to overestimation of the experimental phase shifts by the assumption of the dominance of the peripheral diagrams. The $I=2$ phase shifts are small and negative ($|\delta_0^2|<25°$) and are consistent with the present data.[10]

An interesting quantity is δ_0^0-δ_0^2 at the K-meson mass. In our case this turns out bo be 34°. This is in excellent agreement with the value given by the hard-pion methods[16] and is also consistent with the present

[15] K. V. Vasavada, Phys. Rev. **178**, 2350 (1969); *ibid.* (to be published).

[16] R. Arnowitt, M. H. Friedman, P. Nath, and R. Suitor, Phys. Rev. **175**, 1820 (1968). These authors have given some partial justification for this procedure as follows: In the low-energy region when the phase shifts are small, replacement of $e^{i\delta}\sin\delta$ by $\tan\delta$ can produce only very small errors. Near the poles, violation of crossing symmetry may not be serious. Thus the replacement may be approximately valid in the entire low-energy region. See also L. S. Brown and R. L. Goble, Phys. Rev. Letters **20**, 346 (1968).

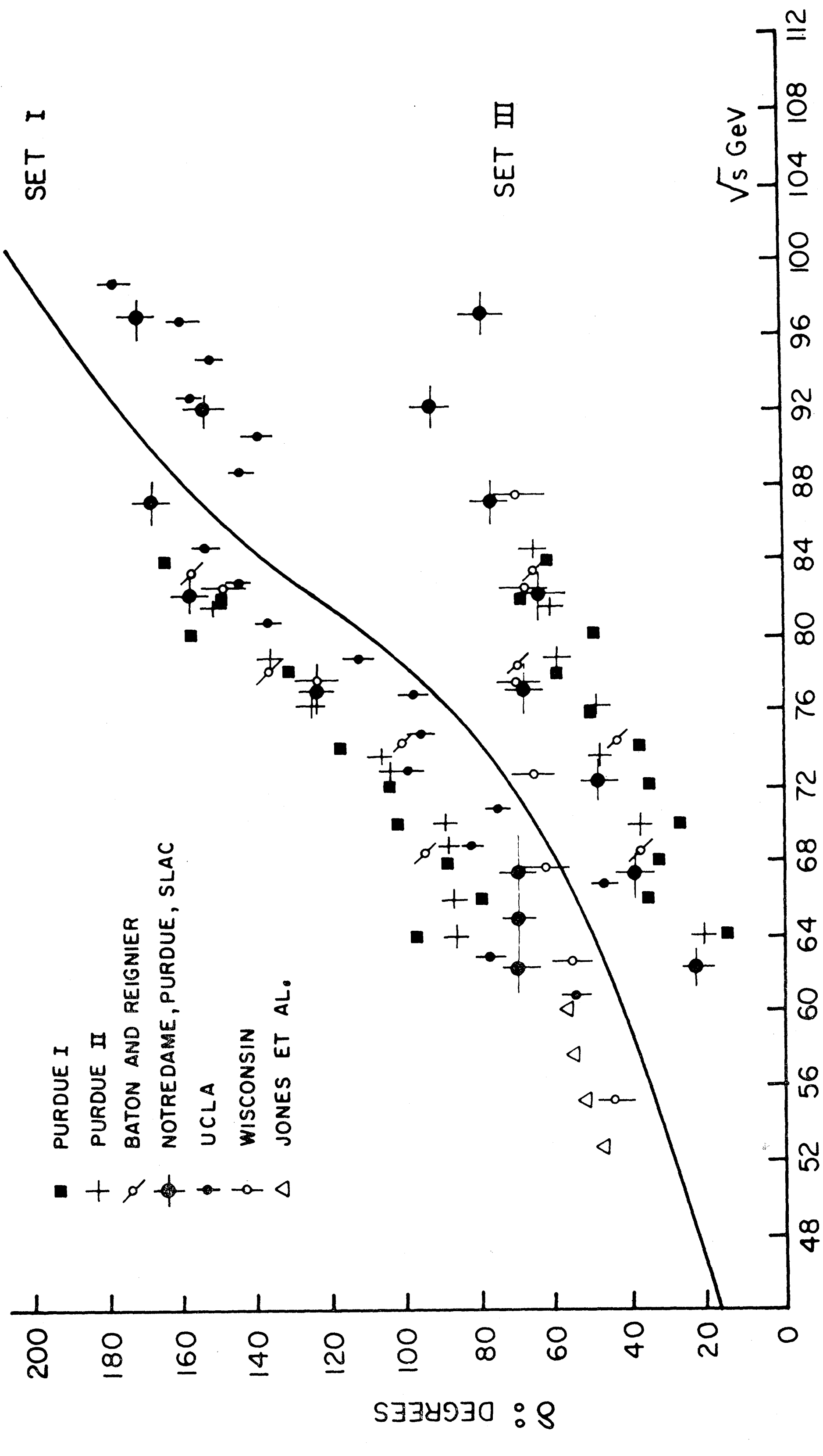

FIG. 1. S-wave phase shifts δ_0^0. Data points are taken from the analysis of Gutay *et al.* (Ref. 10). The solid curve represents our results. The phase shifts for $\alpha_0=0.58$ and 0.56 are essentially identical.

