Space-Time Descriptions of Hadrons. I*

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We investigate the possibility of describing the interactions of composite ("Reggeized") hadrons in space-time. We construct a formalism giving a unified description of the spin states of composite hadrons. A Lorentz family of hadron Regge trajectories allows the introduction of local hadron amplitudes. Investigation of the covariance properties of the local hadron amplitudes with respect to proper complex Lorentz transformations, space and time inversion, charge conjugation, and gauge transformations establishes the principles according to which observed families of hadron states are described in terms of representations of the complex Lorentz group. We find that covariance under these transformations necessarily demands parity doubling ("MacDowell symmetry") for meson as well as fermion trajectories. In a world of parallel Regge trajectories (unbroken Lorentz symmetry), chirality is exactly conserved. We give a manifestly Lorentz-covariant description of the propagation of composite hadrons. Explicit expressions for the composite hadron propagators are given for chiral singlet mesons and chiral doublet fermions. Examples of physical assignments are briefly discussed. It is found in particular that particles of unit baryon number transform according to representations of the complex Lorentz group with Toller quantum number $M=\frac{1}{2}$.

1. INTRODUCTION

In this work we propose to investigate the following problem: Is it possible to give a consistent description of hadron interactions in space-time? The question in this form is empty, since one has to give prescriptions to test the space-time structure of scattering amplitudes by physical measurements. In order to be able to test the space-time development of any system, one has to have suitable objects available which interact locally with the system under consideration, thereby marking out points in space-time.

It is, for example, widely believed (actually, tested by various experiments to quite a high degree of accuracy) that electromagnetic (and possibly weak) interactions have the required property of being local in space-time; therefore they can be used as probes to trace the "history" of the system with which they interact.[1] A simple consequence of this observation is that if one wants to maintain the hypothesis of local electromagnetic interactions, one must be able to construct local-interaction densities (current densities) for virtual, as well as for real, hadrons.

A consistent mathematical description of local elecmagnetic interactions[2] is a highly nontrivial problem. Nevertheless, for systems consisting of "elementary" particles,[3] there exists a well-defined scheme (namely, field theory) which enables one to construct local interactions.

We review briefly the essential steps[4] involved in the construction of a local field theory which describes, say, the interaction of particles of spin s and mass m with photons.

(a) First, one recognizes that a single-particle state, transforming according to the irreducible representation (m,s,ϵ) of the Poincaré group (ϵ stands for the sign of the fourth component of the momentum) is nonlocal, i.e., the amplitude

$$\langle x|\mu m s\epsilon\rangle=(2\pi)^{-3/2}\int\frac{d^3p\,\exp\{-i[\mathbf{p}\cdot\mathbf{x}-(\mathbf{p}^2+m^2)^{1/2}t]\}}{2(\mathbf{p}^2+m^2)^{1/2}}\times\langle p|\mu m s\epsilon\rangle$$

(where $\mathbf{p}$, μ stand for the momentum and spin projection of the particle, respectively) has a spread in space typically of the order of the Compton wavelength of the particle.[5] It is therefore not surprising that an "interaction term" constructed from such amplitudes to describe the emission or absorption of a photon by the particle turns out to be highly nonlocal. (It follows from invariance arguments that the interaction term is proportional to the Clebsch-Gordan coefficient

$$\left(\begin{matrix}m & s & 0 & 1\\ \mathbf{p} & \mu & \mathbf{k} & \lambda\end{matrix}\middle|\begin{matrix}m & s\\ \mathbf{p}' & \mu'\end{matrix}\right)$$

of the Poincaré group, where $\mathbf{k}$, λ stand for the momentum and helicity of the photon, respectively. By taking the Fourier transform, one gets the interaction term in

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[1] The role of the electromagnetic (and weak) interactions in the physical "definition" of space-time has been recognized long ago [G. F. Chew, University of California Radiation Laboratory Report No. UCRL-10891 (unpublished)]. Essentially the same philosophy serves as a basis for any dynamical scheme based on current algebras. [See, e.g., R. Dashen and M. Gell-Mann, Phys. Rev. Letters **17**, 340 (1966); H. Sugawara, Phys. Rev. **170**, 1659 (1968).]

[2] In what follows we shall speak of "electromagnetic interactions" for short instead of "electromagnetic and weak interactions."

[3] There does not seem to exist a consistent physical definition (independent of specific schemes, like field theory, etc.) of what one calls an "elementary" particle. In this work we tentatively call a particle "elementary" if it has no "excited states," i.e., there are no observed states with a more or less sharp mass and spin and the same internal quantum numbers as the "ground state." In this sense, e.g., the leptons and the photon seem to be "elementary," whereas the hadrons are "composite."

[4] Compare, e.g., S. Weinberg, Phys. Rev. **133**, B1318 (1964); G. Feldman and P. T. Matthews, Ann. Phys. (N. Y.) **40**, 19 (1966).

[5] T. D. Newton and E. P. Wigner, Rev. Mod. Phys. **21**, 400 (1949).

coordinate space. The reader can easily verify by considering simple examples—e.g., by taking $s=\frac{1}{2}$—that in this way one indeed arrives at nonlocal interactions.)

(b) A *local* interaction can be constructed only if one describes particles and antiparticles simultaneously, i.e., uses amplitudes $\langle x|\mu,ms\epsilon\rangle$ with both values of ϵ. However, the particular linear combination of amplitudes[4] needed for a local description (a combination of "particle" absorption and "antiparticle" emission amplitudes) has no definite Lorentz transformation properties. Indeed, under a given Lorentz transformation Λ, if the spin projection of the particle absorption amplitude ($\epsilon=1$) undergoes a certain Wigner rotation $R_W(\Lambda)$, the antiparticle emission amplitude ($\epsilon=-1$) undergoes essentially the inverse Wigner rotation.

(c) In order to cure that disease, one has to embed the amplitudes into a "suitable" representation of the homogeneous Lorentz group, i.e., to go over to a spinor basis for the description of spin. Since both the particle absorption and antiparticle emission amplitudes can be embedded into the same representation of the homogeneous Lorentz group, it is possible to obtain in this way amplitudes ("fields") which are both local and have definite Lorentz transformation properties.

(d) For "elementary" particles the choice of the embedding representation of the homogeneous Lorentz group is highly arbitrary, the only essential requirement being that it contain the spin-s representation of the Wigner rotations. "Superfluous" spin states are eliminated either by means of subsidiary conditions or by an economic choice of the representation.[4]

The lesson to be learned from such a construction is that it is apparently impossible to construct a Lorentz-covariant theory with local interactions, unless the spin states of the "particles" can be arranged into representations (not necessarily irreducible ones) of the homogeneous Lorentz group.

In attempting to apply a similar construction to hadrons, the essential complication one has to cope with is that the latter are apparently "composite" objects[6] with relatively low-lying excited states, which therefore play an essential role in physical processes. Indeed, a "Regge description" of virtual (e.g., exchanged) hadrons which takes into account a continuum of "spin" states along a Regge trajectory seems to be an essential element of any reasonable theory of hadrons.

As a result, the embedding processes mentioned in (c) above become nontrivial. First, one has to work, in general, with infinite-dimensional representations of the homogeneous Lorentz group. Second, the choice of the embedding representation is no longer arbitrary, since the eigenvalues of the Casimir operators of the homogeneous Lorentz group are measurable quantities, at least when the four-momentum of the virtual hadron vanishes (and therefore the homogeneous Lorentz group becomes the little group). At nonvanishing four-momenta, analyticity arguments demand that the spin states be still classified according to the—in general reducible—representations of the homogeneous Lorentz group.[7] Fortunately, the analysis of experimental data[8] not only proves quite convincingly that observed hadron states are indeed classified according to representations of the homogeneous Lorentz group, but it also shows that the representations remain "almost irreducible." In other words, the representation mixing is small up to masses of the order of 2 BeV. Therefore, it seems to be a reasonable zeroth approximation to consider "hadron towers" with amplitudes transforming according to *irreducible* representations (IR) of the homogeneous Lorentz group.

An essential element of a local description thus being apparently present, one can try to give a full description of hadron systems in terms of their *local* interactions with the electromagnetic field (and perhaps with each other) and the free propagation of virtual hadrons between those points of the space-time where interactions take place.

Hence, our task is (i) to describe the undisturbed propagation of "hadron towers" between two space-time points where interactions take place, and (ii) to investigate the structure of the interactions themselves.

Assuming that such a twofold task can be accomplished, one still has to take a step further and investigate the behavior of the hadron amplitudes under the group of complex homogeneous Lorentz transformations $L(C)$. Since one knows that covariance under the latter is essential in obtaining crossing-symmetric scattering matrices and also that relativistic local field theories are covariant with respect to $L(C)$, it is reasonable to expect that a theory describing the interactions of composite hadrons should also be covariant with respect to this group. We see later that this is indeed the case, provided that the single-"particle" amplitudes are covariant under $L(C)$ and that the interaction terms are built up appropriately.

The plan of the paper is as follows. In Sec. 2, after a brief review of the structure of the complex Lorentz group, we introduce a convenient basis for the representations of the Lorentz group,[9] which provides a

[6] C. F. Chew and S. C. Frautschi, Phys. Rev. Letters **7**, 394 (1961).

[7] For a concise review of the pertinent arguments the reader may consult the texts of the talks given by M. Toller and the first-named author of the present paper, in *Proceedings of the Nobel Symposium VIII, Aspenasgarden, 1968* (Wiley-Interscience, Inc., New York, 1968). References to earlier works on the subject are given in these papers.

[8] G. Domokos, S. Kovesi-Domokos, and P. Suranyi, Nuovo Cimento **56A**, 233 (1968).

