Theory of the Propagation of Nonstable Waves*

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The general theory of the propagation of pulse disturbances in an unstable, slightly inhomogeneous, and time-dependent medium is derived in complementary fashion to the eikonal theory of the propagation of stable waves. For a dispersion $D(\mathbf{k},\omega;\mathbf{a})$, where $\mathbf{k}$ is the wave number and ω is the (complex) frequency which is related to $\mathbf{k}$ for given values of the auxiliary parameters $\mathbf{a}$ (temperature, density, etc.) by the vanishing of D, the rate of change of propagation velocity of an element of the disturbance is found to be $dV/d\mathbf{a}=\mathrm{Re}(-D_{\mathbf{a}}/kD_\omega)$ in one dimension. A similar equation holds in the case of a general n-dimensional system. This relation makes possible the construction of the complete pulse shape in fairly simple fashion for the case of a general slowly varying medium. The theory is illustrated by application to a particular unstable dispersion.

I. INTRODUCTION

THE description of the response of a general linear medium to an arbitrary disturbance is a fundamental problem whose applications have been of interest to the theoretical physicist of every specialization. For example, almost the entire literature of continuum electrodynamics can be considered such an application, stemming from the solution of a particular integral equation [Eq. (1.10) below] for the most general local, or average nonlocal, linear response. The fact that far-reaching conclusions can be drawn from simple statements regarding general features of the response function, and that the nature of these conclusions can be rendered quantitative as well as qualitative, is perhaps as remarkable as it is well known.

Of particular importance is the description of the propagation of *pulse* disturbances in an arbitrary linear system. By this, we mean the behavior of the response of the system to an initial disturbance which is localized within some finite spatial domain. Normally, as time goes on any such disturbance will move and tend to spread throughout the system. Now, a general description in terms of such pulse disturbances is in fact a complete description, for an arbitrary disturbance can be considered composed of these elementary pulses. A principal utility of this viewpoint, however, lies in the approximations that such a representation allows, in particular with regard to the application of the theory to inhomogeneous or time-dependent systems. For if the medium varies sufficiently slowly in time and space, the pulse can be considered to propagate for each small increment of time Δt as if in a uniform time-independent medium whose properties are just those of the true system at the position $\mathbf{r}$ which is the location of the pulse at time t. Clearly, such a possibility is the basis of the eikonal theory of classical optics,[1] as well as fundamental to the connection between classical and quantum mechanics (via WKB theory).[2] Throughout this paper we shall always consider that the possibility of making such an approximation exists.

A second approximation, or really a second pair of approximations, is also often applicable and indeed is central to the eikonal theory. We shall refer to these approximations collectively as the *eikonal assumption*, which then requires for its validity that (a) the medium be dispersive and (b) that the system be only weakly unstable or weakly damped, with growth or damping essentially independent of wavelength. More specifically, the eikonal theory is fundamentally concerned with the propagation of *phase relationships* among waves of comparable amplitude. Waves of different frequencies will also have corresponding different spatial wavelengths. A wave packet made up of waves whose phases are such that they add constructively in a given region of space will add incoherently elsewhere, and so describe a localized pulse.[3]

On the other hand, it may happen that assumption (b) above is violated, and that the growth or damping rate varies appreciably with the wave number. Then, with the amplitudes of the different waves changing in time at different rates, phase relationships no longer are important. Even if the waves could cancel one another initially over most of space, at a later time only that wave which has grown most rapidly (or decayed the least) will govern the response at a given position.

In order to distinguish between the two situations just described, we will denote as "stable" a collection of waves for which the *variation* of the growth rate (damping rate) is negligible. In contrast, a collection of waves in which the dispersion in the *growth* (or *damping*) is appreciable will be called "nonstable."[4] The theory of propagation of nonstable waves is thus seen to be complementary to the theory of stable waves (eikonal theory). However, while the latter has received great attention in the literature, it is only recently that it has

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[1] Max Born and Emil Wolf, *Principles of Optics* (Pergamon Press, Inc., New York, 1965), 3rd ed., Chap. III.

[2] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Co., New York, 1968), 3rd ed.

[3] Leon Brillouin, *Wave Propagation and Group Velocity* (Academic Press Inc., New York, 1960).

[4] Strictly speaking, we should also make allowance for the case in which neither the real frequency nor the growth (damping) rate depends upon the wave number. But the solution of this problem is trivial and there is no real loss in considering it to be external to the theory of stable waves (the eikonal problem) as well as to the theory of nonstable waves.

become important to understand the theory of nonstable pulses. It is hoped that the present article will provide a useful beginning to the development of a complete theory.

In brief outline we will first establish, in Sec. II, a Huygens principle for the response of a uniform time-independent medium. The form of the result is derived to be both convenient and suggestive for the later extension (in Sec. III) to time-dependent and, subsequently, spatially inhomogeneous media. We then go on in Sec. IV to the most important quantitative result of our theory, the derivation of the equation of propagation, Eq. (4.9),

$$d\mathbf{V}/d\mathbf{a}=\mathrm{Re}(-\mathbf{k}^{-1}D_{\mathbf{a}}/D_{\omega}).$$

This relation yields the rate of change of propagation velocity of an element of a pulse disturbance in an inhomogeneous or time-dependent medium described by the local dispersion $D(\mathbf{k},\omega;\,\mathbf{a})=0$. Here $\mathbf{k}$ and ω are the (complex) wave number and frequency, and $\mathbf{a}$ represents the collection of auxiliary macroscopic parameters, such as temperature or density, whose slow variation expresses the inhomogeneity or time dependence of the system. Our results are then applied, in Sec. V, to a particular example, and in Sec. VI we attempt to pull together a philosophical overview of the theory. The paper concludes, in Sec. VII, with some general statements on the over-all behavior of the results.

Finally, although the theory will have application beyond continuum electrodynamics—specifically, whenever a linear response function can be defined as in Eq. (1.10)—it is perhaps useful to show, briefly, how Eq. (1.10) arises for this important case. We may take the vector Maxwell's equations for the continuum to be

$$\nabla\times\mathbf{B}-\frac{1}{c}\frac{\partial\mathbf{D}}{\partial t}=\frac{4\pi}{c}(\mathbf{j}_C+\mathbf{j}_M+\mathbf{j}_S)\,,\quad \nabla\times\boldsymbol{\mathcal{E}}+\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}=0\,,\quad (1.1)$$

where we have explicitly separated out the conduction current $\mathbf{j}_C$, the magnetization $\mathbf{j}_M$ and a distributed source term $\mathbf{j}_S$. Combining the two Eqs. (1.1) yields the wave equation

$$\nabla\times(\nabla\times\boldsymbol{\mathcal{E}})+\frac{1}{c^2}\frac{\partial^2\mathbf{D}}{\partial t^2}=-\frac{4\pi}{c^2}\left(\frac{\partial\mathbf{j}_C}{\partial t}+\frac{\partial\mathbf{j}_M}{\partial t}+\frac{\partial\mathbf{j}_S}{\partial t}\right),\quad (1.2)$$

which we may now Fourier-analyze in space and Laplace-analyze in time. This corresponds to our stated intention of specifying an initial disturbance of finite spatial extent. Since no signal can propagate with velocity greater than c, such a disturbance will always remain of bounded extent for finite times so that the Fourier transform always exists. Let $\mathbf{k}$ be the spatial wave vector and $-i\omega$ the Laplace parameter with $\mathrm{Im}(\omega)$ sufficiently great to secure convergence, all this corresponding to an elementary resolution of the form $\exp(i\mathbf{k}\cdot\mathbf{r}-i\omega t)$, and let us define the transforms

$$\bar{\psi}(\mathbf{k},t)\equiv\int d\mathbf{r}\;e^{-i\mathbf{k}\cdot\mathbf{r}}\psi(\mathbf{r},t)\,,$$
$$\hat{\psi}(\mathbf{k},\omega)\equiv\int_0^\infty dt\;e^{i\omega t}\bar{\psi}(\mathbf{k},t)\,.\quad (1.3)$$

Then (1.2) becomes

$$-\mathbf{k}\times(\mathbf{k}\times\hat{\mathcal{E}})-\frac{\omega^2}{c^2}\hat{D}+i\omega\bar{D}(\mathbf{k},0)-\bar{D}_t(\mathbf{k},0)=\frac{4\pi i\omega}{c^2}(\hat{\jmath}_C+\hat{\jmath}_M+\hat{\jmath}_S)+\frac{4\pi}{c^2}\bar{\jmath}(0)\,.$$

We then assume that

$$\hat{D}=\epsilon(\mathbf{k},\omega)\cdot\hat{\mathcal{E}}\,,$$
$$\hat{\jmath}_C=\sigma(\mathbf{k},\omega)\cdot\hat{\mathcal{E}}\,,\quad (1.4)$$
$$\hat{\jmath}_M=-\frac{c^2}{i\omega}[\mathbf{k}\times\chi(\mathbf{k},\omega)\times\mathbf{k}]\cdot\hat{\mathcal{E}}\,,$$

corresponding to the most general possible linear development of the constitutive quantities $\hat{D}$, $\hat{\jmath}_C$, and $\hat{\jmath}_M$ in terms of $\hat{\mathcal{E}}$ and $\hat{B}$ [i.e., $(c/\omega)\mathbf{k}\times\boldsymbol{\mathcal{E}}$]. For real values of the frequency ω and wave number k, the dielectric tensor ϵ, the electrical conductivity σ, and the susceptibility χ are each to be considered Hermitian, which serves to determine σ uniquely for any given medium. Conversely, the division of the properties into a susceptibility and a dielectric tensor is somewhat arbitrary, and we may take $\chi=0$ with no loss of generality by simply combining these terms. Doing this, the transformed wave equation takes the final form

$$[\epsilon+(4\pi i/\omega)\sigma+(c^2/\omega^2)(\mathbf{kk}-k^2\hat{\mathbf{I}})]\cdot\hat{\mathcal{E}}=\mathbf{S}'(\mathbf{k},\omega)\,,\quad (1.5)$$

where $\mathbf{S}'(\mathbf{k},\omega)$ represents the combined effects of the initial value terms and the sources, and $\hat{\mathbf{I}}$ is the unit matrix. Equation (1.5) can then be inverted, and if

$$\Delta(\mathbf{k},\omega)\equiv\det[\epsilon+(4\pi i/\omega)\sigma+(c^2/k^2)(\mathbf{kk}-k^2\hat{\mathbf{I}})]\,,\quad (1.6)$$

then

$$\boldsymbol{\mathcal{E}}(\mathbf{r},t)=\left(\frac{1}{2\pi}\right)^4\int d\mathbf{k}\int_{-\infty+ip}^{\infty+ip}d\omega\;e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\frac{\mathbf{S}(\mathbf{k},\omega)}{\Delta(\mathbf{k},\omega)}\,,\quad (1.7)$$

where we have combined in $\mathbf{S}(\mathbf{k},\omega)$ the cofactors of the matrix elements of the original equation with the source term $\mathbf{S}'$.