determination from the K^0-$\bar{K}^0$ decay parameters. The values of this quantity for various models are given in Table I.

In conclusion, we make the following remarks. The original Veneziano-Lovelace model for π-π scattering, although a good first approximation, has many undesirable features. Additional terms, which are theoretically permissible, can remove some of these. Several difficulties do, however, remain. The problem of negative-width ghosts has been already mentioned. Another problem is the position of some of the resonances. As a consequence of the exchange degeneracy, the mass of f^0, for $\alpha_0=0.56$, turns out to be 1384 instead of 1260 MeV. The mass of the second recurrence ($J^P=3^-$) becomes 1810 MeV. If we identify this with the g meson (1650 MeV) as quoted in the compilation by the Particle Data Group,[17] then again there is a discrepancy. Of course, as we go to higher energies, effects of unitarization will be more and more important, and this may shift the mass values in the right direction. Finally, we cannot ascribe any fundamental significance to the seven-term formula at the present time. However, it can be used as a reasonable model amplitude consistent with *all* the generally accepted low-energy properties of the π-π system and with the high-energy phenomenology.

[17] Particle Data Group, Rev. Mod. Phys. **41**, 109 (1969).

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Orbital-Angular-Momentum Structure of the A_1 Meson*

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The orbital angular momentum of the A_1 meson produced from $\pi^-p\to\pi^-\pi^-\pi^+p$ at 16 GeV/c was studied by analyzing the decay angular distributions in the $\rho^0\pi^-$ state. A completely Bose-symmetrized formalism was used to measure the ratio g_1/g_0 of the helicity states of the ρ from the A_1 decay. The value obtained, $|g_1/g_0|=0.48\pm0.13$, indicates a substantial d-wave contribution in the A_1 decay.

I. INTRODUCTION

A NUMBER of theoretical calculations[1] have suggested that the axial-vector mesons, in particular the A_1(1080 MeV) and the B(1210 MeV), do not decay via a single orbital-angular-momentum wave but rather through a mixture of two different waves (s and d). Experimental evidence for this idea was recently presented by Ballam *et al.*[2] for the A_1 and by Ascoli *et al.*[3] for the B meson. In both cases, analysis of the decay distributions indicated that the vector meson from the decay (ρ for the A_1 and ω for the B) was polarized in a significantly differently manner than expected for an axial-vector resonance decaying by s wave.

The possibility of a more complicated orbital-angular-momentum structure in the A_1 is also interesting in that it could indicate that the intuitive idea that the lowest partial wave is dominant is not the best approach to use in spin-parity analyses of multibody final states. This problem is a common occurrence for baryon states and may well occur for meson resonances other than the A_1, such as the recently discovered $Q(K^*\pi)$ and $A_3(\pi f)$, both of which could be of the same spin-parity series as the A_1.

In order to study the orbital-angular-momentum structure of the A_1, we have extended the usual spin-parity techniques to the case of several partial waves and analyzed the 16-GeV/c π^-p data of Ballam *et al.* We have studied all of the independent variables associated with the A_1 decay using a completely Bose-symmetrized formalism and a one-pion-exchange (OPE) model for the background. We conclude that there is positive evidence for the presence of both orbital-angular-momentum states in the A_1. Although it is difficult to discriminate conclusively between $J^P=1^+$ or 2^- for the A_1, a 1^+ assignment would indicate that the A_1 has a large d-wave component.

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[1] F. J. Gilman and H. Harari, Phys. Rev. **165**, 1803 (1968); H. J. Lipkin, *ibid.* **176**, 1709 (1968); H. J. Schnitzer and S. Weinberg, *ibid.* **164**, 1828 (1967); S. G. Brown and G. B. West, MIT and Harvard University Report, 1969 (unpublished).

[2] J. Ballam *et al.*, Phys. Rev. Letters **21**, 934 (1968).

[3] G. Ascoli *et al.*, Phys. Rev. Letters **20**, 1411 (1968).