[9] The standard references on the subjects discussed in this section are as follows: R. F. Streater and A. S. Wightman, *PCT, Spin and Statistics, and All That* (W. A. Benjamin, Inc., New York, 1964); M. A. Naimark, *Linear Representations of the Lorentz Group* (The Macmillan Co., New York, 1964); Am. Math. Soc. Transls. **36**, 100 (1964); I. M. Gelfand, M. I. Graev, and N. Y. Vilenkin, *Generalized Functions* (Academic Press Inc. New York, 1966), Vol. 5.

labeling for the "single-particle" amplitudes. We then work out the representation theory in that basis in a form which proves useful later on. This section is mainly paedagogical in nature; the material presented is available in the mathematical literature, although not always in a form which is convenient for physical applications. In Sec. 3 we investigate the transformation properties of *local* hadron amplitudes involving one virtual hadron under discrete transformations (space inversion, charge conjugation, time reversal) as well as under gauge transformations. Section 4 contains the construction of the propagator of "Reggeized" virtual hadrons in a manifestly covariant form. In Sec. 5 the results of the paper are briefly summarized and discussed. Some calculations of a purely technical nature are collected in the Appendices.

2. COMPLEX LORENTZ GROUP REPRESENTATIONS

A. Structure of $L(C)$

We recall that an infinitesimal, *real* Lorentz transformation[10] can be written in the form

$$S=1+i\alpha_j m_j+i\beta_j n_j\,, \qquad (2.1)$$

where α_i, β_j ($j=1,2,3$) are real parameters, and the generators satisfy the well-known commutation relations (CR)

$$\begin{aligned}[][m_i,m_j]&=i\epsilon_{ijk}m_k\,,\\ [m_i,n_j]&=i\epsilon_{ijk}n_k\,,\\ [n_i,n_j]&=-i\epsilon_{ijk}m_k\,.\end{aligned} \qquad (2.2)$$

(ϵ_{ijk} is the Levi-Civita tensor in three dimensions, and repeated indices are to be summed over.) The CR (2.2) are decoupled by introducing the operators $j_k^{(1)}$ and $j_i^{(2)}$ by

$$m_k=j_k^{(1)}+j_k^{(2)},\quad n_k=i(j_k^{(1)}-j_k^{(2)})\,, \qquad (2.3)$$

to obtain

$$[j_i^{(A)},j_j^{(B)}]=i\epsilon_{ijk}\delta_{AB}j_k^{(A)}\quad (A=1,2)\,. \qquad (2.4)$$

Any representation of $SL(2,C)$ is characterized by the eigenvalues $j_1(j_1+1)$ and $j_2(j_2+1)$ of the Casimir operators $j_i^{(A)}j_i^{(A)}$, the only restriction being that $2|j_1-j_2|=$integer. Sometimes we find it convenient to write $j_1=\frac{1}{2}(\sigma+M)$, $j_2=\frac{1}{2}(\sigma-M)$, with $M=0,\pm\frac{1}{2},\pm1,\ldots,\sigma$ arbitrary. In terms of the operators (2.3), Eq. (2.1) is rewritten as

$$S=1+i(\alpha_k+i\beta_k)j_k^{(1)}+i(\alpha_k-i\beta_k)j_k^{(2)}\,. \qquad (2.1')$$

A convenient parametrization of the *complex Lorentz transformations* is obtained by letting α_k, β_k become complex numbers, say, $\alpha_k=\alpha_{1k}+i\alpha_{2k}$, $\beta_k=\beta_{1k}-i\beta_{2k}$. Equation (2.1′) then becomes

$$S^C=1+i[(\alpha_{1k}+\beta_{2k})+i(\alpha_{2k}+\beta_{1k})]j_k^{(1)}+i[(\alpha_{1k}-\beta_{2k})+i(\alpha_{2k}-\beta_{1k})]j_k^{(2)}\,.$$

This is one of the conventional definitions of complex Lorentz transformations according to which "dotted and undotted spinor indices undergo independent complex rotations." Equivalently, one can define $L(C)$ as the group of all complex 4×4 matrices which leaves the scalar product of two complex vectors $x_\mu y^\mu=x_iy_i-x_0y_0$ invariant. In order to clarify the structure of $L(C)$, we define the generators

$$m_k^{(1)}\equiv m_k\,,\ n_k^{(1)}\equiv n_k\,,\ m_k^{(2)}=-in_k\,,\ n_k^{(2)}=im_k\,, \qquad (2.5)$$

satisfying obviously the following CR:

$$\begin{aligned}&[m_i^{(1)},m_j^{(1)}]=i\epsilon_{ijk}m_k^{(1)}\,,\qquad [m_i^{(2)},m_j^{(2)}]=i\epsilon_{ijk}m_k^{(1)}\,,\\ &[m_i^{(1)},n_j^{(1)}]=i\epsilon_{ijk}n_k^{(1)}\,,\qquad [m_i^{(2)},n_j^{(2)}]=i\epsilon_{ijk}n_k^{(1)}\,,\\ &[n_i^{(1)},n_j^{(1)}]=-i\epsilon_{ijk}m_k^{(1)}\,,\qquad [n_i^{(2)},n_j^{(2)}]=-i\epsilon_{ijk}m_k^{(1)}\,,\\ &[m_i^{(1)},m_j^{(2)}]=i\epsilon_{ijk}m_k^{(2)}\,,\\ &[m_i^{(1)},n_j^{(2)}]=i\epsilon_{ijk}n_k^{(2)}\,,\\ &[m_i^{(2)},n_j^{(1)}]=i\epsilon_{ijk}n_k^{(2)}\,,\\ &[n_i^{(2)},n_j^{(1)}]=-i\epsilon_{ijk}m_k^{(2)}\,.\end{aligned} \qquad (2.6)$$

It follows that the operators

$$\begin{aligned}M_i^{(\pm)}&=\tfrac{1}{2}(m_i^{(1)}\pm m_i^{(2)})\,,\\ N_i^{(\pm)}&=\tfrac{1}{2}(n_i^{(1)}\pm n_i^{(2)})\end{aligned} \qquad (2.7)$$

satisfy independent CR, viz.,

$$\begin{aligned}[][M_i^{(a)},M_j^{(b)}]&=i\delta_{ab}\epsilon_{ijk}M_k^{(a)}\,,\\ [M_i^{(a)},N_j^{(b)}]&=i\delta_{ab}\epsilon_{ijk}N_k^{(a)}\,,\\ [N_i^{(a)},N_j^{(b)}]&=-i\delta_{ab}\epsilon_{ijk}M_k^{(a)}\end{aligned} \qquad (2.8)$$

(a, $b=\pm$; no summation over a, b). Hence, $L(C)\sim SL(2,C)\otimes SL(2,C)$ and all the representations of $L(C)$ are obtained by letting the direct factors run through their representations.

For reasons that become clear later, we call the independent Lorentz transformations generated by the operators (2.7) *chiral* Lorentz transformations.

B. Choice of Basis: z Representation

Conventionally, vectors spanning an IR of $SL(2,C)$ are labeled by the eigenvalues of $\mathbf{M}^2$ and M_3. This is not convenient for our purposes, since these eigenvalues (the square of the total angular momentum and its projection) have a physical meaning only when the momentum of a particle is timelike. In order to obtain a uniform labeling of the states insensitive to the structure of the little group, we transform to a new pair of labels according to the following procedure.

Consider the row spinor (z_1,z_2) with arbitrary com-

[10] It is always understood that we actually consider the covering group $SL(2,C)$ of the Lorentz group.

plex components. Let

$$g=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix},\quad \alpha\delta-\beta\gamma=1,$$

so that $g\in SL(2,C)$. g transforms (z_1,z_2) into

$$(z_1',z_2')=(\alpha z_1+\gamma z_2,\ \beta z_1+\delta z_2).\tag{2.9}$$

The spinor (z_1,z_2) is the simplest example of the homogeneous functions $f(z_1,z_2,z_1^*,z_2^*)$ with the property that

$$f(\lambda z_1,\lambda z_2,\lambda^* z_1^*,\lambda^* z_2^*)=\lambda^{2j_1}\lambda^{*2j_2}f(z_1,z_2,z_1^*,z_2^*).\tag{2.10}$$

The functions (2.10) realize the IR (j_1,j_2) of $SL(2,C)$, and under g they transform according to

$$T(g)f(z_1,\ldots,z_2^*)=f(z_1',\ldots,z_2'^*),\tag{2.11}$$

where $z_1',\ldots,z_2'^*$ are given by (2.9). The homogeneity condition (2.10) can be used to obtain another realization of $SL(2,C)$ in the space of functions

$$\varphi(z)\equiv f\left(\frac{z_1}{z_2},1,\frac{z_1^*}{z_2^*},1\right),\quad z=z_1/z_2$$

which transform under g according to

$$\begin{aligned}T(g)\varphi(z)&=(\beta z+\delta)^{2j_1}(\beta^*z^*+\delta^*)^{2j_2}\varphi(z'),\\ z'&=(\alpha z+\gamma)/(\beta z+\delta).\end{aligned}\tag{2.12}$$

The complex variable z can be used instead of j, m to label vectors within an IR of $SL(2,C)$, the transformation coefficients between the j, m representation and the z representation being given by the equation (see Appendix A)

$$\begin{aligned}(\sigma M jm|z)&=\left(\frac{2j+1}{8\pi 3}\right)^{1/2}(1+zz^*)^{\sigma}\left(\frac{z}{z^*}\right)^{(M+m)/2}\\ &\times d_{mM}{}^{j}\left(\frac{zz^*-1}{zz^*+1}\right)\quad(\sigma=j_1+j_2,\ M=j_1-j_2).\end{aligned}\tag{2.13}$$

The spinor (z_1,z_2) itself (and hence the variable z) can be parametrized by a Lorentz transformation, for example, by performing the Lorentz transformation

$$L(\gamma\beta\xi\alpha)=e^{-i\gamma M_3}e^{-i\beta M_2}e^{-i\xi N_3}e^{-i\alpha M_3}\tag{2.14}$$

on the standard spinor

$$\begin{pmatrix}1\\0\end{pmatrix}.$$

Noting that the spinor

$$\begin{pmatrix}z_1\\z_2\end{pmatrix}$$

transforms according to the IR $(\frac{1}{2}0)$, so that in terms of the Pauli matrices, $M_j=\frac{1}{2}\sigma_j$, $N_j=-\frac{1}{2}i\sigma_j$, one is led to the following parametrization: $z=\cot\frac{1}{2}\beta\,\exp(-i\gamma)$. We shall find this parametrization often useful.