Now, so far we have been reasonably general in our analysis, and have only assumed that a linear relation exists connecting the constitutive quantities and the electric field $\boldsymbol{\mathcal{E}}$, e.g.,

$$\mathbf{D}(\mathbf{r},t)=\int d\mathbf{r}'\int_0^\infty dt'\epsilon^0(\mathbf{r}-\mathbf{r}',\,t-t')\cdot\boldsymbol{\mathcal{E}}(\mathbf{r}',t')\,,\quad (1.8)$$

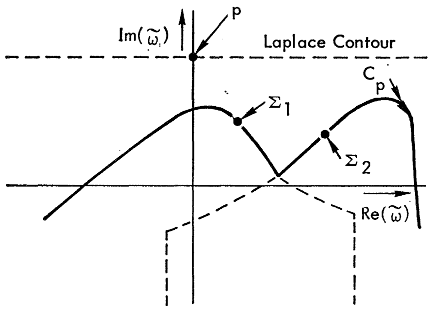

FIG. 1. Variation as a function of $\mathbf{V}$ of the singularities of $F(\tilde{\omega};\mathbf{V};\mathbf{a})$ along the proper contour C_p, for the case of two branches of C_p.

where

$$\epsilon^0(\mathbf{r},t)=\left(\frac{1}{2\pi}\right)^4\int d\mathbf{k}\int_{-\infty+ip}^{\infty+ip}d\omega\; e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\epsilon(\mathbf{k},\omega). \tag{1.9}$$

Strictly speaking, however, we have assumed homogeneity of ϵ and σ in space and time, as is clear from Eq. (1.8) and the dependence there only on the differences $(\mathbf{r}-\mathbf{r}')$ and $(t-t')$. This restriction may be relaxed somewhat to allow slow temporal and spatial dependences. In this case, Eq. (1.7) remains unchanged to lowest order except that $\mathbf{S}(\mathbf{k},\omega)$ and $\Delta(\mathbf{k},\omega)$ are allowed to depend slightly on $(\mathbf{r},t)$. Thus Eq. (1.7) is a special case of the general (local or average nonlocal) linear response equation[5]

$$\psi(\mathbf{r},t)=\int d\mathbf{k}\int_{-\infty+ip}^{\infty+ip}d\omega\; e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\frac{S(\mathbf{k},\omega;\,\mathbf{r},t)}{D(\mathbf{k},\omega;\,\mathbf{a})}. \tag{1.10}$$

Here $\mathbf{a}=\mathbf{a}(\mathbf{r},t)$ is a collection of macroscopic parameters assumed to depend only very slowly upon t and $\mathbf{r}$ (the n-dimensional "position" vector). The meaning of the time and space dependence appearing explicitly in S will require appropriate discussion, since S can no longer be looked upon as the mere specification of initial data. For example, one can show, as in the Appendix, that for an ordinary medium of slowly varying refractive index, S is such that Eq. (1.10) reduces to the WKB result: $\psi\sim\exp(i\int\mathbf{k}\cdot d\mathbf{r}-i\omega t)$. The analogous result for non-stable waves is obtained in Secs. II and III. For the present, all we need recognize is that some auxiliary space-time dependence (which we choose to incorporate in S) can be expected to occur when $\mathbf{a}$ is allowed to vary.

II. HUYGENS' PRINCIPLE

Before attempting to analyze the general case, let us first examine Eq. (1.10) in the limit of a homogeneous, time-independent dispersion ($\mathbf{a}$=const). Then if we introduce for times $t>0$ the normalized position $\mathbf{V}$ and associated Doppler-shifted frequency $\tilde{\omega}$,

$$\mathbf{r}=\mathbf{V}t,\quad \tilde{\omega}=\omega-\mathbf{k}\cdot\mathbf{V}, \tag{2.1}$$

the response becomes

$$\psi(t)=\int_{-\infty+ip}^{\infty+ip}d\tilde{\omega}\,\exp(-i\tilde{\omega}t)F(\tilde{\omega};\,\mathbf{V};\,\mathbf{a}), \tag{2.2}$$

where

$$F(\tilde{\omega};\,\mathbf{V};\,\mathbf{a})=\lim\int d\mathbf{k}\times\exp(i\mathbf{k}\cdot\boldsymbol{\varrho})\frac{S(\mathbf{k},\,\tilde{\omega}+\mathbf{k}\cdot\mathbf{V})}{D(\mathbf{k},\,\tilde{\omega}+\mathbf{k}\cdot\mathbf{V};\,\mathbf{a})}\quad\text{as}\quad\boldsymbol{\varrho}\rightarrow 0. \tag{2.3}$$

By the theory of Fourier-Laplace transforms, F is a well-defined analytic function of $\tilde{\omega}$ on and above the Laplace contour [$\mathrm{Im}(\tilde{\omega})=p$]. Furthermore, we may analytically continue F below the Laplace contour, and to make its definition in this region precise, we choose to let all necessary branch cuts run vertically downward from the associated branch point to $-i\infty$.

Now asymptotically as a function of time, the greatest contribution to the response will occur at the uppermost singularity of F in the ω plane. That is, if $\gamma=\mathrm{Im}(\tilde{\omega}^0)$, where $\tilde{\omega}^0$ is this uppermost singularity, then for sufficiently large times the amplitude of the response is proportional to $e^{\gamma t}$.

In general, the position Σ of a singularity of F in the $\tilde{\omega}$ plane will depend upon both $\mathbf{V}$ and $\mathbf{a}$,

$$\Sigma=\Sigma(\mathbf{V},\mathbf{a}). \tag{2.4}$$

For fixed $\mathbf{a}$, the behavior might be as diagrammed in Fig. 1. Here we have explicitly indicated the possibility of more than one branch of Eq. (2.4), and we have also indicated the possibility that a given branch may abruptly terminate, e.g., as Σ passes through the branch cut of a more rapidly growing mode.[6]

Let us now assume that no external sources are present after $t=0$, so that S is independent of ω. (There is no real loss of generality here, a superposition of ψ's for different elapsed times t would re-allow anything that is lost hereby.) Then, if we denote by $S(\mathbf{k})$ the Fourier transform of the initial response (which we take to be given), we have

$$F(\tilde{\omega};\,\mathbf{V};\,\mathbf{a})=\lim\int d\mathbf{k}\,\frac{\exp(i\mathbf{k}\cdot\boldsymbol{\varrho})}{\mathfrak{D}(\mathbf{k},\,\tilde{\omega}+\mathbf{k}\cdot\mathbf{V};\,\mathbf{a})}\quad\text{as}\quad\boldsymbol{\varrho}\rightarrow 0, \tag{2.5}$$

where we now introduce the normalized dispersion

$$\mathfrak{D}(\mathbf{k},\,\tilde{\omega}+\mathbf{k}\cdot\mathbf{V};\,\mathbf{a})\equiv D(\mathbf{k},\,\tilde{\omega}+\mathbf{k}\cdot\mathbf{V};\,\mathbf{a})\times\lim_{t\to 0}\int_{-\infty+ip}^{\infty+ip}d\tilde{\omega}\,\exp(-i\tilde{\omega}t)/D(\mathbf{k},\,\tilde{\omega}+\mathbf{k}\cdot\mathbf{V};\,\mathbf{a}), \tag{2.6}$$

which is defined as in Eq. (2.6) for $\mathbf{k}$ real, and as its analytic continuation if it be needed elsewhere.

It is of importance to our later analysis that the response, Eqs. (2.2) and (2.5), satisfies a form of Huygens'

[5] The homogeneous, time-independent form of Eq. (1.10) has been analyzed by L. S. Hall and W. Heckrotte, Phys. Rev. **166**, 120 (1968), which analysis is basic to the present paper. For additional work, see also the references cited therein.

[6] See Hall and Heckrotte, Ref. 5. For such an occurrence to happen as diagrammed, the variable $\mathbf{V}$ at the position at which the lower lying singularity disappears would have to match up in value with $\mathbf{V}$ at the point on the other curve directly above.

principle. Let $\{\tilde{\omega}_\alpha\}$ be the set of zeros of $\mathfrak{D}$:

$$\mathfrak{D}(\mathbf{k},\,\tilde{\omega}_\alpha+\mathbf{k}\cdot\mathbf{V};\,\mathbf{a})=0\,,\quad \tilde{\omega}_\alpha=\tilde{\omega}_\alpha(\mathbf{k};\,\mathbf{V};\,\mathbf{a})\,, \tag{2.7}$$

and number the zeros for any $\mathbf{k}$ according to their altitude in the $\tilde{\omega}$ plane, the uppermost being $\tilde{\omega}_0$. We then introduce the renormalization

$$\mathfrak{D}^0(\mathbf{k},\,\tilde{\omega}+\mathbf{k}\cdot\mathbf{V};\,\mathbf{a})=-2\pi i\frac{\mathfrak{D}(\mathbf{k},\,\tilde{\omega}+\mathbf{k}\cdot\mathbf{V};\,\mathbf{a})}{\mathfrak{D}_\omega(\mathbf{k},\,\tilde{\omega}_0+\mathbf{k}\cdot\mathbf{V};\,\mathbf{a})}\,. \tag{2.8}$$

For dispersion with but a single mode $\tilde{\omega}_0$, $\mathfrak{D}^0=\mathfrak{D}$ by Eq. (2.6). Let us consider this special case first. Then define

$$\psi(t-t',\,t')\equiv\int d\mathbf{k}\;e^{i\mathbf{k}\cdot 0}\int_{-\infty+ip}^{\infty+ip}d\tilde{\omega}\times\frac{\exp[-i\tilde{\omega}(t-t')]}{\mathfrak{D}^0(\mathbf{k},\,\tilde{\omega}+\mathbf{k}\cdot\mathbf{V};\,\mathbf{a})}\bar{\psi}(\mathbf{k},t')\,, \tag{2.9}$$

where $e^{i\mathbf{k}\cdot 0}$ signifies the limiting process of Eq. (2.3), and where

$$\bar{\psi}(\mathbf{k},t')\equiv\int_{-\infty+ip}^{\infty+ip}d\tilde{\omega}\,\frac{\exp(-i\tilde{\omega}t')S(\mathbf{k})}{\mathfrak{D}^0(\mathbf{k},\,\tilde{\omega}+\mathbf{k}\cdot\mathbf{V};\,\mathbf{a})} \tag{2.10}$$

is the Fourier transform of the response at time t'.

Huygens' principle then says that

$$\psi(t)=\psi(t-t',\,t')\,; \tag{2.11}$$

for the present case, the result is exact, as may be easily proved by immediately performing the $\tilde{\omega}$ integrals. A corollary of this result is obtained by averaging the bitemporal response over t', i.e.,

$$\psi(t)=\langle\psi\rangle\equiv\frac{1}{t}\int_0^t dt'\psi(t-t',\,t')\,. \tag{2.12}$$

An exact form of Huygens' principle can also be written for the case of an arbitrary number of modes $\tilde{\omega}_\alpha$, but in this event we would be required to break up the ω integrals into separate parts which pick up each zero $\tilde{\omega}_\alpha$ individually, whereas we wish to keep to the original Laplace contour at this stage of the problem. Nevertheless, since the branch points of $\tilde{\omega}(\mathbf{k})$ have no physical meaning in themselves,[7] one should expect to be able to write an approximate form of Huygens' principle using the definition (2.9) for the general case. Indeed, when t may be taken sufficiently great, Eq. (2.12) remains satisfied and provides a useful approximate principle.