We therefore label the amplitudes of hadrons transforming according to the IR (j_1j_2) by the variables z and z^*.

C. Invariants, Contravariant Representations

Consider two amplitudes[11] transforming according to the IR $[\sigma M]$ and $[\sigma' M']$, respectively. The quantity

$$I=\int d^2z\,\varphi^{[\sigma M]}(z)\varphi^{[\sigma' M']}(z)\tag{2.15}$$

($d^2z=d\,\mathrm{Re}z d\,\mathrm{Im}z$, $-\infty<\mathrm{Re}z<\infty$, $-\infty<\mathrm{Im}z<\infty$) is an invariant provided $\sigma+\sigma'+2=0$, $M+M'=0$. Indeed,

$$\begin{aligned}T(g)I&=\int d^2z\,T(g)\varphi^{[\sigma M]}(z)T(g)\varphi^{[\sigma' M']}(z)\\ &=\int d^2z(\beta z+\delta)^{\sigma+\sigma'+M+M'}(\beta^*z^*+\delta^*)^{\sigma+\sigma'-M-M'}\\ &\qquad\times\varphi^{[\sigma M]}(z')\varphi^{[\sigma' M']}(z')\\ &=\int d^2z'\,(-\beta z'+\alpha)^{-\sigma-\sigma'-M-M'-2}\\ &\qquad\times(-\beta^*z'^*\alpha^*)^{-\sigma-\sigma'+M+M'-2}\\ &\qquad\times\varphi^{[\sigma M]}(z')\varphi^{[\sigma' M']}(z')=I,\end{aligned}\tag{2.16}$$

provided that $\sigma'=-\sigma-2$, $M'=-M$. Hence, we call the IR $[-\sigma-2\ -M]$ *contravariant* to $[\sigma M]$. The *contravariant* representation $[-\sigma-2\ -M]$ is equivalent to $[\sigma M]$. Indeed, the reader can verify immediately that $\psi(z)$ defined by

$$\psi(z)=\int d^2z_1\,(z-z_1)^{-\sigma-M-2}(z^*-z_1^*)^{-\sigma+M-2}\varphi^{[\sigma M]}(z_1)$$

transforms according to the IR $[-\sigma-2\ -M]$ if φ transforms according to $[\sigma M]$. Thus the quantity

$$G^{[\sigma M]}(z_1,z_2)=(z_1-z_2)^{-\sigma-M-2}(z_1^*-z_2^*)^{-\sigma+M-2}\tag{2.17}$$

plays the role of the metric tensor which transforms the "covariant" vectors of the IR $[\sigma M]$ into their contravariant counterparts. Obviously, Eq. (2.15) can be generalized to an arbitrary number of amplitudes, viz.,

$$K=\int d^2z\,\psi^{[\sigma_1M_1]}(z)\cdots\varphi^{[\sigma_nM_n]}(z)\tag{2.18}$$

is an invariant if (and only if)

$$\sum\sigma_i+2=0,$$
$$\sum M_i=0.$$

We end this section by proving a simple but very useful lemma.

[11] When characterizing an IR of SL(2,C) by the labels (j_1,j_2), we shall write them in parentheses, whereas the labels $[\sigma M]$ employ square brackets.

Lemma. Consider a function $K(z_1, \ldots, z_n)$ such that

$$K \equiv \int d^2z_1 \cdots d^2z_n \varphi^{[\sigma_1 M_1]}(z_1) \cdots \varphi^{[\sigma_n M_n]}(z_n) \times K(z_1, \ldots, z_n) = \text{invariant}. \quad (2.19)$$

Then $\mathbf{K}$ is a function of the "coordinate differences" $z_i - z_k$ only.

Proof. Every $g \in SL(2,C)$ can be composed of the following matrices:

$$\alpha \equiv \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \quad r \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad a \equiv \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$$

(a, α are complex numbers). Take the matrix $g = a$; then from (2.12) and (2.19),

$$T(a)K = \int d^2z_1 \cdots d^2z_n \varphi^{[\sigma_1 M_1]}(z_1 + a) \cdots \varphi^{[\sigma_n M_n]}(z_n + a) \times K(z_1, \ldots, Z_n)$$

$$= \int d^2z_1' \cdots d^2z_n' \varphi^{[\sigma_1 M_1]}(z_1') \cdots \varphi^{[\sigma_n M_n]}(z_n'),$$

$$K(z_1' - a, \ldots, z_n' - a) = K.$$

Since the functions φ are arbitrary, it follows that $K(z_1, \ldots, z_n) = K(z_1 - a, \ldots, z_n - a)$, which is possible only if K is a function of the differences $z_i - z_k$ only. Q. E. D.

We now indicate the significance of the lemma just proved. As we see in Paper II, scattering matrix elements, vertices, etc., in the z representation always appear in the form (2.19), where the functions φ are single-hadron amplitudes. According to the lemma, the kernels $K(z_1 \cdots z_n)$ are translationally invariant functions in z space. We therefore introduce Fourier transforms of these functions.

We assume that $K(z_1, \ldots, z_n)$ is some vertex function and introduce *quasimomenta* by the equations

$$\varphi^{[\sigma_i M_i]}(z) = \frac{1}{2\pi} \int d^2P \, e^{i \operatorname{Re} Pz} \tilde{\varphi}^{[\sigma_i M_i]}(P),$$

$$K(z_1, \ldots, z_n) = \frac{1}{(2\pi)^n} \int d^2P_1 \cdots d^2P_n e^{i \operatorname{Re}(P_1 z_1 + \cdots + P_n z_n)} \times \tilde{K}(P_1 \cdots P_n) \delta^{(2)}(P_1 + \cdots + P_n). \quad (2.20)$$

Here the quasimomenta P_i are complex variables (two-dimensional vectors); the two-dimensional δ function, defined by

$$\delta^{(2)}(P) = \frac{1}{(2\pi)^2} \int d^2z \, e^{-i \operatorname{Re} Pz},$$

takes care of the translation invariance. The invariant (2.19) can thus be rewritten as

$$K = \int d^2P_1 \cdots d^2P_n \tilde{\varphi}^{[\sigma_1 M_1]}(-P_1) \cdots \tilde{\varphi}^{[\sigma_n M_n]}(-P_n) \times \tilde{K}(P_1 \cdots P_n) \delta(P_1 + \cdots + P_n). \quad (2.19')$$

Expressed in words, the *quasimomenta* are conserved at each vertex. Evidently, the quasimomenta can be used instead of the variables z or j, m to label the spin states of hadron amplitudes. Because of the conservation law expressed by Eq. (2.19′), the use of the P representation greatly simplifies the calculations.

3. REFLECTIONS AND GAUGE TRANSFORMATIONS: PHYSICAL ASSIGNMENTS

A. Local Hadron Amplitudes

We want to establish the rules according to which one can assign hadron towers with parallel Regge trajectories to IR of $L(C)$. To this end one has to establish the behavior of hadron amplitudes under space and time inversion, charge conjugation (P,T,C), and gauge transformations. Consider a "single-hadron amplitude" for the process in which a virtual single hadron[12] goes over into a system of physical hadrons (Fig. 1).

Let $\mathbf{p}$, W, j, and m be, respectively, the total momentum, mass, total angular momentum, and its projection of the hadron system. Let ξ be the collection of all other quantum numbers necessary to characterize the state. This amplitude is—at least in principle—measurable, e.g., in high-energy processes with a single-hadron exchange. The amplitude $\langle \sigma M \mathbf{p} W jm | \xi \rangle$ can be written as follows (see Appendix B):

$$\langle \sigma M \mathbf{p} W jm | \xi \rangle = \sum_{s\mu} D_{jms\mu}{}^{\sigma M}(L(\mathbf{p})) A_{s\mu}{}^{\sigma M}(\mathbf{p}, \xi). \quad (3.1)$$

Here, $L(\mathbf{p})$ stands for the boost $(\mathbf{p}, \epsilon\omega) = L(\mathbf{p})(000, \epsilon W)$, where $\omega = (\mathbf{p}^2 + w^2)^{1/2}$, $\epsilon = p_0/|p_0|$; in the fundamental representation $[\frac{1}{2}\frac{1}{2}]$ of $SL(2,C)$, we have

$$L(\mathbf{p}) = [2W(W+\omega)]^{-1/2}[-W - \omega + \epsilon \boldsymbol{\sigma} \cdot \mathbf{p}]. \quad (3.2)$$

We call $A_{s\mu}{}^{\sigma M}(\mathbf{p}, \xi)$ a "reduced" amplitude. We use the "canonical" (rotation-free) boost convention. Henceforth, the variable W is suppressed; it is also understood that σ varies with W along the "Lorentz trajectory."