To establish the validity of (2.12) in the general case, we evaluate the integrals over $\tilde{\omega}$ immediately by contour integration:

$$\begin{aligned}
\frac{1}{t}\int_0^t dt'\int_{-\infty+ip}^{\infty+ip}d\tilde{\omega}'\int_{-\infty+ip}^{\infty+ip}d\tilde{\omega}''&\frac{\exp[-i\tilde{\omega}'(t-t')-i\tilde{\omega}''t']}{\mathfrak{D}^0(\mathbf{k},\,\tilde{\omega}'+\mathbf{k}\cdot\mathbf{V};\,\mathbf{a})\mathfrak{D}^0(\mathbf{k},\,\tilde{\omega}''+\mathbf{k}\cdot\mathbf{V};\,\mathbf{a})}\\
&=\sum_\alpha\sum_\beta\left(\frac{\mathfrak{D}_{\omega 0}}{\mathfrak{D}_{\omega\alpha}}\right)\left(\frac{\mathfrak{D}_{\omega 0}}{\mathfrak{D}_{\omega\beta}}\right)\exp(-i\tilde{\omega}_\alpha t)\left(\frac{\exp[i(\tilde{\omega}_\alpha-\tilde{\omega}_\beta)t]-1}{i(\tilde{\omega}_\alpha-\tilde{\omega}_\beta)t}\right)\\
&=\sum_\alpha\sum_\beta\left(\frac{\mathfrak{D}_{\omega 0}}{\mathfrak{D}_{\omega\alpha}}\right)\left(\frac{\mathfrak{D}_{\omega 0}}{\mathfrak{D}_{\omega\beta}}\right)\exp[-\tfrac{1}{2}i(\tilde{\omega}_\alpha+\tilde{\omega}_\beta)t]\frac{\sin[\frac{1}{2}(\tilde{\omega}_\alpha-\tilde{\omega}_\beta)t]}{\frac{1}{2}(\tilde{\omega}_\alpha-\tilde{\omega}_\beta)t}\\
&=\sum_\alpha\left(\frac{\mathfrak{D}_{\omega 0}}{\mathfrak{D}_{\omega\alpha}}\right)^2\exp(-i\tilde{\omega}_\alpha t)\left[1+\sum_{\beta>\alpha}\left(\frac{\mathfrak{D}_{\omega\alpha}}{\mathfrak{D}_{\omega\beta}}\right)\exp[\tfrac{1}{2}i(\tilde{\omega}_\alpha-\tilde{\omega}_\beta)t]\frac{\sin[\frac{1}{2}(\tilde{\omega}_\alpha-\tilde{\omega}_\beta)t]}{\frac{1}{4}(\tilde{\omega}_\alpha-\tilde{\omega}_\beta)t}\right].
\end{aligned}$$

By the ordering of the zeros, $\mathrm{Im}(\tilde{\omega}_\alpha-\tilde{\omega}_\beta)>0$ for $\beta>\alpha$, so that the term in the second sum vanishes as $t\to\infty$. Similarly, in the sum over α, all subsequent terms are exponentially small in comparison to the first, whence

$$\langle\psi\rangle=\int d\mathbf{k}\;e^{i\mathbf{k}\cdot 0}S(\mathbf{k})\,\exp[-i\tilde{\omega}_0(\mathbf{k})t]\,. \tag{2.13}$$

Similarly, ψ may be evaluated from Eqs. (2.2) and (2.5), and found to be equal within exponentially small order to the leading term of $\langle\psi\rangle$. Q.E.D.

Our Huygens' principle may then be reapplied continuously over successive intervals Δt, for sufficiently great Δt, in which case we have

$$\psi(t+\Delta t)=\int d\mathbf{k}\;e^{i\mathbf{k}\cdot 0}\frac{1}{\Delta t}\int_0^{\Delta t}dt'\int_{-\infty+ip}^{\infty+ip}d\tilde{\omega}\times\frac{\exp[-i\tilde{\omega}(\Delta t-t')]\bar{\psi}(\mathbf{k},\,t+t')}{\mathfrak{D}^0(\mathbf{k},\,\tilde{\omega}+\mathbf{k}\cdot\mathbf{V};\,\mathbf{a})}\,, \tag{2.14}$$

$$\bar{\psi}(\mathbf{k},\,t+t')=\int_{-\infty+ip}^{\infty+ip}d\tilde{\omega}\frac{\exp(-i\tilde{\omega}t')\bar{\psi}(\mathbf{k},t)}{\mathfrak{D}^0(\mathbf{k},\,\tilde{\omega}+\mathbf{k}\cdot\mathbf{V};\,\mathbf{a})}\,,$$

[7] Cf. K. B. Dysthe, Nucl. Fusion **6**, 215 (1966), or Hall and Heckrotte, Ref. 5. The reason for desiring to keep to the original Laplace contour is to enable us to more freely deform the integration contour, or to exchange orders of integration indiscriminately, at various points in our subsequent analysis (Sec. III).

where $\bar{\psi}(\mathbf{k},t)$ is the Fourier transform of the response at time t, so that

$$\psi(t)=\int d\mathbf{k}\; e^{i\mathbf{k}\cdot 0}\bar{\psi}(\mathbf{k},t).$$

The response can also be rewritten in a form analogous to Eq. (2.2) by an exchange of the orders of integration, viz.,

$$\psi(t+\Delta t)=\frac{1}{\Delta t}\int_0^{\Delta t}d\tau\int_{-\infty+ip}^{\infty+ip}d\tilde{\omega}'\int_{-\infty+ip}^{\infty+ip}d\tilde{\omega}''\times\exp\{-i[\tilde{\omega}'\tau+\tilde{\omega}''(\Delta t-\tau)]\}F(\tilde{\omega}',\tilde{\omega}'';\mathbf{V};\mathbf{a};t),\quad(2.15)$$

where

$$F(\tilde{\omega}',\tilde{\omega}'';\mathbf{V};\mathbf{a};t)\equiv\int d\mathbf{k}\;e^{i\mathbf{k}\cdot 0}\frac{\bar{\psi}(\mathbf{k},t)}{\mathfrak{D}^0(\mathbf{k},\,\tilde{\omega}+\mathbf{k}\cdot\mathbf{V};\,\mathbf{a})\mathfrak{D}^0(\mathbf{k},\,\tilde{\omega}''+\mathbf{k}\cdot\mathbf{V};\,\mathbf{a})}.\quad(2.16)$$

Having obtained Huygens' principle either in the form Eq. (2.14) or Eq. (2.15) for the case of homogeneous, time-independent dispersion, we may now turn our attention to the development of the problem for the more general case, as we shall do in Sec. III. As a last point before doing so, however, we may remark that in one sense Eq. (2.14), or, equivalently, Eq. (2.15), is in fact exact—or, more precisely, that the one case in which they are exactly true, namely, when only one mode $\tilde{\omega}_\alpha$ exists, is in principle sufficient to the complete analysis. For, the total response may be written *in general* as the sum of responses to the individual dispersion

$$\mathfrak{D}_\alpha=-2\pi i[\tilde{\omega}-\tilde{\omega}_\alpha(\mathbf{k};\,\mathbf{V};\,\mathbf{a})],$$

with $\mathfrak{D}^0$ replaced successively by each $\mathfrak{D}_\alpha$. Often, however, it may be very inconvenient to make this separation explicitly. Our analysis here shows that this is not really necessary anyway, especially since we will want to make approximations similar to those made above in our application of these ideas to the case of time-dependent or inhomogeneous dispersion. We now go on to this application in Sec. III.

III. EQUATION OF ADIABATIC RESPONSE

Let us return to a consideration of the response, Eq. (2.2), as a function of the singularities of $F(\tilde{\omega};\,\mathbf{V};\,\mathbf{a})$, Eq. (2.5). These singularities may be broadly categorized as belonging to one or the other of two general classes: (a) *fixed singularities* which have their origin in the non-analyticities of either $S(\mathbf{k})$ or of $D(\mathbf{k},\omega)$ thought of as a function of $\mathbf{k}$, and (b) *variable singularities* which have their origin in the zeros of D. In physical applications, the meaningful behavior of the response stems entirely from the variable singularities, so named because in general their positions in $\mathbf{k}$ space vary with $\mathbf{V}$ [reflecting the frequency dependence of the zeros, $\mathbf{k}_n(\omega)$ of D]. The fixed singularities, so designated because they occur normally only at specific fixed values of $\mathbf{k}$, will not ordinarily contribute directly to the response. They cannot be overlooked, however, since they do affect the *topology* of the Riemann sheet containing the Laplace integration contour (the physical sheet). Thus it is possible, for example, for a variable singularity to escape to a nonphysical sheet by passing through a branch cut arising from a fixed singularity.

Particular examples of fixed singularities are not difficult to find. One such case is the branch point of $D(k,\omega)$ that is introduced at $k=0$ in many dispersion relations by the consideration of thermal effects.[8] Another example arises from the presence of wavelength restrictions such as might be imposed by the finite geometry of a given experimental system. Here, one may be required to set $S(\mathbf{k})=0$ for a given range or ranges of $\mathbf{k}$, introducing thereby a nonanalyticity into F. In both of these cases, the resulting fixed singularities may be viewed alternatively as end-point singularities[9] arising from integrating $\mathbf{k}$ (or one of its components) over a domain with finite boundaries (since in each of these instances the singularity occurs for real $\mathbf{k}$).[10]

Let us now focus our attention upon the variable singularities of $F(\tilde{\omega};\,\mathbf{V};\,\mathbf{a})$, introducing the "proper contour" C_p, as in Fig. 1, which is the locus of the uppermost singularity in the $\tilde{\omega}$ plane as a function of $\mathbf{V}$ (for fixed $\mathbf{a}$). The original Laplace integration contour may always be deformed downward to lie just above C_p. If we next consider a section of C_p determined by the variable singularity Σ, as in Fig. 2, for any given velocity $\mathbf{V}$ the contour may be further deformed to lie along the branch cut from Σ to $-i\infty$. Consider now the effect of

[8] See, e.g., R. J. Briggs, *Electron-Stream Interaction with Plasmas* (M.I.T. Press, Cambridge, Mass., 1964), or L. S. Hall, W. Heckrotte, and T. Kammash, Phys. Rev. **139**, A1117 (1965), for applications in which D is obtained for thermal plasmas.

[9] Consideration of the singularities of functions of several complex variables has also been of interest to researchers in the theory of elementary particles. Notable are a number of articles by R. J. Eden and J. C. Polkinghorne and their collaborators, who have analyzed end-point singularities as well as the "pinching" singularities (discussed in the text) such as arise from the zeros of D. See, e.g., *Homology and Feynman Integrals*, edited by R. C. Hwa and V. L. Teplitz (W. A. Benjamin, Inc., New York, 1966).