Notice the important properties of the boost (3.2):

$$L(-\mathbf{p}) = L(\mathbf{p})^{-1} = -i\sigma_2 L(\mathbf{p})^* i\sigma_2 \equiv rL(\mathbf{p})^* r^{-1}, \quad (3.3)$$

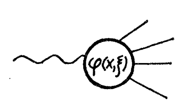

FIG. 1. Single-hadron amplitude.

[12] Here and in what follows, by a "single hadron" we always mean a tower of hadron states, spanning a representation of $SL(2,C)$.

where the matrix r was already defined in Sec. 2. Similarly, we write for the antiparticle amplitude

$$\langle \sigma M \mathbf{p} W jm | \xi^c \rangle = \sum_{s\mu} D_{jms\mu}{}^{\sigma M}(L(\mathbf{p})) B_{s\mu}{}^{\sigma M}(\mathbf{p},\xi). \quad (3.1')$$

We go over to the z representation by multiplying (3.1) and (3.1′) with $\langle z|\sigma M jm\rangle$ and summing over jm. Thus we get

$$\langle \sigma M \mathbf{p} W z | \xi \rangle = \sum_{s\mu} \varphi_{s\mu}{}^{\sigma M}(\mathbf{p},z) A_{s\mu}{}^{\sigma M}(\mathbf{p},\xi), \quad (3.4)$$

and similarly for the antiparticle amplitude.[13] Here,

$$\varphi_{s\mu}{}^{\sigma M}(\mathbf{p},z) = \sum_{jm} \langle z|\sigma M jm\rangle D_{jms\mu}{}^{\sigma M}(L(\mathbf{p})) = T(L(\mathbf{p}))\langle z|\sigma M s\mu\rangle. \quad (3.5)$$

We now combine (3.4) with the complex conjugate of the reduced antiparticle amplitude (i.e., the antiparticle emission amplitude) and take their Fourier transforms to obtain the local[14] hadron amplitude

$$\varphi^{\sigma M}(x,z,\xi) = (2\pi)^{-3/2} \sum_{s\mu} \int \frac{d^3p}{2\omega} \{ \varphi_{s\mu}{}^{\sigma M}(\mathbf{p},z) A_{s\mu}{}^{\sigma M}(\mathbf{p},\xi) e^{ip\cdot x} + (-)^{s-\mu} \varphi_{s-\mu}{}^{\sigma M}(\mathbf{p},z) [B_{s\mu}{}^{\sigma-M}(\mathbf{p},\xi)]^* e^{-ip\cdot x} \}, \quad (3.6)$$

where $p\cdot x = \mathbf{p}\cdot\mathbf{x} - \omega x_0$.

In order to establish a connection with the representations of $L(C)$, we notice that a Dirac spinor, say, ψ, spans the fundamental representation of $L(C)$. Indeed, combining the 12 real infinitesimal parameters of $L(C)$ into two antisymmetric tensors, say, $\omega^{\mu\nu}$ and $\tau^{\mu\nu}$, we find that ψ transforms under an infinitesimal complex Lorentz transformation as follows:

$$\delta\psi = \tfrac{1}{4} i\omega^{\mu\nu}\sigma_{\mu\nu}(1+\gamma_5)\psi + \tfrac{1}{4} i\tau^{\mu\nu}\sigma_{\mu\nu}(1-\gamma_5)\psi.$$

The chiral components of ψ are exhibited by writing

$$\psi = \tfrac{1}{2}(1+\gamma_5)Z + \tfrac{1}{2}(1-\gamma_5)U,$$

where Z and U are two-component spinsors which transform, respectively, according to the IR $[\frac{1}{2}\frac{1}{2}]$ and $[\frac{1}{2}\ -\frac{1}{2}]$ of $SL(2,C)$, and thus according to the irreducible representations $[\frac{1}{2}\frac{1}{2}00]$ and $[00\frac{1}{2}-\frac{1}{2}]$ of $L(C)$. On identifying Z with the two-component spinor used in Sec. 2 to build up IR of $SL(2,C)$, we find that the amplitude (3.6) transforms according to the IR $[\sigma M00]$ of $L(C)$.

Using the spinor U instead of Z, we can build up a local amplitude similar to (3.6), viz.,

$$\chi^{\sigma-M}(x,u,\xi) = (2\pi)^{-3/2} \sum_{s\mu} \int \frac{d^3p}{2\omega} \{ \chi_{s\mu}{}^{\sigma-M}(\mathbf{p},u) A_{s\mu}{}^{\sigma-M}(\mathbf{p},\xi) e^{ip\cdot x} + (-)^{s-\mu} \chi_{s-\mu}{}^{\sigma-M}(\mathbf{p},u) [B_{s\mu}{}^{\sigma M}(\mathbf{p},\xi)]^* e^{-ip\cdot x} \}, \quad (3.7)$$

which transforms according to the IR $[00\sigma-M]$ of $L(C)$. In order to understand the structure of $\chi_{s\mu}{}^{\sigma-M}(\mathbf{p},u)$, we observe that under $SL(2,C)$, $U\sim rZ^*$ (U is a dotted contravariant spinor). Hence, in order to obtain χ, take

$$\begin{aligned} T(r)\varphi_{s\mu}{}^{\sigma M}(\mathbf{p},z)^* &= T(r)T(L(\mathbf{p})^*)\langle z|\sigma M s\mu\rangle^* \\ &= T(rL(\mathbf{p})^* r^{-1})T(r)\langle z|\sigma M s\mu\rangle^* \\ &= (-)^{s-M} T(L(-\mathbf{p}))\langle z|\sigma - M s\mu\rangle, \end{aligned} \quad (3.8a)$$

where the last line follows from (3.3) and (2.13).

We now define within $SL(2,C)$

$$\chi_{s\mu}{}^{\sigma-M}(\mathbf{p},u) = T(L(\mathbf{p}))\langle u|\sigma - M s\mu\rangle; \quad (3.8b)$$

the transformation property under $L(C)$ follows from the assignment of Z and U to IR of $L(C)$.

B. Reflections and Gauge Transformations

The transformation properties of the local hadron amplitudes (3.6) and (3.7) under P, C, and T follow immediately from those of the Dirac spinors and the construction of the transformation functions $\varphi_{s\mu}{}^{\mu-M}(\mathbf{p},u)$. The transformation properties of the chiral components of the Dirac spinors under P, C, and T are well known; for later reference we summarize them with the conventional choice of phase in Table I. As an example, we work out the transformation of $\varphi^{\sigma M}(xz\xi)$ under space inversion in detail; the derivation of the transformation properties of the amplitudes under the other discrete operations follows the same pattern.

We observe that the generators (2.7) of $L(C)$ transform under space inversion as follows:

$$\begin{aligned} PM_i{}^{(\pm)}P^{-1} &= M_i{}^{(\mp)}, \\ PN_i{}^{(\pm)}P^{-1} &= -N_i{}^{(\mp)}. \end{aligned} \quad (3.9)$$

This follows from the definition (2.7) and from the well-known transformation properties of the generators of the real Lorentz group. Also, looking at the parametrization (2.14) of the spinors, we see that under parity the variable z goes over into a variable transforming as $-z^{*-1}$, i.e., space inversion is represented by

Table I. Reflection properties of Dirac spinors.

$PZ=iU$	$PU=iZ$
$CZ=i\sigma_2U^*$	$CU=-i\sigma_2Z^*$
$TZ=i\sigma_2Z^*$	$TU=i\sigma_2U^*$

[13] The transformation function entering, e.g., (3.4) is the matrix element of the boost $L(\mathbf{p})$ in a "mixed" basis: The initial state is specified by the angular momentum s and its projection μ, whereas the final state is specified by the variable z. $\varphi_{s\mu}{}^{\sigma M}(\mathbf{p},z)$ is the analog of the "single-particle wave functions" used in a conventional field theory.

[14] Strictly speaking, we do not know at this point whether the amplitude we construct is truly local in the sense of having zero spread in spacelike directions for any value of σ. We use the term because of the analogy with the field-theoretic construction sketched in Sec. 1.

a complex conjugation followed by a unimodular transformation with the matrix r.

Thus, Eq. (3.8a) tells us that

$$T(P)\varphi_{s\mu}{}^{\sigma M}(\mathbf{p},z)=(-)^{s-M}\varphi_{s\mu}{}^{\sigma-M}(-\mathbf{p},z) \quad \text{or} \quad (-)^{s-M}\chi_{s\mu}{}^{\sigma-M}(-\mathbf{p},u).$$

The ambiguity is removed by looking at the first line of Table I [resulting from the transformation property (3.9)], so that

$$T(P)\varphi_{s\mu}{}^{\sigma M}(\mathbf{p},z)=(-)^{s-M}\chi_{s\mu}{}^{\sigma-M}(-\mathbf{p},u). \quad (3.10)$$

Inserting Eq. (3.10) into (3.6), and using the relations

$$\begin{aligned} A_{s\mu}{}^{\sigma-M}(-\mathbf{p},\xi^p)&=(-)^{s-M}H_pA_{s\mu}{}^{\sigma M}(\mathbf{p}\xi^p),\\ B_{s\mu}{}^{\sigma M}(-\mathbf{p},\xi^p)&=(-)^{s+M}\bar{H}_pB_{s\mu}{}^{\sigma-M}(\mathbf{p},\xi)\end{aligned} \quad (3.11)$$

derived in Appendix B and the phase conventions fixed by Table I, we finally obtain

$$T(P)\varphi^{\sigma M}(\mathbf{x},t,z,\xi)=(-)^MH_p\chi^{\sigma-M}(-\mathbf{x},t,z\xi^p), \quad (3.12)$$

provided the particle and antiparticle inversion phases satisfy the familiar relation $H_p\bar{H}_p=(-)^{2M}=(-)^{2s}$. In Table II we summarize the inversion properties of the amplitudes.