[10] We might interject at this point a cautionary remark in that the presence of end-point singularities within a range of *unstable* real wave numbers may indicate the inappropriateness of a description of the instabilities of the system by the methods of this paper, when instead a normal-mode analysis is called for. Essentially, the sharp k cutoff contains within itself the idealization of perfectly reflecting boundaries separated by the associated half-wavelength (or integral multiple thereof) and carries with it the possibility of establishing a standing wave which grows (absolutely) in time at the rate $\mathrm{Im}[\omega(k)]$. This is not an incorrect result. Nevertheless, introduction of perfectly reflecting boundaries at the cutoff wavelength may be an overidealization, particularly if the effects of instability are undesirable in the given application so that efforts are made experimentally to eliminate or substantially reduce reflections from the boundary. On the other hand, the *presence* of sharp boundaries is not incompatible with a pulse description if, in addition, there are slow spatial dependences in the intervening region. While the variety of circumstances which might occur is too extensive and would take us too far afield to allow a complete analysis in the present paper, the appropriate modifications appear to be as straightforward in the context of the theory of nonstable waves as they are in ordinary optics.

making a slight change $\delta\mathbf{a}$ in the auxiliary parameters $\mathbf{a}$. Concomitantly, there will be a distortion induced in the proper contour, as indicated in Fig. 3. In general, it may also occur that a small change $\delta\mathbf{V}$ in $\mathbf{V}$ is to be associated with $\delta\mathbf{a}$. In any event, a point Σ' on C_p' is somehow associated with another point Σ'' on C_p''. Indeed as we shall see, one of the problems to be resolved is that of making any such identification unique. Once we are able to do this, however, we should be able to expect, as mentioned in the Introduction, that an adequate representation of the true response would be to suppose that each element of the pulse be considered to propagate for a small increment of time Δt as if in a time-independent uniform medium whose properties are just those of the true system at the position $\mathbf{r}$ of the pulse and at the time t under consideration.

Going on now to the restatement of the uniform time-independent problem in terms of the average bitemporal response, in particular as we have written the results in the form Eqs. (2.15) and (2.16), the singularities of $F(\tilde{\omega}',\tilde{\omega}''; \mathbf{V},\mathbf{a}; t)$ will fall along the *same* curve C_p in both the $\tilde{\omega}'$ plane and the $\tilde{\omega}''$ plane. However, in light of the preceding discussion, we are now prepared to make the central assertion of our theory, which is to say that for a slowly time-dependent change of macroscopic parameters, the true response at time $t+\Delta t$ is approximately given (suppressing the superscript zero on $\mathfrak{D}$) by

$$\psi(t+\Delta t)=\frac{1}{\Delta t}\int_0^{\Delta t} d\tau \int_{-\infty+ip}^{\infty+ip} d\tilde{\omega}' \int_{-\infty+ip}^{\infty+ip} d\tilde{\omega}'' \times\exp\{-i[\tilde{\omega}'\tau+\tilde{\omega}''(\Delta t-\tau)]\}\int d\mathbf{k} \times e^{i\mathbf{k}\cdot 0}\frac{\bar{\psi}(\mathbf{k},t)}{\mathfrak{D}(\mathbf{k},\, \tilde{\omega}'+\mathbf{k}\cdot\mathbf{V}';\, \mathbf{a}')\mathfrak{D}(\mathbf{k},\, \tilde{\omega}''+\mathbf{k}\cdot\mathbf{V}'';\, \mathbf{a}'')}, \quad (3.1)$$

where $\mathbf{a}''=\mathbf{a}'+\delta\mathbf{a}$ and $\mathbf{V}''=\mathbf{V}'+\delta\mathbf{V}$, the relationship between $\delta\mathbf{V}$ and $\delta\mathbf{a}$ as yet to be specified. (Here $\mathbf{a}'$ is the true value of $\mathbf{a}$ at time t, and $\mathbf{a}''$ is the value at $t+\Delta t$.) Equation (3.1) is the same form as Eq. (2.15) if we replace $F(\tilde{\omega}',\tilde{\omega}''; \mathbf{V},\mathbf{a}; t)$ by the integral

$$F=\int d\mathbf{k}\, e^{i\mathbf{k}\cdot 0}\frac{\bar{\psi}(\mathbf{k},t)}{\mathfrak{D}(\mathbf{k},\, \tilde{\omega}'+\mathbf{k}\cdot\mathbf{V}';\, \mathbf{a}')\mathfrak{D}(\mathbf{k},\, \tilde{\omega}''+\mathbf{k}\cdot\mathbf{V}'';\, \mathbf{a}'')}. \quad (3.2)$$

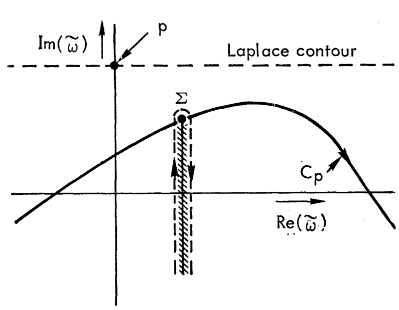

FIG. 2. Distortion of the Laplace contour allowed for the case of a particular variable singularity $\sum(\mathbf{V};\mathbf{a})$.

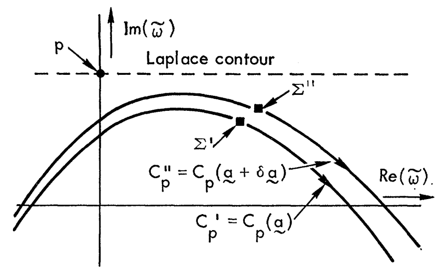

FIG. 3. Distortion of C_p induced by a small change $\delta\mathbf{a}$ in the auxiliary parameters.

Thus the singularities of F now occur on two different curves, contour C_p' of Fig. 3 in the $\tilde{\omega}'$ plane and contour C_p'' in the $\tilde{\omega}''$ plane.

Continuing our investigation of the effects of *variable* singularities, which we have defined to be those associated with the zeros of $\mathfrak{D}$, we can perform the integration over τ, change the variables of integration $(\tilde{\omega}',\tilde{\omega}'')$ to $(\tilde{\omega},\delta\tilde{\omega})$, where $\tilde{\omega}=\frac{1}{2}(\tilde{\omega}'+\tilde{\omega}'')$, $\delta\tilde{\omega}=\tilde{\omega}''-\tilde{\omega}'$, and integrate over $\tilde{\omega}$ to reduce (3.1) to

$$\psi=\int d\mathbf{k}\, e^{i\mathbf{k}\cdot 0}\int_{-\infty}^{\infty} d\delta\tilde{\omega}\frac{\sin(\frac{1}{2}\delta\tilde{\omega}\Delta t)}{\frac{1}{2}\delta\tilde{\omega}\Delta t} \times\left(\frac{\exp[\frac{1}{2}i(\delta\tilde{\omega}\Delta t)]\bar{\psi}_u'(k,\, t+\Delta t)}{\mathfrak{D}[\mathbf{k},\, \tilde{\omega}'(\mathbf{k})+\mathbf{k}\cdot\mathbf{V}'+(\delta\tilde{\omega}+\mathbf{k}\cdot\delta\mathbf{V});\, \mathbf{a}'+\delta\mathbf{a}]} +\frac{\exp[-\frac{1}{2}i(\delta\tilde{\omega}\Delta t)]\bar{\psi}_u''(\mathbf{k},\, t+\Delta t)}{\mathfrak{D}[\mathbf{k},\, \tilde{\omega}''(\mathbf{k})+\mathbf{k}\cdot\mathbf{V}''-(\delta\tilde{\omega}+k\cdot\delta\mathbf{V});\, \mathbf{a}''-\delta\mathbf{a}]}\right), \quad (3.3)$$

where $\tilde{\omega}'(\mathbf{k})$ is the solution to $\mathfrak{D}=0$ for $\mathbf{V}=\mathbf{V}'$, $\mathbf{a}=\mathbf{a}'$ and $\tilde{\omega}''(\mathbf{k})$ is similarly defined. Likewise, we have introduced the "unperturbed" responses,

$$\bar{\psi}_u'(\mathbf{k},\, t+\Delta t)=\int_{-\infty+ip}^{\infty+ip} d\tilde{\omega}'\frac{\exp(-i\tilde{\omega}'\Delta t)\bar{\psi}(\mathbf{k},t)}{\mathfrak{D}(\mathbf{k},\, \tilde{\omega}'+\mathbf{k}\cdot\mathbf{V};\, \mathbf{a}')}, \quad (3.4)$$

with a similar definition for $\bar{\psi}_u''$.

Next, perform the integration over $\delta\tilde{\omega}$. In general, the changes $(\delta\mathbf{a},\delta\mathbf{V})$ produce a shift in the zeros of $\mathfrak{D}$ thought of as a function of $\tilde{\omega}$ for fixed $\mathbf{k}$. Such a shift can always be considered, for fixed $\mathbf{k}$, to be associated with a definite sign of $\mathrm{Im}(\tilde{\omega})$, with the possible exclusion of a countable number of points $\mathbf{k}$ for which $\mathrm{Im}(\tilde{\omega})$ vanishes.[11] Let us assume, tentatively, that the imaginary frequency shift is positive.

We can now perform the integral over $\delta\tilde{\omega}$ for the first term in Eq. (3.3) by closing the contour in the upper half-plane [if necessary, introducing the convergence factor $\exp(i\delta\tilde{\omega}0^+)$]; in similar fashion, the second term is evaluated by closing the contour in the lower half $\delta\tilde{\omega}$ plane. But because the imaginary frequency shift is positive, only the first of these contributes, yielding

[11] If $\mathrm{Im}(\tilde{\omega})$ vanishes for a continuum of values of real $\mathbf{k}$, one can always introduce a slight distortion of the $d\mathbf{k}$ integration into complex $\mathbf{k}$ space so that our supposition about the sign of $\mathrm{Im}(\omega)$ is satisfied.

a factor

$$[\exp(i\delta\tilde{\omega}'\Delta t)-1]/i\delta\tilde{\omega}'\Delta t,$$

where $\delta\tilde{\omega}'$ is given from the requirement

$$0=[(\delta\tilde{\omega}'+\mathbf{k}\cdot\delta\mathbf{V})\mathfrak{D}_{\tilde{\omega}}+\delta\mathbf{a}\cdot\mathfrak{D}_{\mathbf{a}}]|_{(\mathbf{k},\tilde{\omega}'+\mathbf{k}\cdot\mathbf{V}';\mathbf{a}')} \quad (3.5)$$

and $\text{Im}(\delta\tilde{\omega}')>0$. Hence the absolute value of the numerator is less than 2, and the greatest contribution to the response comes when $|\delta\tilde{\omega}'|$ is as small as possible.

If, on the other hand, we had assumed that the changes $(\delta\mathbf{a},\delta\mathbf{V})$ produced an imaginary frequency shift less than zero, only the second term within the large parentheses in the integral of Eq. (3.3) would have contributed and the resulting integrand would have contained the factor

$$[1-\exp(-i\delta\tilde{\omega}''\Delta t)]/i\delta\tilde{\omega}''\Delta t,$$

with $\delta\tilde{\omega}''$ given by Eq. (3.5) with the primes replaced by double primes. In this case $\text{Im}(\delta\tilde{\omega}'')<0$ by hypothesis, so that again the greatest contribution occurs when $|\delta\tilde{\omega}''|$ is minimal. Referring again to Fig. 3, this means that $\delta\mathbf{V}$ must be so chosen that Σ' and Σ'' lie along the normal to the proper contour in the $\tilde{\omega}$ plane. This provides the needed relationship between $\delta\mathbf{V}$ and $\delta\mathbf{a}$, so that Eq. (3.1) is now rendered definite.