The Dirac spinors also have definite transformation properties under fermion number and chiral gauge transformations, viz., $\psi\to e^{i\gamma}\psi$ and $\psi\to e^{i\delta\gamma_5}\psi$, respectively. It is remarkable that any basis of an IR of $L(C)$ automatically supports these transformations as well.

The transformation rule (2.12) is applicable (although the gauge transformations are *not* unimodular) and we get

fermion gauge:

$$\begin{aligned}\varphi^{\sigma M}(x,z,\xi)&\to e^{2i\gamma M}\varphi^{\sigma M}(x,z,\xi),\\ \chi^{\sigma-M}(x,z,\xi)&\to e^{2i\gamma M}\chi^{\sigma-M}(x,z\xi);\end{aligned} \quad (3.13)$$

chiral gauge:

$$\begin{aligned}\varphi^{\sigma M}(x,z,\xi)&\to e^{2i\delta M}\varphi^{\sigma M}(x,z,\xi),\\ \chi^{\sigma-M}(x,z,\xi)&\to e^{-2i\delta M}\chi^{\sigma-M}(x,z\xi).\end{aligned}$$

The gauge transformations are outside $L(C)$; in particular, the eigenvalues of the generator of chiral gauge transformation can be used to label the $SL(2,C)$ subgroup entering as direct factors of $L(C)$, since they commute with all the generators of the latter. We anticipated this result in Sec. 2 in calling the direct factors

TABLE II. Transformation of the local hadron amplitudes under P, C, T.

$T(P)\varphi^{\sigma M}(x,t,z,\xi)=(-)^MH_P\chi^{\sigma-M}(-x,t,u,\xi^P)$
$T(P)\chi^{\sigma-M}(x,t,u,\xi)=(-)^M\bar{H}_P\varphi^{\sigma M}(-x,t,z,\xi^P)$
$T(C)\varphi^{\sigma M}(x,z,\xi)=H_CT(r^{-1})\chi^{\sigma-M}(x,u,\xi^C)^*$
$T(C)\chi^{\sigma-M}(x,u,\xi)=(-)^{2M}\text{A}_CT(r)\varphi^{\sigma M}(x,z,\xi^C)^*$
$T(T)\varphi^{\sigma M}(x,t,z,\xi)=H_TT(r^{-1})\varphi^{\sigma M}(x,-t,z,\xi^T)^*$
$T(T)\chi^{\sigma-M}(x,t,u,\xi)=\bar{H}_TT(r^{-1})\chi^{\sigma-M}(x,-t,u,\xi^T)^*$
$H_P^*=H_P,\quad H_C^*=H_C,\quad H_T^*=H_T$

chiral Lorentz groups; indeed, the generators (2.7) are labeled by the chiralities. More important, we learn that *as long as $L(C)$ is a good symmetry*, as it is in a world with parallel Regge trajectories (and not just a covariance group of the theory), *chirality is also a good quantum number*.

C. Physical Assignments

The transformation properties established are—at least in principle—sufficient to assign hadron towers to the representations of $L(C)$. The task can be easily accomplished in the approximation where the $L(C)$ symmetry is considered exact even for $W\neq0$, since then the assignments can be deduced from the properties of physical single-particle states. Here we restrict ourselves to a few remarks; the question of physical assignments is investigated in more detail elsewhere.

(*a*) *Baryons*. All states with baryon numbers$=1$ must have $|M|=\frac{1}{2}$, because of the gauge transformation property (3.13); this immediately fixes the chiral transformation property as well. Baryons necessarily occur in parity doublets, a well-known result (MacDowell symmetry). Baryons form chiral doublets; the parity eigenstates are linear combinations of the form $\varphi\pm\chi$; therefore they do not transform simply under $L(C)$ or $SL(2,C)$.

(*b*) *Mesons*. Mesons belonging to the representations of the type $[\sigma M00]\oplus[00\sigma-M]$, investigated in this paper [they are certainly *not* the most general representations of $L(C)$], must necessarily have $M=0$, since they carry baryon number zero. According to (3.13), these mesons must form chiral *singlets*. The neutral members of a meson multiplet are eigenstates of C; hence, we must have

$$A_{s\mu}{}^{\sigma0}(\mathbf{p},\xi)=\pm B_{s\mu}{}^{\sigma0}(\mathbf{p},\xi). \quad (3.14)$$

Inserting this into (3.6) and (3.7), we learn that

$$\varphi^{\sigma0}(x,z,\xi)\cong\pm T(r)\varphi^{\sigma0}(x,z,\xi)^*,$$

and similarly for χ. Parity eigenstates are formed by taking $\varphi^{\sigma0}\pm\chi^{\sigma0}$. These again do not transform simply under $L(C)$; however, since $M=0$, they transform irreducibly under $SL(2,C)$ (on identifying the variables z and u, which means restriction to real Lorentz transformations). Depending on the choice of signs in (3.14), we have $P=\pm C$. There is one candidate among the known meson families for each choice of signs. On choosing the $(+)$ sign in (3.14), we get $P=C$, which can be identified with the exchange-degenerate vector-tensor family. The $P=C=-1$ members are $\rho(1^{--})$, $g(3^{--})$, etc., while $P=C=+1$ corresponds to $A_{2H}(2^{++})$, etc. The other choice in (3.14) gives $P=-C$; this apparently can be identified with an exchange-degenerate pseudoscalar–axial-vector family, the first members of them being $\pi(0^{-+})$ and $B(1^{+-})$, respectively. Clearly, other families do exist; they can be accommodated by various choices of the inversion phases H_P and H_C.

4. PROPAGATION OF VIRTUAL HADRONS

We describe the propagation of virtual hadrons between two points in space-time where interactions take place (Fig. 2). The "interactions" are described by the local hadron amplitudes introduced in Sec. 3. Because of the chiral invariance of the theory it is sufficient to consider the propagation of hadrons with definite chirality.

A. $M=0$ Propagator

We start with the simplest case, that of a meson propagator with $M=0$. Clearly, the function describing the propagation of the virtual meson between the space-time points x_1 and x_2, where interactions take place, must be a function of x_1-x_2 only; it also depends on the spin variables z_1 and z_2 carried by the hadron amplitudes. Taking for the sake of simplicity an "elastic" process (the hadron amplitudes on both ends of Fig. 2 are the same), the structure of the invariant scattering matrix element corresponding to the process of Fig. 2 must be

$$T=\int d^2z_1d^2z_2\int d^4x_1d^4x_2\ \varphi^{\sigma0}(x_1,z_1,\xi_1)^*$$
$$\times\Delta^{\sigma0}(x_1-x_2,\ z_1,\ z_2)\varphi^{\sigma0}(x_2,z_2,\xi_2)$$
$$\pm\int d^2u_1d^2u_2\int d^4x_1d^4x_2\ \chi^{\sigma0}(x_1,z_1,\xi_1)^*$$
$$\times\Delta^{\sigma0}(x_1-x_2,\ u_1,\ u_2)\chi^{\sigma0}(x_2,u_2,\xi_2).\quad(4.1)$$

On going over to momentum space, the last equation reads

$$T=\frac{1}{(2\pi)^4}\int d^4k\int d^2z_1d^2z_2\ \varphi^{\sigma0}(k,z_1\xi_1)^*$$
$$\times\Delta^{\sigma0}(k,z_1,z_2)\varphi^{\sigma0}(k,z_2,\xi_2)\pm(\varphi\rightarrow\chi),\quad(4.2)$$

where

$$\varphi^{\sigma0}(x,z,\xi)=(2\pi)^{-4}\int d^4k\ \varphi^{\sigma0}(k,z,\xi)e^{ik\cdot x},$$

$$\Delta^{\sigma0}(x,z_1,z_2)=(2\pi)^{-4}\int d^4k\ \Delta^{\sigma0}(k,z_1,z_2)e^{ik\cdot x}.$$

We assume that σ is real ("narrow-resonance approximation"); thus both $\varphi^{\sigma0}(k,z,\xi)$ and $\varphi^{\sigma0}(k,z,\xi)^*$ transform according to the IR $[\sigma000]$ of $L(C)$; likewise, $\chi^{\sigma0}$ and $\chi^{\sigma0*}$ transform according to the IR $[00\sigma0]$.

It follows that

$$\int d^2z_1d^2z_2\varphi^{\sigma0}(k,z_1,\xi_1)^*\Delta^{\sigma0}(k,z_1,z_2)\varphi^{\sigma0}(k,z_2,\xi_2)$$

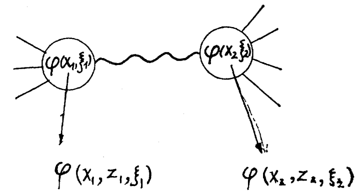

FIG. 2. Hadron scattering amplitude with one virtual intermediate hadron.

is an invariant. Invariance of the scattering matrix element under the discrete transformations (Sec. 3) requires that $\Delta^{\sigma0}$ be a function of k^2 only, and that $\Delta^{\sigma0}(k^2,z_1,z_2)$ and $\Delta^{\sigma0}(k^2,u_1,u_2)$ have the same functional forms.