It is often inconvenient in applications to specify a particular point Σ on C_p in terms of the variable $\mathbf{V}$, since the description changes with each choice of $\mathbf{a}$, and, in any event, the position is a one-parameter specification and we would have to change its nature, depending on the component of $\mathbf{V}$ to be specified. Thus we may introduce the *proper frequency* ν_p, which is simply a measure of distance along C_p. However, in order that ν_p be unchanged upon moving orthogonal to C_p, we shall have to introduce a scaling which takes account of the stretching of C_p in going from $\mathbf{a}'$ to $\mathbf{a}''$. This stretching is proportional to the radius of curvature of C_p in the $\tilde{\omega}$ plane,

$$\frac{1}{R}=\left|\frac{d^2\tilde{\omega}}{ds^2}\right|,$$

where s is the distance along C_p. Thus we take

$$\nu_p=R_0\int_{s0}^{s}\left|\frac{d^2\tilde{\omega}}{ds^2}\right|ds, \quad (3.6)$$

where s_0 is some convenient initial position and R_0 a typical radius of curvature. In the limit of small growth or damping, C_p lies very nearly along the real $\tilde{\omega}$ axis, and ν_p becomes just the (real) frequency.

Finally, since we are now able to associate a unique velocity $\mathbf{V}$ with the propagation of a given element of the disturbance, i.e., that element associated with the increment $\Delta\nu_p$ about ν_p, we may extend the theory of time-dependent dispersion to include the effects of spatial gradients in $\mathbf{a}$, since these may be reduced to purely temporal changes by moving into the velocity frame $\mathbf{V}$. Thus, interpreted as we have discussed above, Eq. (3.1) is the equation for the response in the (adiabatic) limit of sufficiently slowly varying macroscopic parameters associated with the local (or average nonlocal) dispersion $D(\mathbf{k},\omega;\mathbf{a})$.

In summary, *we have shown that the spatial position at which an initial excitation of a specified proper frequency dominates the nonstable response moves so as to keep the proper frequency constant.* That is, the position in the laboratory at which the amplitude of the elementary excitation ν_p dominates all other amplitudes is governed by the requirement that the position of the associated variable singularity in the $\tilde{\omega}$ plane lies on a given normal to the proper contour.

IV. EQUATION OF PROPAGATION

As we have previously stated, the variable singularities of F, Eq. (3.2) or, as is sufficient for the moment, Eq. (2.3), are closely associated with the zeros of D. For the case of a single spatial dimension, we have

$$F=\int_\alpha^\beta dk\, e^{ik0}\frac{S(k)}{D(k,\,\tilde{\omega}+kV)}.$$

In this circumstance, for $\tilde{\omega}$ on the Laplace contour, no zeros of D lie along that portion of the real k axis between α and β, and the end points $k=\alpha$ and $k=\beta$ are fixed singularities of F. The analytical continuation of F into the region below the Laplace contour can always be carried out provided we do not permit any zeros of D to cross the integration contour. When they attempt to do so, the integration contour must be appropriately distorted away from the real axis in the complex k plane. If, however, a pair of zeros of D can (as a function of ω) be made to collide across the integration contour, no distortion to avoid a zero-crossing is possible. This "pinching" then gives rise to a branch point of F.[5] The necessary condition for such an eventuality is that there be a double zero of $D(k,\,\tilde{\omega}+kV;\,\mathbf{a})$, in which case the pair of equations

$$D=0,\quad \frac{\partial D}{\partial k}+V\frac{\partial D}{\partial\omega}=0,$$

must be simultaneously satisfied. Provided such a double zero [saddle-point of $\tilde{\omega}(k)$] is also of the pinching variety, a variable singularity in F occurs. Hall and Heckrotte[5] have shown that upon variation of V, the position of the saddle point varies according to the equations

$$\left.\frac{\delta\tilde{\omega}}{\delta V}\right|_{\mathbf{a}}=-k,\quad \left.\frac{\delta k}{\delta V}\right|_{\mathbf{a}}=\frac{1}{d^2\tilde{\omega}/k^2}=\frac{1}{d^2\omega/dk^2}, \quad (4.1)$$

where $d^2\omega/dk^2$ is taken along $D[k,\omega(k)]=0$. Equations (4.1) therefore define the proper contour over the range in which this singularity is dominant.

In the general multidimensional case, $\mathbf{k}$ is to be integrated over an n-dimensional volume in the space of real wave vectors, or an n-dimensional hypersurface in the $2n$-dimensional complex wave-vector space. Let the first of these integrals be performed for $\mathbf{k}_1$, for example. Then if a pinch develops across the k_1 contour for all $\mathbf{K}_1$ (the component of $\mathbf{k}$ orthogonal to $\mathbf{k}_1$) within a nonvanishing domain of integration over $\mathbf{K}_1$, F will have a singular point. Clearly, then, one must require the existence of a point in the $(2n+2)$-dimensional complex $(\mathbf{k},\omega)$ space, satisfying the conditions

$$D(\mathbf{k},\tilde{\omega}+\mathbf{k}\cdot\mathbf{V};\mathbf{a})=0\,,\quad \frac{\partial D}{\partial\mathbf{k}}+\mathbf{V}\frac{\partial D}{\partial\omega}=0\,, \tag{4.2}$$

for only then is the singularity of D^{-1} nonintegrable.[12] Equation (4.2), however, is again only a necessary condition for the appearance of a variable singularity of F. Relaxing the second of Eqs. (4.2) one component at a time, to trace $\mathbf{K}_1$ back to the surface of real wave numbers, and then relaxing the final restriction $(\partial D/\partial k_1)+V_1(\partial D/\partial\omega)=0$ to ascertain a pinch in standard one-dimensional fashion[5] could, for example, be used as a particular algorithm for verification of the significance of the given saddle point. If the procedure breaks down at any point, or if no pinch is achieved, there is no singularity. Otherwise, the saddle point corresponds to a variable singularity of F.

Once the relevance of a saddle point is established, we need to know its variation with $\mathbf{V}$ in order to determine the proper contour. The multidimensional analogs of Eq. (4.1) may be straightforwardly obtained:

$$\begin{aligned}\delta\tilde{\omega}|_{\mathbf{a}}&=-\mathbf{k}\cdot\delta\mathbf{V}|_{\mathbf{a}}\,,\\ \delta\mathbf{V}|_{\mathbf{a}}&=(d^2\omega/d\mathbf{k}d\mathbf{k})\cdot\delta\mathbf{k}|_{\mathbf{a}}\,,\end{aligned} \tag{4.3}$$

where

$$\frac{d^2\omega}{d\mathbf{k}d\mathbf{k}}=-\left(\frac{\partial D}{\partial\omega}\right)^{-1}\left(\frac{\partial^2 D}{\partial\mathbf{k}\partial\mathbf{k}}+\mathbf{V}\frac{\partial^2 D}{\partial\omega\partial\mathbf{k}}+\frac{\partial^2 D}{\partial\mathbf{k}\partial\omega}\mathbf{V}+\mathbf{V}\frac{\partial^2 D}{\partial\omega^2}\mathbf{V}\right). \tag{4.4}$$

Equations (4.3) are then the differential equations for the proper contour C_p. In addition, we shall need the contours of constant ν_p, or the curves orthogonal to C_p. Consider first the one-dimensional case, for which

$$\begin{aligned}\Delta\tilde{\omega}&=\Delta\mathbf{a}\cdot\left(\frac{\delta\tilde{\omega}}{\delta\mathbf{a}}\bigg|_V+\frac{dV}{d\mathbf{a}}\frac{\delta\tilde{\omega}}{\delta V}\bigg|_{\mathbf{a}}\right),\\ \Delta k&=\Delta\mathbf{a}\cdot\left(\frac{\delta k}{\delta\mathbf{a}}\bigg|_V+\frac{dV}{d\mathbf{a}}\frac{\delta k}{\delta V}\bigg|_{\mathbf{a}}\right).\end{aligned} \tag{4.5}$$

Then, together with the requirement that $\Delta\tilde{\omega}$ be orthogonal to C_p, Eqs. (4.5) determine $dV/d\mathbf{a}$.

Now

$$\frac{\delta\tilde{\omega}}{\delta\mathbf{a}}\bigg|_V=-\frac{D_{\mathbf{a}}}{D_\omega}\,,\quad \frac{\delta k}{\delta\mathbf{a}}\bigg|_V=\frac{D_{\mathbf{a}}Q_\omega-Q_{\mathbf{a}}D_\omega}{D_\omega(Q_k+VQ_\omega)}\,, \tag{4.6}$$

[12] See also, K. B. Dysthe, Nucl. Fusion 6, 215 (1966), for a discussion from the point of view of relevant saddle points.

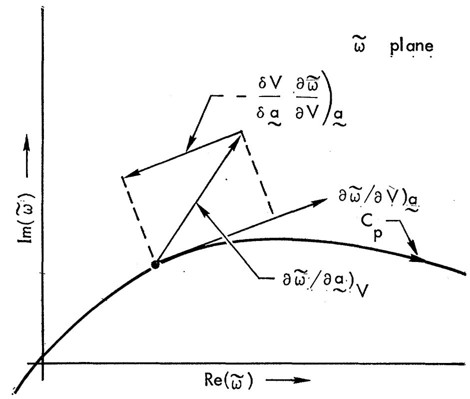

FIG. 4. Relationship of various differential quantities associated with the proper contour C_p.

where $Q\equiv D_k+VD_\omega$. Referring to Fig. 4, the component of $\delta\tilde{\omega}/\delta\mathbf{a}|_V$ along $\delta\tilde{\omega}/\delta V|_{\mathbf{a}}$, that is, along C_p, is just

$$\mathrm{Re}\left(\frac{\delta\tilde{\omega}}{\delta\mathbf{a}}\bigg|_V\,\frac{\delta\tilde{\omega}}{\delta V}\bigg|_{\mathbf{a}}^{*}\right)\bigg/\left|\frac{\delta\tilde{\omega}}{\delta V}\bigg|_{\mathbf{a}}\right|.$$

But this must equal $-(dV/d\mathbf{a})|\delta\tilde{\omega}/\delta V|_{\mathbf{a}}|$, whence

$$\frac{dV}{d\mathbf{a}}=\mathrm{Re}\left(\frac{-D_{\mathbf{a}}}{kD_\omega}\right). \tag{4.7}$$

Equation (4.7) is the chief result of this section, for the one-dimensional case. It can be supplemented either by Eqs. (4.6) or by the original saddle-point equations $D=Q=0$ to determine the contours of constant ν_p. In straightforward fashion, the multidimensional analogs of Eqs. (4.6) are

$$\begin{aligned}\delta\tilde{\omega}/\delta\mathbf{a}|_V&=-D_{\mathbf{a}}/D_\omega\,,\\ D_\omega(D_{\mathbf{kk}}+\mathbf{V}D_{\omega\mathbf{k}}+D_{\mathbf{k}\omega}\mathbf{V}+\mathbf{V}D_{\omega\omega}\mathbf{V})\cdot(\delta\mathbf{k}/\delta\mathbf{a})_{\mathbf{V}}&=D_{\mathbf{a}}\mathbf{Q}_\omega-\mathbf{Q}_{\mathbf{a}}D_\omega\,,\end{aligned} \tag{4.8}$$

where $\mathbf{Q}=D_{\mathbf{k}}+\mathbf{V}D_\omega$. Similarly, the general form of the important equation (4.7) is, simply,

$$d\mathbf{V}/d\mathbf{a}=\mathrm{Re}(-\mathbf{k}^{-1}D_{\mathbf{a}}/D_\omega)\,. \tag{4.9}$$

We are now prepared to follow the propagation of a nonstable disturbance through a general inhomogeneous time-dependent medium. For clarity, let us consider at this point a one-dimensional system with only one variable singularity in order to avoid the tedium of a too-general example. (The reader is by now familiar with the modifications required for the general case.)

At time $t=0$ let there be a δ-function disturbance of the response at $x=0$. We shall of course assume that the medium varies only slowly in space and/or time, so that the response will very quickly (on the macroscopic scale) develop its local asymptotic form. Indeed, if $\tilde{\omega}_\Sigma$ is the position of the variable singularity (saddle point) in the $\tilde{\omega}$ plane, the plot of $\gamma=\mathrm{Im}(\tilde{\omega}_\Sigma)$ versus V, as in Fig. 5, is in fact a normalized snapshot of the pulse itself.[5]

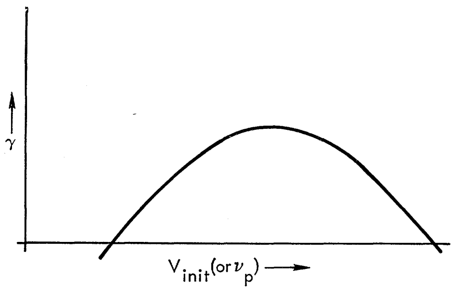

FIG. 5. Normalized initial asymptotic amplitude of a typical convectively unstable pulse.

For if we multiply both ordinate and abscissa by the elapsed time t, we have a plot of the logarithm of the pulse amplitude versus spatial position. Each point on the normalized pulse amplitude, Fig. 5, corresponds to a different proper frequency ν_p, which we can compute explicitly. Or, if we prefer, we may simply carry along ν_p implicitly in terms of the initial velocity.

As time goes on, the macroscopic parameters will change, either because of their own explicit time dependence, or because parts of the pulse move convectively into regions of differing parameters. To trace the fate of the whole pulse, we then look at the behavior at each frequency ν_p, adding up the results in the end.

The response itself according to our method of analysis is not an explicit function of position. Nevertheless, this is implicitly the case through the dependence of the response upon ν_p. That is, specification of ν_p tells us not only the initial V, but also, through the equation of propagation Eq. (4.7), the value of V at all subsequent times. More precisely,

$$x=\int_0^t V(t')dt',$$

$$V(t)=V(0)+\int_0^t \frac{dV}{dt'}dt' \tag{4.10}$$

$$=V(0)+\int_0^t \frac{dV}{d\mathbf{a}}\cdot[\mathbf{a}_t(x(t'),t')+V(t')\mathbf{a}_x(x(t'),t')]dt'.$$

Moreover, the response, at each position x so determined, is just

$$\mathrm{const}\cdot\exp\left(-i\int_0^t \tilde{\omega}_\Sigma(t')dt'\right),$$

where the time integral is along the trajectory, Eq. (4.10). Hence, a snapshot of the pulse analogous to Fig. 5 would be obtained by plotting

$$\bar{\gamma}\equiv\frac{1}{t}\int_0^t \gamma(t')dt'$$

versus $\bar{V}\equiv x/t$.

If, as a function of parameters, V passes through zero at some point along the trajectory, the wave of this proper frequency will be reflected back into the system. Thus, if there are a pair of such reflection points bounding the region of instability, such a wave would be reflected back and forth, growing in some regions and damping in others. If T is the total time for a complete traversal of the system, then for the case of an inhomogeneous but time-independent system, $\mathbf{a}=\mathbf{a}(x)$, there will be unbounded growth in the region between reflection points if $\bar{\gamma}>0$, averaged over the period T:

$$\bar{\gamma}=\frac{1}{T}\oint\gamma(t)dt=\frac{1}{T}\oint\frac{\gamma(x)}{|V(x)|}dx,$$

$$T=\oint dt=\oint\frac{d(x)}{|V(x)|},$$

the integrals being performed over the trajectory associated with the given proper frequency ν_p.

The presence of such an instability as discussed above, where the otherwise convective mode grows and remains confined within a finite domain due to reflections in the inhomogeneous medium, is as disastrous to an experimental system as the presence of an absolute instability for which the spreading of a pulse is sufficient to keep it growing at a given fixed lab position, in spite of its convective velocity. Both such circumstances may therefore be lumped into a single category which we shall call *unconditional instability*. The contrary circumstance, *conditional instability*, will normally correspond to the situation in which there is a bound on the amplitude of that portion of the pulse remaining within a bounded spatial domain. On the other hand, the circumstance in which the growth rate vanishes as $V\to 0$ but no more rapidly than as V^p for $p<1$, can lead to unbounded growth just as in the homogeneous medium when this behavior occurs for convective modes.[5] It seems convenient, therefore, to include within the definition of unconditional instability the circumstance in which $\oint\gamma dt\to+\infty$. Since this also implies that $\oint dt$ is infinite, the wave can be considered trapped within an unstable spatial domain, even though true reflection never occurs.

V. EXAMPLE

As an example of the type of calculation to which our theory may be applied, consider the dispersion relation

$$D=\omega^2-k^2(\omega-a). \tag{5.1}$$

This dispersion has been obtained by Kennel and Sagdeev[13] in the discussion of the finite cyclotron radius "firehose" instability of Alfvén waves. For this problem, a is the normalized pressure anisotropy and one may take, as a function of position in the typical "mirror" geometry of the earth's magnetic field,

$$a=1-x^2/L^2, \tag{5.2}$$

[13] C. F. Kennel and R. Z. Sagdeev, J. Geophys. Res. 72, 3303 (1967).

where x is the distance from the equator and L is the scale length. Instability occurs for $a>0$, i.e., in the vicinity of the magnetic equator. We assume that a is itself time-independent.[14]

For a given value of a, solving for k or ω gives

$$k=\pm\omega/(\omega-a)^{1/2}, \quad \omega=\tfrac{1}{2}k^2\pm(\tfrac{1}{4}k^2-a)^{1/2},$$

so that reflection points can occur only at $k=0$ or $k=\infty$, and the range of unstable k is $0<k^2<4a$. Maximum $\mathrm{Im}(\omega)$ occurs versus real k for $k=\pm k_0$, where $k_0=(2a)^{1/2}$. The propagation velocity of the peak of the pulse is $V_0=k$ at this point, and the frequency is $\omega_0=(1+i)a$ for the unstable mode.

The variable singularity is determined by

$$D=0, \quad Q\equiv D_k+VD_\omega=0,$$

and the combination of these equations shows that

$$(\omega/kV)^2-(\omega/kV)+\tfrac{1}{2}(k/V)=0.$$

Thus, we define

$$\lambda\equiv[(2k/V)-1]^{1/2}, \tag{5.3}$$

with the branch cut of the square root taken just below the negative real axis, so that either $\mathrm{Re}(\lambda)>0$ or $\mathrm{Re}(\lambda)=0$ and $\mathrm{Im}(\lambda)>0$. Then

$$\omega=\tfrac{1}{2}kV(1+i\lambda), \tag{5.4}$$

and we have chosen the sign of $i\lambda$ to correspond to the unstable mode. If we now substitute this result back into Eq. (5.1), we find

$$-i\lambda(1+i\lambda)^2=\alpha, \quad \alpha\equiv 4a/V^2. \tag{5.5}$$

Equation (5.5) is a cubic equation for $\lambda(\alpha)$, and clearly the desired root is that for which $\lambda\to i$ as $\alpha\to 2$, since $k\to V$ and $\frac{1}{2}V^2\to a$ at the point corresponding to the propagation of peak growth.

The roots of Eq. (5.5) are given, for $\alpha>4/27$, by

$$\begin{aligned} -i\lambda&=\tfrac{2}{3}+A+B,\\ -i\lambda&=\tfrac{2}{3}-\tfrac{1}{2}(A+B)-\tfrac{1}{2}i\sqrt{3}(A-B),\\ -i\lambda&=\tfrac{2}{3}-\tfrac{1}{2}(A+B)+\tfrac{1}{2}i\sqrt{3}(A-B),\end{aligned} \tag{5.6}$$

where

$$\begin{aligned} A&=[\tfrac{1}{2}\alpha-1/27+(\tfrac{1}{4}\alpha^2-\alpha/27)^{1/2}]^{1/3},\\ B&\equiv[\tfrac{1}{2}\alpha-1/27-(\tfrac{1}{4}\alpha^2-\alpha/27)^{1/2}]^{1/3}.\end{aligned} \tag{5.7}$$

[14] An equation very similar to Eq. (5.1), $D=a\omega^2-k_{\|}^2(\omega-1)$, also arises in the high-density cold-electron limit of the "Harris" instability. [See E. G. Harris, Phys. Rev. Letters **2**, 34 (1959); J. Nucl. Energy **C2**, 138 (1961).] Here $k_{\|}$ is the wave number along the magnetic field and instability occurs again for positive a. In this case, however, a is proportional to

$$\langle P_l\rangle=\langle 2l\beta^{-1}J_l(\beta)J_l'(\beta)\rangle,$$

where the J_l are ordinary Bessel functions; $\beta=k_\perp\rho$, where ρ is the cyclotron radius of the ions; and the angular brackets denote the average over the distribution. [See Hall *et al.*, Ref. 8, Eq. (47).] Typically, for fixed $k_\perp$ (i.e., the one-dimensional problem), $k_\perp\rho$ tends to be of order l if and when $\langle P_l\rangle>0$, with $\langle P_l\rangle$ becoming negative for smaller perpendicular energies. Thus in a mirror geometry, just the reverse behavior to Eq. (5.2) would be expected, with a tending to negative values near the midplane and becoming positive near the extremeties. The results of the present section, through the point where $\mathrm{Im}(\omega)$ and dV^2/da are explicitly obtained as functions of α (Fig. 6), can be carried over directly to this other problem by a simple renormalization of ω.

It is the second of Eqs. (5.6) that has the desired correspondence $\lambda\to 1$ as $\alpha\to 2$. However, as α decreases to $4/27$, the second and third roots merge and the situation becomes ambiguous. Indeed, since $\alpha=4/27$ corresponds to a quadruple zero of $D(k,\tilde{\omega}+kV)$, thought of as a function of k, there is a possibility that the pinch dissolves at this point (by a re-pairing of the zeros across the integration contour in the k plane in the expression for the response). Actually, this turns out not to be the case, and if we solve the cubic equation for the range $0<\alpha<4/27$ in the form

$$\begin{aligned} -i\lambda&=\tfrac{2}{3}\{1+\cos[\tfrac{1}{3}(\pi-\psi)]\},\\ -i\lambda&=\tfrac{2}{3}\{1+\cos[\tfrac{1}{3}(\pi+\psi)]\},\\ -i\lambda&=\tfrac{2}{3}[1-\cos\tfrac{1}{3}\psi],\end{aligned} \tag{5.8}$$

where

$$\alpha=(2/27)(1-\cos\psi), \quad 0<\psi<\pi, \tag{5.9}$$

it is the last of the roots given by Eqs. (5.8) which pinches.