Thus [cf. (2.15)–(2.17)] $\Delta^{\sigma0}(k^2,z_1,z_2)$ must be proportional to the metric tensor $G^{[\sigma0]}(z_1,z_2)$, i.e.,

$$\Delta^{\sigma0}(k^2,z_1,z_2)=f(k^2,\sigma(k^2))|z_1-z_2|^{-2\sigma(k^2)-4},\quad(4.3)$$

where $\sigma(k^2)$ is the trajectory of the Lorentz pole, and $f(k^2,\sigma(k^2))$ is an invariant function. We determine f by the following normalization condition. Evidently, the inverse propagator is proportional to the contravariant metric tensor $G^{[-\sigma-2\ 0]}$, and by definition we must have

$$\int d^2z\ \Delta^{\sigma0}(k^2,z_1,z)\Delta^{\sigma0}(k^2z,z_2)^{-1}=\delta^{(2)}(z_1-z_2).$$

Inserting (4.3) and the corresponding one with σ replaced by $-\sigma-2$, the resulting integral can be readily evaluated (e.g., by introducing polar coordinates in the complex z plane). As a result we obtain

$$\int d^2z\Delta^{\sigma0}(k^2,z_1,z)\Delta^{\sigma0}(k^2,z,z_1)$$
$$=-\pi^2\frac{f(k^2,\sigma)f(k^2,\ -\sigma-2)}{(\sigma+1)^2}\delta^{(2)}(z_1-z_2),$$

so that by choosing $f(k^2,\sigma(k^2))=-\pi^{-1}[\sigma(k^2)+1]$ the normalization condition is satisfied.[15] Thus we get the final expression for the propagator

$$\Delta^{\sigma0}(k^2,z_1,z_2)=(-1/\pi)[\sigma(k^2)+1]|z_1-z_2|^{-2\sigma(k^2)-4}.\quad(4.4)$$

It will be useful to write the propagator (4.4) in the quasimomentum representation. The corresponding transformation is carried out in Ref. 9 and is reproduced in Appendix C. The result reads

$$\Delta^{\sigma0}(k^2,K)=\frac{2^{-2\sigma(k^2)-2}}{2\pi}\frac{\Gamma(-\sigma(k^2))}{\Gamma(\sigma(k^2)+1)}K^{\sigma(k^2)+1}K^{*\sigma(k^2)+1}.\quad(4.5)$$

(We have introduced here a convenient notation to be

[15] Evidently, the normalization condition does not fix f uniquely. The function chosen in the main text could be multiplied by any invariant function, say, $F(k^2,\sigma)$, which satisfies the functional equation $F(k^2,\sigma)F(k^2,\ -\sigma-2)=1$. However, any F other than the trivial choice ($F\equiv1$) seems to give rise to a propagator with wrong physical properties (e.g., cuts or extra poles in σ).

used throughout the paper: When a function is written down in momentum and quasimomentum representation, the quasimomenta and momenta are denoted by the same letter; lower case is used for the momentum, capitals for the quasimomentum variables.)

B. $M\neq 0$ Propagator

The physical reasoning that leads to the form of the propagators for $M\neq 0$ is exactly the same as that described for $M=0$ in Sec. 4 A. This time we start from the corresponding invariant for $M\neq 0$:

$$\int d^2z_1d^2z_2\ \varphi^{\sigma M}(k,z_1,\xi_1)^*\Delta^{\sigma M}(k,z_1,z_2)\varphi^{\sigma M}(k,z_2,\xi_2).$$

The essential complication is that now $\varphi^{\sigma M}$ and $\varphi^{\sigma M*}$ transform according to inequivalent representations of $L(C)$, viz., $\varphi^{\sigma M*}$ transforms as $[\sigma\ -M\ 0\ 0]$. The amplitudes φ and φ^* can be coupled to an invariant together with an irreducible tensor transforming as $[2M\ 0]$ formed of the momentum k. {Example: propagator of a Weyl field. This reads $\Delta^{\frac{1}{2}\frac{1}{2}}(k)=k^{-2}[k_0+\boldsymbol{\sigma}\cdot\mathbf{k}]$.} The latter is evidently equal to the transformation function $\varphi_{00}{}^{2M\ 0}(k,z)$ multiplied by W^{2M}.

Thus we write

$$\Delta^{\sigma M}(k,z_1,z_2)=NW^{2M}\int d^2z_3\begin{pmatrix}\sigma-M & \sigma M & -2M-2\ 0\\ z_1 & z_2 & z_3\end{pmatrix}\times\varphi_{00}{}^{-2M-2\ 0}(k,z_3).\quad(4.6)$$

(Evidently, we have to use the contravariant form $\varphi^{-2M-2\ 0}$ of the tensor $\varphi^{2M\ 0}$.) The first factor under the integral is a symmetric Clebsch-Gordan coefficient of $SL(2,C)$, while N is a normalization coefficient. Inserting the explicit expression for the Clebsch-Gordan coefficient from Appendix D and for the function $\varphi_{00}{}^{-2M-2\ 0}(k,z)$, after some elementary manipulations we obtain the expression

$$\Delta^{\sigma M}(k,z_1,z_2)=N|z_1-z_2|^{-2(\sigma+2+M)}A^M(k),\quad(4.7)$$

where

$$A^M(k)=(b^*z_1{}^*+d^*)^{2M}(bz_2+d)^{2M}W^{2M}$$
$$\times\int d^2z_3'(z_2'-z_3')^{2M}(z_3'^*-z_1'^*)^{2M}(1+z_3'z_3'^*)^{-2M-2}.\quad(4.8)$$

In expression (4.8), $z_i'=(az_i+c)(bz_i+d)^{-1}$ $(i=1,\ 2)$, and $a,\ \ldots,\ d$ denote the elements of the 2×2 boost matrix (3.2):

$$L(\mathbf{k})\equiv\begin{pmatrix}a & b\\ c & d\end{pmatrix}.$$

The integral in (4.8) can be evaluated in an elementary way for every value of M, for example, by introducing polar coordinates. We state the result for the most important case, $M=\frac{1}{2}$:

$$A^{1/2}(k)=-\tfrac{1}{2}\pi\epsilon[(k_3+k_0)z_1{}^*z_2+z_2(k_1-ik_2)+z_1{}^*(k_1+ik_2)k_0-k_3],$$

which can be rewritten in the concise form

$$A^{1/2}(k)=-\tfrac{1}{2}\pi\epsilon(z_2,1)(-k_0+\boldsymbol{\sigma}\cdot\mathbf{k})\begin{pmatrix}z_1{}^*\\ 1\end{pmatrix}.$$

The propagator can be normalized again to a δ function. The final result reads

$$\Delta^{\sigma\frac{1}{2}}(k,z_1,z_2)=\frac{-(\sigma+\frac{3}{2})}{\pi}\epsilon(z_2,1)(-k_0+\boldsymbol{\sigma}\cdot\mathbf{k})\begin{pmatrix}z_1{}^*\\ 1\end{pmatrix}\Big/(z_1-z_2)^{\sigma+5/2}(z_1{}^*-z_2{}^*)^{\sigma+5/2}.\quad(4.9)$$

The propagator of the opposite chirality state can be derived in an entirely analogous way. We simply quote the result:

$$\Delta^{\sigma-\frac{1}{2}}(k,u_1,u_2)=\frac{(\sigma+\frac{3}{2})}{\pi}\epsilon(u_2{}^*,1)(k_0+\boldsymbol{\sigma}\cdot\mathbf{k})\begin{pmatrix}u_1\\ 1\end{pmatrix}\Big/(u_1-u_2)^{\sigma+5/2}(u_1{}^*-u_2{}^*)^{\sigma+5/2}.\quad(4.10)$$

In both equations σ is the position of the Lorentz pole, $\sigma=\sigma(k^2)$.

In order to transform the propagator to the quasimomentum representation, one has to observe that (4.9) and (4.10) are not translationally invariant expressions in the z space. Therefore write

$$\Delta^{\sigma\frac{1}{2}}(k,K_1,K_2)=\frac{1}{(2\pi)^2}\int d^2z_1d^2z_2\,e^{i\,\mathrm{Re}(K_1z_1-K_2z_2)}\Delta^{\sigma\frac{1}{2}}(k,z_1,z_2).$$

Using the results of Appendix C, the propagators in the quasimomentum representation take the form

$$\Delta^{\sigma\frac{1}{2}}(k,K_1,K_2)=\frac{2^{-2\sigma-2}}{4\pi}\frac{\Gamma(-\sigma-\frac{1}{2})}{\Gamma(\sigma+\frac{5}{2})}\left(i\frac{\partial}{\partial K_2},1\right)(-k_0+\boldsymbol{\sigma}\cdot\mathbf{k})\times\begin{pmatrix}-i(\partial/\partial K_1{}^*)\\ 1\end{pmatrix}\delta^{(2)}(K_1-K_2)\left|\frac{K_1+K_2}{2}\right|^{2\sigma+3}.\quad(4.11)$$

The reader is reminded that the expression of the two-dimensional δ function entering the last equations reads

$$\delta^{(2)}(K_1-K_2)=-2i\delta(K_1-K_2)\delta(K_1{}^*-K_2{}^*).$$

5. DISCUSSION

The main purpose of this work was to set the stage for a local description of composite hadrons. Clearly, the classical example of a local, Lorentz-invariant theory is quantum field theory, so we model our construction after it. As already mentioned, the essential physical and technical complication in treating *composite* hadrons is that they lie along Regge trajectories. In this respect the basic result contained in this paper is the introduction of the z representation (and of the quasimomentum representation), which gives a compact and unified description of the infinitely many spin states contained in a "Lorentz family." The connection of virtual hadron states with physical particle states is established through the introduction of the local hadron amplitudes; this in turn allows one to study the symmetry properties of the hadron towers. It is very important that the local hadron amplitudes can support representations of $L(C)$; indeed, covariance with respect to the latter group is essential for obtaining crossing-symmetric scattering amplitudes.