The growth rate $\gamma=\mathrm{Im}(\tilde{\omega})$, or its normalized value $\Gamma=\gamma/a$, is given from Eqs. (5.3) and (5.4):

$$\tilde{\omega}=\omega-kV=-\tfrac{1}{2}kV(1-i\lambda)$$

or

$$\Gamma=-\alpha^{-1}\,\mathrm{Im}[(1+\lambda^2)(1-i\lambda)]. \tag{5.10}$$

Thus Γ vanishes for $\alpha<4/27$, and for greater values of α,

$$\Gamma=\tfrac{3}{2}\sqrt{3}\frac{(A-B)(A+B-\frac{2}{3})}{\alpha}. \tag{5.11}$$

Plotting Γ versus $\alpha^{-1/2}$ gives the normalized pulse shape, Fig. 6.

The equation of propagation, Eq. (4.7), together with Eqs. (5.3)–(5.5), can be put in the form

$$\begin{aligned} d\ln V^2/d\ln a&=-\mathrm{Im}[\lambda(\alpha)]\\ &=-\tfrac{2}{3}[1-\cos(\tfrac{1}{3}\psi)], \quad \alpha<4/27\\ &=\tfrac{1}{2}(A+B)-\tfrac{2}{3}, \quad \alpha>4/27.\end{aligned} \tag{5.12}$$

Figure 6 also shows a plot of the right-hand side of Eq. (5.12).

For an initial pulse disturbance at $x=0$, a will decrease monotonically from unity. Consider, therefore, the fate of each wave making up the unstable pulse. For all those proper frequencies for which $\alpha<2$ initially, that is, all of the pulse to the right of the peak in Fig. 6, the sign of Eq. (5.12) is initially negative. Hence on decreasing a, V^2 increases and α decreases as a result of both changes. But then the sign of Eq. (5.12) never changes and V^2 increases monotonically to some limiting value as the wave of the given proper frequency propagates out of the system. No reflection occurs.

Next, for $2<\alpha<36$, although V decreases initially with a, the changes are such that in ratio α also decreases, the change in a surpassing the effect of the change in V. That is, $d\ln\alpha/d\ln a<0$ in this range, and α tends to decrease toward 2. If this value is reached before $V^2\to 0$, there is a minimum in V^2 at the point

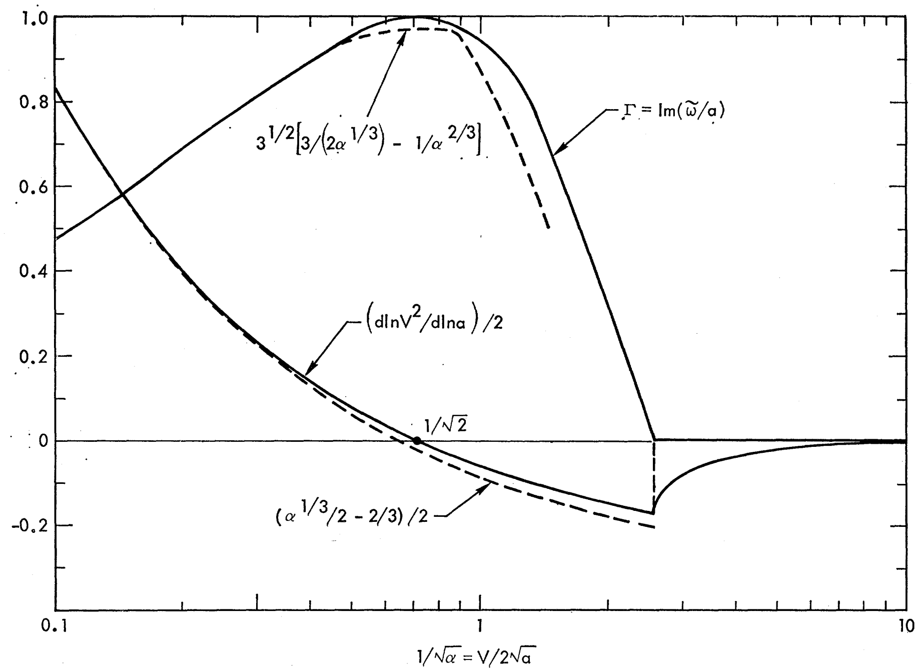

FIG. 6. Growth rate and rate of change of V versus $\alpha^{-1/2}$. Approximate expressions for these quantities, used in the text for the range $\alpha^{-1/2}<2^{-1/2}$, are also shown for comparison.

$\alpha=2$, and thereafter V^2 increase monotonically. Here again the wave escapes without reflection. For some proper frequencies, however, it may happen that V vanishes before the minimum is reached, and in this case reflection would occur. Finally, for those proper frequencies for which $\alpha\geq 36$ initially, d^2V/da is always positive and V must vanish at some point $|x|<L$.

Since the growth rate is always positive for those (slower) waves which *can* reflect, and since V does vanish for some of these waves, the system appears to be unconditionally unstable to those proper frequencies for which $V\rightarrow 0$. This is partly true. On the other hand, our particular example will illustrate—as is often the case in such problems—the unwiseness of being too hasty in drawing conclusions about physical behavior in a field in which nature seems to make the most out of mathematical subtlety.

For the waves for which $V\rightarrow 0$, we can approximate the equation of propagation, Eq. (5.12), by

$$d\ln V^2/d\ln\alpha \approx \tfrac{1}{2}\alpha^{1/3}-\tfrac{2}{3},$$

the usefulness of this approximation being illustrated in Fig. 6. Thus

$$d\ln\alpha/d\ln a \approx (5/3)-\tfrac{1}{2}\alpha^{1/3}.$$

Integrating the latter equation, we find

$$5/(3\alpha^{1/3})-\tfrac{1}{2}\approx[5/(3\alpha_1^{1/3})-\tfrac{1}{2}]a^{-5/9},$$

where $\alpha_1=\alpha(a=1)=4/V_1^2$. Substituting for α, we obtain

$$V\approx 2a^{1/2}\{0.3+a^{-5/9}[(V_1/2)^{2/3}-0.3]\}^{3/2}. \quad (5.13)$$

Hence, V vanishes for those proper frequencies having initial velocity $V_1<2(3/10)^{3/2}$, for which case the reflection point is

$$a_R=[1-(10/3)(V_1/2)^{2/3}]^{9/5}. \quad (5.14)$$

Now, the characteristic growth rates can be computed through use of a as a generalized position variable. Thus,

$$x=L(1-a)^{1/2},$$

$$t=\int_0^x \frac{dx}{V}=\tfrac{1}{2}L\int_a^1 \frac{ds}{(1-s)^{1/2}V}, \quad (5.15)$$

$$\bar{\gamma}t=\int_0^t \gamma dt=\tfrac{1}{2}L\int_a^1 \frac{s\Gamma ds}{(1-s)^{1/2}V}.$$

Using the approximate expression derived above for $V(a)$ and a similarly valid approximation for Γ, namely,

$$\Gamma\approx 3^{1/2}\left(\frac{3}{2\alpha^{1/3}}-\frac{1}{\alpha^{2/3}}\right),\quad \alpha^{-1/3}=\frac{3}{10}\left[1-\left(\frac{a_R}{a}\right)\right]^{5/9} \quad (5.16)$$

(see Fig. 6), we obtain

$$\bar{\gamma}t=\tfrac{3}{8}L(10)^{1/2}\int_a^1 \frac{s^{1/2}ds}{(1-s)^{1/2}}\left\{\left[1-\left(\frac{a_R}{s}\right)^{5/9}\right]^{-1/2} -\frac{1}{5}\left[1-\left(\frac{a_R}{s}\right)^{5/9}\right]^{1/2}\right\}, \quad (5.17)$$

$$t=\tfrac{1}{3}L\left(\frac{10}{3}\right)^{3/2}\int_a^1 \frac{ds}{s^{1/2}(1-s)^{1/2}}\left[1-\left(\frac{a_R}{s}\right)^{5/9}\right]^{-3/2}.$$

It is of interest that although the expression for $\bar{\gamma}t$ remains finite as $a\rightarrow a_R$, the expression for t diverges at the lower limit, showing that an infinite time is required to reach the reflection point. Hence, most of the time must be spent in a region of vanishing growth near the point where $V=0$. The net asymptotic growth factor for $a=a_R$ is shown as a function of a_R in Fig. 7.

The exceptional case occurs for $\alpha_1=36$, or $V_1=\frac{1}{3}$ and $a_R=0$. In this instance α is constant and $V=\frac{1}{3}a^{1/2}$. The integrals, Eqs. (5.15), become

$$\bar{\gamma}t=\tfrac{3}{2}L\Gamma_c\int_0^1\frac{s^{1/2}ds}{(1-s)^{1/2}}=\tfrac{3}{4}\pi L\Gamma_c.$$

$$t=\tfrac{3}{2}L\int_0^1\frac{ds}{s^{1/2}(1-s)^{1/2}}=\tfrac{3}{2}\pi L,$$

whence $\bar{\gamma}=\frac{1}{2}\Gamma_c$, where $\Gamma_c=\Gamma(\alpha=36)=0.6285$. In this case, therefore, reflection really does occur within finite time. Our system, then, is unconditionally unstable only to that wave whose proper frequency corresponds to the initial velocity $V_1=\frac{1}{3}$.

VI. NATURE OF THEORY

Having completed the development and illustration of the theory of nonstable waves, it seems profitable at this stage to depart from our quantitative analysis to recapitulate and collect at one point the various qualitative aspects or philosophical stands which have marked the approach taken here. Thus we will again contrast the goals of the present theory, which seeks to trace as a function of time the location at which a given wave dominates the response because of its growth (amplitude), with the objectives of an eikonal theory which seeks to describe the propagation of positions of phase coherence among waves of *comparable* amplitude. Indeed, within the approximate development achieved in this paper, phase relationships are completely unimportant. Thus, for example, the treatment of reflections can proceed on a much simpler basis than is possible in the complementary limit. One need only know that reflection takes place at a given point, and questions regarding the nature of the phase change upon reflection—a principal cause of difficulties in the eikonal theory—need not be asked at all. This is an enormous simplification, and it is because of this that the condition $V\rightarrow 0$ can be handled with no more trouble than in treating any other velocity.

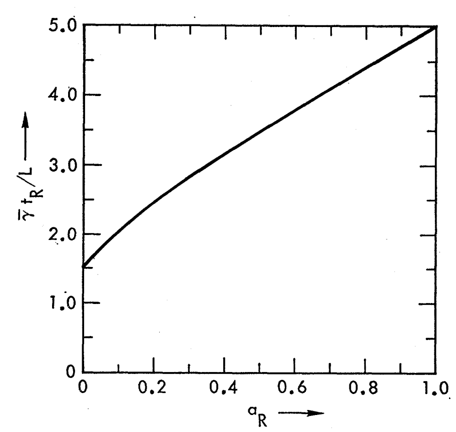

FIG. 7. Asymptotic growth factor as a function of the reflection point parameter.