In studying the transformation properties of the hadron amplitudes under P, C, and T, one realizes that the latter transform in exactly the same way as local vertices do in a field theory of elementary particles. A somewhat unexpected result is that the hadron amplitudes have definite transformation properties under gauge transformations as well; in particular, chirality is conserved in a world of parallel Regge trajectories. [This result is not so surprising if one remembers that the present model, using unbroken $L(C)$ symmetry, in a sense is an extrapolation of the symmetry of a massless world to nonvanishing masses. A massless world of elementary particles has long been known to be chiral-invariant.] Another physically interesting consequence of the study of symmetry properties is that covariance under $L(C)$ requires parity doubling (MacDowell symmetry) for mesons as well as fermions. This fact (well known experimentally), to our knowledge, had no rational explanation before.

Finally, we achieved a unified description of the propagation of Reggeized hadron towers between two space-time points where they interact. The simple expressions obtained are already sufficient to demonstrate the advantages of the z representation.

Evidently, in order to complete the scheme, one has to study the structure of local hadron amplitudes involving several virtual hadrons. This is a highly nontrivial problem, mainly because vertices with several virtual hadrons contain essentially dynamical information. The next part of this work is devoted almost entirely to this problem.

APPENDIX A: GENERALIZED SPHERICAL FUNCTIONS ON LIGHT CONE: TRANSFORMATION FROM (jm) TO z REPRESENTATION

Let $\langle\sigma M jm|$ be a basis vector of the IR $[\sigma M]$ of $SL(2,C)$ ($j=|M|, |M|+1, \ldots, -j\leq m\leq j$). We want to transform this basis vector to the representation introduced in Sec. 2. First we transform to the spinor basis $\langle\sigma M z_1z_2|$. On parametrizing the spinor (z_1,z_2) with the Lorentz transformation (2.14), we get

$$z_1=e^{\xi/2}\cos\tfrac{1}{2}\beta e^{-i(\alpha+\gamma)/2},\quad z_2=e^{\xi/2}\sin\tfrac{1}{2}\beta e^{-i(\alpha-\gamma)/2}. \tag{A1}$$

Clearly,

$$\langle\sigma M z_1z_2|=\sum_{jm}\langle\sigma M z_1z_2|\sigma M jm\rangle\langle\sigma M jm|.$$

The transformation function $\langle\sigma M z_1z_2|\sigma M jm\rangle$ is an eigenfunction of the operators $M_1^2+M_2^2+M_3^2$ and M_3; also, because of (2.10), it is a homogeneous function of the variables z_1 and z_2; hence it is proportional to $\exp\sigma\xi$.

In the variables α, β, and γ, we find:

$$-M^2=\frac{\partial^2}{\partial\beta^2}+\cot\beta\frac{\partial}{\partial\beta}+\frac{1}{\sin^2\beta}\frac{\partial^2}{2\alpha^2}+\frac{\partial^2}{2\gamma^2}-2\cos\beta\frac{\partial^2}{\partial\alpha\partial\beta};$$

hence (in a somewhat condensed notation)

$$\langle z_1z_2|\sigma M jm\rangle=Ne^{\sigma\xi}D_{Mm}{}^{j}(\alpha\beta\gamma), \tag{A2}$$

where N is a normalization constant. In order to fix the latter, we normalize (A2) on the principal series ($\sigma=-1+i\lambda$). On choosing

$$N=\left(\frac{2j+1}{8\pi^3}\right)^{1/2}$$

we find

$$\int_0^\infty d\xi\,\sinh^2\xi\int_0^{2\pi}d\alpha\int_0^{2\pi}d\gamma\int_0^{\pi}\sin\beta d\beta\langle\sigma'M'j'm'|z_1z_2\rangle \times\langle z_1z_2|\sigma M jm\rangle=\delta(\lambda-\lambda')\delta_{jj'}\delta_{MM'}\delta_{mm'}.$$

We use the definition of $D_{Mm}{}^{j}(\alpha\beta\gamma)$ as given by Edmonds.[16]

The functions (A2) can be called generalized spherical functions of $SL(2,C)$ on the light cone. Indeed, for $M=0$ the function (A2) is independent of α; it can be regarded as a function of the lightlike vector

$$v_\mu=(z_1{}^*z_2{}^*)\sigma_\mu\begin{pmatrix}z_1\\z_2\end{pmatrix}\quad(v_\mu v^\mu=0), \tag{A3}$$

with components

$$v_0=e^\xi,\quad v_3=e^\xi\cos\beta,\quad v_2=e^\xi\sin\beta\sin\gamma,\quad v_1=e^\xi\sin\beta\cos\gamma.$$

The functions (A2) form a complete orthogonal set; any square integrable function of $(\xi,\alpha,\beta,\gamma)$ (or, equivalently, of z_1, z_2) can be expanded into irreducible representations of $SL(2,C)$ as follows:

$$F(z_1z_2)=\int_{-1-i\infty}^{-1+i\infty}d\sigma\sum_{M=-\infty}^{\infty}\sum_{jm}\langle z_1z_2|\sigma M jm\rangle F_{jm}{}^{\sigma M},$$

with

$$F_{jm}{}^{\sigma M}=\int d^2z_1d^2z_2\langle\sigma M jm|z_1z_2\rangle F(z_1,z_2),$$

where we used the abbreviation

$$d^2z_1d^2z_2=\sinh^2\xi d\xi d\alpha d\gamma\,\sin\beta d\beta.$$

[16] A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, N. J., 1947).

As we pointed out in Sec. 2, if one stays within one IR of $SL(2,C)$, the single variable $z=z_1/z_2$ is sufficient to label the states.

Using (A1), we find

$$z=\cot\tfrac{1}{2}\beta\, e^{-i\gamma};$$

the transformation functions $\langle z|\sigma Mjm\rangle$ are defined by

$$\langle z_1z_2|\sigma Mjm\rangle=z_2^{\sigma+M}z_2^{*\sigma-M}\langle z|\sigma Mjm\rangle, \quad \text{(A4)}$$

which immediately leads to (2.13).

It is useful to record the expression for the differential d^2z in terms of the Euler angles. A short calculation gives

$$d^2z=4(\sin\beta d\beta d\gamma)/(1-\cos\beta)^2.$$

APPENDIX B: HADRON AMPLITUDES

Consider the (improper) state vector of a single hadron $\langle\sigma Mjm\mathbf{p}\xi|$. (The notation is explained in Sec. 3.) Let $|p_1s_1m_1\rangle$, $|p_1s_1m_1,p_2s_2m_2\rangle$, $\ldots$, be *physical* single-particle, two-particle, $\ldots$, states, with momentum p_i, spin s_i, and spin projection m_i. *Formally*, we can write

$$\langle\sigma Mjm\mathbf{p}\xi|=\sum\langle\sigma Mjm\mathbf{p}\xi|p_1s_1m_1\cdots p_ns_nm_n\rangle \times\langle p_1s_1m_1\cdots p_ns_nm_n|. \quad \text{(B1)}$$

[Indeed, expansion (B1) serves as a *definition* of the improper state $\langle\sigma Mjm\mathbf{p}\xi|$.] Here, $\mathbf{p}=\mathbf{p}_1+\mathbf{p}_2+\cdots+\mathbf{p}_n$. Clearly,

$$\begin{aligned}\langle\sigma Mjm\mathbf{p}\xi|p_1s_1m_1\cdots p_ns_nm_n\rangle &=\langle\sigma Mjm\mathbf{p}\xi|U(I(\mathbf{p}))U(I(\mathbf{p}))^{-1}\\ &\quad\times|p_1s_1m_1\cdots p_ns_nm_n\rangle, \quad \text{(B2)}\end{aligned}$$

where $L(\mathbf{p})$ is the boost (3.2).

$$U(L(\mathbf{p}))^{-1}|p_1s_1m_1\cdots p_ns_nm_n\rangle$$

is the multiparticle state in its center-of-mass (c.m.) system.

Inserting a complete set of states into the identity (B2), one has (in a somewhat condensed notation)

$$\begin{aligned}\langle\sigma Mjm\mathbf{p}\xi|p_1s_1m_1\cdots p_ns_nm_n\rangle &=\sum_{s\mu}\langle\sigma Ms\mu\mathbf{p}\xi|U(L(\mathbf{p}))|\sigma Ms\mu\rangle\\ &\quad\times\langle\sigma Ms\mu\mathbf{p}\xi|U(L(\mathbf{p}))^{-1}|p_1s_1m_1\cdots\rangle\\ &\equiv\sum_{s\mu}D_{jms\mu}{}^{\sigma M}(L(\mathbf{p}))A_{s\mu}{}^{\sigma M}(\mathbf{p},\xi). \quad \text{(B3)}\end{aligned}$$

Here $D_{jms\mu}{}^{\sigma M}(L(\mathbf{p}))$ is the matrix element of the Lorentz transformation in the IR $[\sigma M]$; $s\mu$ are the total angular momentum and its projection in the c. m. system. Equation (B3) serves as a definition of the reduced amplitudes $A_{s\mu}{}^{\sigma M}(\mathbf{p},\xi)$.

In order to derive relations of the type (3.11), one notices that, for example, under space inversion,

$$P|\sigma Mjm\rangle=(-)^{j-M}\eta_P|\sigma-Mjm\rangle. \quad \text{(B4)}$$

[This can be obtained by observing that under P, the Euler angles (A1) transform as $\alpha\to\pi-\alpha$, $\beta\to\pi-\beta$, $\gamma\to\pi+\gamma$, $\xi\to\xi$ and by use of (A2).]

Also,

$$P|p_1s_1m_1\cdots p_ns_nm_n\rangle =\prod_{i=1}^{n}\eta_{iP}|-p_1s_1m_1\cdots -p_ns_nm_n\rangle. \quad \text{(B5)}$$

In these equations, η_P and η_{iP} are intrinsic parities.