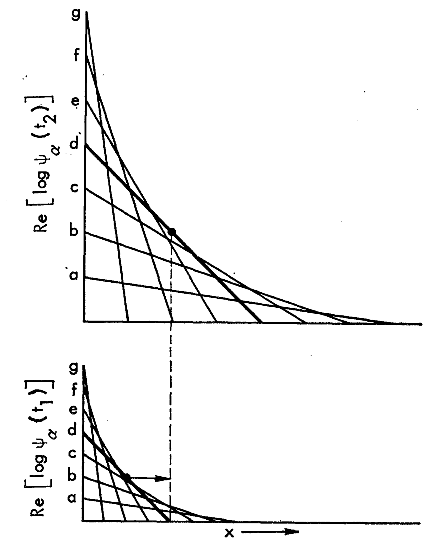

FIG. 8. Motion of the position of amplitude dominance of wave d for a response $\Psi(t)=\sum\Psi_\alpha(t)$, a sum over seven temporally growing, spatially decaying waves. The pulse shape at a given time is sensibly the envelope of the positions of amplitude dominance.

It is also very important that the meaning of velocity be understood as it is used here. In effect, V is a *position* variable. That is, associated with each proper frequency there is a given trajectory, with t as the parameter. Specifying t specifies a unique position. And just as at time t each significant ν_p has associated with it a given position x, there is at x a unique velocity V, a unique frequency ω (in general, complex), and a unique wave number k (also generally complex). Of course, waves of a broad band of proper frequencies contribute at a given instant to the response over the whole region disturbed by the pulse. Nevertheless, it is at position x that the wave ν_p *dominates* the response (see Fig. 8). Seen in this light, there should be no confusion about the fact that we can at once consider a precise wave number *and* a precise position, or a precise frequency *and* a precise time.

It is also useful at this juncture to consider a little more concretely the range of validity of our phase-neglect approximation. If we compare the relative growth of modes ν_p and ν_p', then the response at ν_p' may be neglected in comparison to that at ν_p, provided $\Delta\bar{\gamma}t\equiv(\bar{\gamma}-\bar{\gamma}')t>\theta^2$, where θ (typically of order unity) is a measure of the accuracy. Let us, for simplicity, neglect the variation of γ along the trajectory so that $\bar{\gamma}\approx\gamma$, and consider a range of proper frequencies near the peak of the pulse. Here $d\gamma/dV$ vanishes and $\Delta\gamma t>\theta^2$

is hardest to satisfy. Then if ΔV_i is the spread of initial velocities,

$$\Delta\gamma = -\tfrac{1}{2}\Delta V_i^2(d^2\gamma/dV^2)_0 = \tfrac{1}{2}\Delta V_i^2 \operatorname{Im}(1/\omega_0''), \quad (6.1)$$

and we have used the subscript zero to denote evaluation at the velocity of propagation of the peak of the pulse. In obtaining the above, we have used the formulas of Hall and Heckrotte[5] for the variation of a variable singularity,

$$d\tilde{\omega}/dV = -k, \quad dk/dV = 1/\omega'',$$

where $\omega'' = d^2\omega/dk^2$ along $D=0$. If, further, we suppose dV^2/da is approximately constant and that a varies quadratically near $x=0$ $[a=a_0(1\pm x^2/L^2)]$, then

$$dV^2/dx = -2\alpha^2 L^{-2} x,$$

where $\alpha^2>0$ for reflection. Hence, $V^2 = V_i^2 - \alpha^2 x^2/L^2$, and with $V=dx/dt$, the result integrates to

$$x = (V_i L/\alpha)\sin(\alpha t/L). \quad (6.2)$$

In this case, the position of amplitude dominance moves back and forth in simple harmonic fashion. For short times, the spreading in position due to the spread of initial velocities is clearly proportional to ΔV_i. For longer times, we must take into account changes in dV^2/da, which produces a dependence of α on V_i. Thus,

$$\Delta x = [(L/\alpha)(1-\epsilon)\sin(\alpha t/L) + \epsilon t\cos(\alpha t/L)]\Delta V_i, \quad (6.3)$$

where $\epsilon \equiv (V_i/\alpha)(d\alpha/dV_i)$. Hence, $\Delta x = \eta t \Delta V_i$, where η is of order unity for smaller times and becomes of order ϵ on a long-time scale. Substituting for ΔV_i in terms of Δx in Eq. (6.1), our validity condition becomes

$$|\Delta x| > [2/\operatorname{Im}(\omega_0'')]^{-1/2}\theta\eta t^{1/2}. \quad (6.4)$$

Now, the approach we have taken in terms of a pulse response is meaningful when, on the *macroscopic* scale, neighboring elements of the pulse are such that the wave contributing the most to the response at one point at a given time t is of too small an amplitude to contribute appreciably at the neighboring point. Hence $|\Delta x|$ must be small compared to macroscopic dimensions, and, in particular, $|\Delta x| \ll L$. This provides an upper limit on the time for which our analysis is valid. For times in which $|\Delta x|$ from Eq. (6.4) becomes comparable to the system dimensions, the pulse spreads throughout the system and one is forced to consider the effects of phase coherence. An example of a theory valid for these longer times is the WKB normal-mode analysis currently under development by theorists at Livermore and Princeton, with application to the effects of inhomogeneity upon convectively unstable plasma.[15]

We should be very clear about the complementarity of our theory with that of the normal-mode approach. While in our case the spreading of a pulse goes as $t^{1/2}$, both the total distance traveled by the pulse and the total growth are proportional to t. Thus many traversals of the system can be accomplished and the instability can grow to destructive amplitudes long before the limits of Eq. (6.4) apply. Clearly, our methods are particularly well adapted to the study of "quasi-modes," the situation in which the normal-mode approach predicts eventual damping (through phase mixing) even though disturbances can grow to large amplitude initially, and at rates characterizing convective growth.[16]

Although the decision as to which theory is applicable is a *quantitative* question, depending entirely upon the specific application, we note some particular advantages of our approach. In the first place, the particular configurational details—the nature of the inhomogeneity—enter only at a very late stage of solution of the problem. In common with eikonal or ray-optical theories, we can solve for the properties associated with propagation (γ, V) in terms of the macroscopic parameters alone—solving the basic problem but once. The results can then be applied to particular configurations by simple graphical or numerical techniques in terms of those graphs or tables supplied as the solution of the initial problem. One does not have to repeat what are often very difficult numerical calculations for each specific application. Moreover, the compactness of such presentation often illustrates very effectively the trends and the nuances of the particular dispersion and is thereby of great benefit in the visualization of physical behavior.

As a last point, if we exclude such esoteric phenomena as the destabilization of negative-energy waves by radiation (effectively a dissipative destabilization),[17] the additional feature provided by the normal-mode analysis at long times is simply a stabilization due to the phase mixing of unstable convective modes. Hence the normal-mode approach should show instability only if it is also indicated by the theory of nonstable pulse propagation. (The converse, of course would not be true.) An initial analysis from our point of view, therefore, not only gives important information in its own right, but also tells us where a phase-sensitive approach must be utilized. Such information may be of importance in restricting the domain within which often more complicated calculations must be performed.

VII. CONCLUSION

We have shown that the response to a δ-function pulse disturbance at the origin of coordinates of a nonstable medium grows in amplitude at the rate $\exp[\int_0^t \gamma(t')dt']$ at the position $\mathbf{r} = \int_0^t \mathbf{V}(t')dt'$, where the time integrals are taken along a trajectory of constant proper frequency according to the equation of propagation, Eq.

[15] H. L. Berk, T. K. Fowler, L. D. Pearlstein, R. F. Post, J. D. Callen, C. W. Horton, and M. N. Rosenbluth, in *Proceedings of the Conference on Plasma Physics and Controlled Nuclear Fusion, Novosibirsk, 1968* (International Atomic Energy Agency, Vienna, 1969), Vol. II, p. 151.

[16] B. Coppi, G. Laval, R. Pellat, and M. N. Rosenbluth, Nucl. Fusion **6**, 261 (1966).

[17] H. L. Berk, L. D. Pearlstein, J. D. Callen, C. W. Horton, and M. N. Rosenbluth, Phys. Rev. Letters **22**, 876 (1969).

(4.9). Provided $\gamma>0$ at some point in the region of interest for some proper frequency ν_p, such a system will be either conditionally or unconditionally unstable. For unconditional instability to occur, there has to be a correspondence with one or the other of two possible situations. Either there is (a) *absolute instability*, in which the spreading of the pulse is more rapid than its convection, or there is (b) *convective instability with reflection and average growth*, for which there must exist some proper frequency such that the region of instability is bounded by reflection points $\mathbf{V}=0$ and for which $\oint \gamma dt>0$ along the associated closed trajectory. If conditions (a) or (b) are not met, the system is conditionally unstable and the maximum growth experienced by the pulse will be bounded. A necessary requirement for unconditional instability is the presence of a reflection point $\mathbf{V}=0$ for some unstable ν_p within the system. In turn, therefore, the pair of equations

$$D=0\,,\quad \partial D/\partial\mathbf{k}=0$$

must be simultaneously satisfied for some unstable proper frequency.

ACKNOWLEDGMENTS

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APPENDIX: PROPAGATION OF STABLE WAVES

While the approach taken in this paper is perhaps less convenient to the derivation of the behavior of stable waves than is conventional theory (when one is to proceed from first principles alone), such an eventuality is hardly surprising for two theories directed toward the examination of entirely different physical regimes. Nevertheless, the classical WKB result can be obtained directly from Eq. (1.10) using a Huygens principle, just as we have done for nonstable waves, if we also make the auxiliary assumption (implicit in conventional derivations) that one can label an elementary excitation by its frequency and that this frequency is constant during propagation. Thus, if V is the propagation velocity of the disturbance (in this case, just the ordinary group velocity), the response along the trajectory at frequency ω is

$$\begin{aligned}
\psi(t)&=\int dk\frac{\exp\{i[kV(t)-\omega]\Delta t\}}{\mathfrak{D}[k,\omega;\,x(t)]}\psi(t-\Delta t)\\
&=\int dk'\cdots\int dk\frac{\exp\{i[kV(t)-\omega]\Delta t\}}{\mathfrak{D}[k,\omega;\,x(t)]}\\
&\qquad\cdots\frac{\exp\{i[k'V(\Delta t)-\omega]\Delta t\}}{\mathfrak{D}[k',\omega;\,x(\Delta t)]}\psi(0)\\
&\sim\exp\{i\sum_{j=1}^{t/\Delta t}k[x(j\Delta t)]V(j\Delta t)\Delta t-i\omega t\}\\
&\rightarrow\exp\left(i\int k[x(t)]V(t)dt-i\omega t\right)\\
&=\exp\left(i\int k(x)dx-i\omega t\right).
\end{aligned}$$

Of course, the validity of the constant-frequency assumption and factors associated with the assertion regarding the propagation of dominant response at the group velocity should be carefully examined for this case if we wished to develop this approximation from first principles, or if we had no WKB theory to guide us. The analogous problem in the theory of nonstable waves, associated with the introduction of the concept of proper frequency, is a principal topic of Secs. II and III.