Further, using (B4),

$$\begin{aligned}D_{jms\mu}{}^{\sigma M}(L(-\mathbf{p}))&=\langle\sigma Mjm|U(L(-\mathbf{p}))|\sigma Ms\mu\rangle\\ &=\langle\sigma Mjm|P^{-1}U(L(\mathbf{p}))P|\sigma Ms\mu\rangle\\ &=(-)^{j+s}D_{jms\mu}{}^{\sigma-M}(L(\mathbf{p})).\end{aligned}$$

Combining these results with (B1), one gets

$$A_{s\mu}{}^{\sigma-M}(-\mathbf{p}\xi^P)=H_PA_{s\mu}{}^{\sigma M}(\mathbf{p},\xi)(-)^{s+M}, \quad \text{(B6)}$$

where $H_P=\eta_P^*\prod_i\eta_{iP}$; ξ^P is the collection of quantum numbers characterizing the space-reflected state. Since an amplitude necessarily transforms according to a single-valued representation of the Lorentz group, H_P is real. Hence $H_P=\pm1$; also, $(\xi^P)^P=\xi$. The symmetry properties of the reduced amplitudes under C and T are derived in the same way.

One finds in addition to (B6)

$$\begin{aligned}B_{s\mu}{}^{\sigma-M}(-\mathbf{p},\,\xi^P)&=\bar{H}_PB_{s\mu}{}^{\sigma M}(\mathbf{p},\xi)(-)^{s+M},\\ A_{s\mu}{}^{\sigma M}(-\mathbf{p},\,\xi^T)&=H_TA_{s-\mu}{}^{\sigma-M}(\mathbf{p},\xi)(-)^{s+\mu},\\ B_{s\mu}{}^{\sigma M}(-\mathbf{p},\,\xi^T)&=\bar{H}_TB_{s-\mu}{}^{\sigma-M}(\mathbf{p},\xi)(-)^{s+\mu},\\ A_{s\mu}{}^{\sigma M}(\mathbf{p},\,\xi^C)&=H_CB_{s\mu}{}^{\sigma M}(\mathbf{p},\xi),\\ B_{s\mu}{}^{\sigma M}(\mathbf{p},\xi^C)&=\bar{H}_CA_{s\mu}{}^{\sigma M}(p,\xi).\end{aligned}$$

The reflection properties of the local amplitudes listed in Sec. 3 require

$$\begin{aligned}\bar{H}_P&=(-)^{2M}H_P,\\ \bar{H}_C&=H_C,\\ \bar{H}_T&=H_T.\end{aligned}$$

APPENDIX C: FOURIER TRANSFORM OF HOMOGENEOUS FUNCTIONS

We calculate the Fourier transform

$$F^{(j_1j_2)}(P)=\frac{1}{2\pi}\int d^2z\,e^{-i\,\mathrm{Re}Pz}z^{2j_1}z^{*2j_2} \quad [2(j_1-j_2)=\text{integer}]. \quad \text{(C1)}$$

We introduce polar coordinates

$$P=pe^{i\psi},\quad z=re^{i\varphi},\quad d^2z=rdrd\varphi\quad(0\leq r<\infty,\,0\leq\varphi\leq2\pi).$$

With this, (C1) becomes

$$F^{(j_1j_2)}(P)=\frac{e^{-2i\psi(j_1-j_2)}}{2\pi}\int_0^\infty rdr\, r^{2(j_1+j_2)}$$

$$\times\int_0^{2\pi} d\chi\, e^{-ipr\cos\chi}e^{i\chi 2(j_1-j_2)}\,,$$

where $\chi=\varphi+\psi$. Apart from some coefficients, the integral over χ is a standard integral representation of the Bessel function $J_{2|j_1-j_2|}(pr)$; thus

$$F^{(j_1j_2)}(P)=e^{-2i\psi(j_1-j_2)}e^{-i\pi(j_1-j_2)}$$

$$\times\int_0^\infty rdr\, r^{2(j_1+j_2)}J_{2|j_1-j_2|}(pr)\,,$$

so the radial integral is a Hankel transform of integer index. [Compare Eq. (14) in Sec. 6.561 of Gradshtein and Rishik.[17]]

Finally, we obtain

$$F^{(j_1j_2)}(P)=2^{2(j_1+j_2)+1}e^{-i\pi(j_1-j_2)}e^{-2i\psi(j_1-j_2)}p^{-2(j_1+j_2)-2}$$

$$\times\frac{\Gamma(j_1+j_2+|j_1-j_2|)}{\Gamma(-j_1-j_2+|j_3-j_2|)}$$

$$=2^{2(j_1+j_2)+1}e^{-i\pi(j_1-j_2)}\frac{\Gamma(j_1+j_2+1+|j_1-j_2|)}{\Gamma(-j_1-j_2+|j_1-j_2|)}$$

$$\times P^{-j_1-j_2-|j_1-j_2|-1}P^{*-j_1-j_2+|j_1-j_2|-1}. \qquad \text{(C2)}$$

APPENDIX D: CLEBSCH-GORDAN COEFFICIENTS OF $SL(2,C)$

We define the symmetric Clebsch-Gordan coefficients (CGC) of $SL(2,C)$ in the z representation by the equation

$$\int d^2z_1d^2z_2d^2z_3\; \varphi^{[\sigma_1M_1]}(z_1)\varphi^{[\sigma_2M_2]}(z_2)\varphi^{[\sigma_3M_3]}(z_3)$$

$$\times\begin{pmatrix}\sigma_1M_1 & \sigma_2M_2 & \sigma_3M_3\\ z_1 & z_2 & z_3\end{pmatrix}=\text{invariant}. \qquad \text{(D1)}$$

It is sufficient to check the invariance of (D1) with respect to the three basic transformations α, a, and r introduced in Sec. 2. The functions $\varphi^{[\sigma_iM_i]}(z_i)$ transform according to (2.12); otherwise they are arbitrary.

[17] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic Press Inc., New York, 1965).

The "translation" a gives

$$\begin{pmatrix}\sigma_1M_1 & \sigma_2M_2 & \sigma_3M_3\\ z_1 & z_2 & z_3\end{pmatrix}=\begin{pmatrix}\sigma_1M_1 & \sigma_2M_2 & \sigma_3M_3\\ z_1+a & z_2+a & z_3+a\end{pmatrix},$$

so that the CGC depends on z_i-z_j (i, $j=1$, 2, 3) only.

The "dilatation" α gives

$$\begin{pmatrix}\sigma_1M_1 & \sigma_2M_2 & \sigma_3M_3\\ z_1 & z_2 & z_3\end{pmatrix}=\alpha^{-2(\sigma_1+M_1)-2}\cdots\alpha^{*-2(\sigma_3-M_3)-2}$$

$$\times\begin{pmatrix}\sigma_1M_1 & \sigma_2M_2 & \sigma_3M_3\\ \alpha^{-2}z_1 & \alpha^{-2}z_2 & \alpha^{-3}z_3\end{pmatrix}. \qquad \text{(D2)}$$

Thus we try the ansatz

$$\begin{pmatrix}\sigma_1M_1 & \sigma_2M_2 & \sigma_3M_3\\ z_1 & z_2 & z_3\end{pmatrix}$$

$$=(z_1-z_2)^{-\alpha_1}(z_2-z_3)^{-\alpha_2}(z_3-z_1)^{-\alpha_3}$$

$$\times(z_1^*-z_2^*)^{-\beta_1}(z_2^*-z_3^*)^{-\beta_2}(z_3^*-z_1^*)^{-\beta_3}. \qquad \text{(D3)}$$

Equation (D2) restricts the powers α_i, β_i by

$$\begin{aligned}\alpha_1+\alpha_2+\alpha_3&=\sigma_1+\sigma_2+\sigma_3+M_1+M_2+M_3+3\,,\\ \beta_1+\beta_2+\beta_3&=\sigma_1+\sigma_2+\sigma_3-M_1-M_2-M_3+3\,.\end{aligned} \qquad \text{(D4)}$$

Finally, r (rotation by π around the y axis) gives the condition

$$\begin{pmatrix}\sigma_1M_1 & \sigma_2M_2 & \sigma_3M_3\\ z_1 & z_2 & z_3\end{pmatrix}=z_1^{-2(\sigma_1+M_1)-2}\cdots z_3^{*-2(\sigma_3-M_3)-2}$$

$$\times\begin{pmatrix}\sigma_1M_1 & \sigma_2M_2 & \sigma_3M_3\\ -1/z_1 & -1/z_2 & -1/z_3\end{pmatrix}.$$

Inserting the ansatz (D3), we get six independent linear equations for α_i, β_i, (D4) being automatically satisfied.

The solution reads

$$\begin{aligned}\alpha_1&=\tfrac{1}{2}(\sigma_1+\sigma_2-\sigma_3+M_1+M_2-M_3+2)\,,\\ \beta_1&=\tfrac{1}{2}(\sigma_1+\sigma_2-\sigma_3-M_1-M_2+M_3+2)\,,\\ \alpha_2&=\tfrac{1}{2}(\sigma_2+\sigma_3-\sigma_1+M_2+M_3-M_1+2)\,,\\ \beta_2&=\tfrac{1}{2}(\sigma_2+\sigma_3-\sigma_1+M_1-M_3-M_2+2)\,,\\ \alpha_3&=\tfrac{1}{2}(\sigma_1+\sigma_3-\sigma_2+M_1+M_3-M_2+2)\,,\\ \beta_3&=\tfrac{1}{2}(\sigma_1+\sigma_3-\sigma_2-M_1-M_3+M_2+2)\,.\end{aligned}$$

A numerical factor in front of (D3) clearly remains arbitrary. We set it equal to unity.