Slowly rotating black holes in degenerate higher order scalar-tensor theories

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We study slowly rotating black hole solutions within degenerate higher order scalar tensor (DHOST) theories. Starting from a static, spherically symmetric metric solution of a DHOST theory, we employ the Hartle-Thorne *Ansatz* to model a slowly rotating spacetime. We show that the differential equation governing the frame-dragging function ω (which is supposed to depend on the radial coordinate only) is integrable for any DHOST theory allowing us to obtain its explicit form. We also consider angular dependence in ω and show that, for specific DHOST theories, regularity at the horizon and at infinity forbids it, as in general relativity. As an illustration of the formalism introduced here, we study the slowly rotating version of black hole solutions with primary hair obtained recently, examining the influence of the rotation on the innermost stable circular orbit (ISCO) and on the circular light trajectories in the equatorial plane.

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I. INTRODUCTION

General relativity (with a cosmological constant) is the unique theory that satisfies the assumptions of Lovelock's theorem in four dimensions [1]. However, there are many reasons to consider theories beyond general relativity, whether phenomenological or more theoretical. One of the simplest ways to construct alternatives to Einstein's gravity is to assume that the metric field comes with a scalar field to mediate the gravitational interaction. Scalar-tensor theories have been extensively studied in recent years, leading to the classification of degenerate higher order scalar-tensor (DHOST) theories [2–4] (and even more generally U-DHOST theories which are degenerate in the unitary gauge [5]), which currently represent the most general class of theories propagating only three degrees of freedom. In physically relevant cases, these correspond to the two tensorial degrees of freedom of the metric field and one additional scalar degree of freedom. DHOST theories have been applied to cosmological and astrophysical contexts (see the review [6]), such as black holes or compact objects.

In general relativity (GR), black holes are known to have "no hair" [7]. The only stationary and axisymmetric solution of the vacuum Einstein equations is the Kerr metric [8], described by two parameters, its mass M and its angular momentum J. In the context of scalar-tensor theories, no hair theorems have also been derived for some subclasses of DHOST theories, see for instance [9,10], but their assumptions are generally more restrictive compared to GR. In the recent years, many hairy static black hole solutions have been found within DHOST theories, breaking the hypotheses of these theorems, as illustrated in [11–13] for instance. These solutions exhibit a nontrivial scalar hair, and the spacetime metric can differ from the Schwarzschild solution expected in GR. However, very few solutions describing a rotating black hole are known to date. One of the first rotating solutions are the so-called "stealth Kerr" solutions which were found in [11,14] by assuming the spacetime metric to be that of Kerr, and then solving for the scalar field. Starting from a stealth Kerr solution, one can apply a disformal transformation to the metric to obtain a new, nonstealth, solution, as shown in [15–19]. Among its notable features, the resulting metric is generically noncircular [20]. These solutions currently constitute the only fully analytic examples of rotating black holes with scalar hair in DHOST theories. By contrast, nondisformally related Kerr solutions have been obtained only numerically, for instance in the cubic Galileon theory [21,22] and in scalar-Gauss-Bonnet theories [23–25].

The limited number of analytical rotating solutions stems from the fact that solving the equations of motion becomes considerably more challenging when the spherical symmetry assumption of the static case is abandoned. However, analytical approximate solutions can still be obtained and approximate rotating black hole solutions in some specific scalar-tensor theories have already been found. An analytical slowly rotating black hole solution in a scalar-Gauss-Bonnet theory was presented in [26] and further analyzed in [27]. Moreover, for a specific subclass of DHOST theories, a no-hair theorem at first order in the angular momentum J has been derived in [28].

In this paper, we present general solutions for slowly rotating black holes in DHOST theories. Specifically, we make use of the well-known Hartle-Thorne *Ansatz* [29–31], originally introduced by Lense[1] and subsequently by Hartle and Thorne to describe the gravitational field of a rotating relativistic star at first order in the angular momentum J within the framework of general relativity. We show that, at first order in J, the equations of motion exhibit a remarkable integrability property, which allows us to derive a very general expression for slowly rotating black hole metrics in DHOST theories in terms of the corresponding static solutions. Notably, this integrability property is likely not exclusive to scalar-tensor theories, and the method presented here could potentially be extended to other modified gravity frameworks. When specializing this general result to an interesting subclass of theories possessing shift and parity symmetry, we find that the first-order rotational term is always identical to that of Kerr, even when the static solution differs from Schwarzschild. In order to illustrate these results, we study the slowly rotating version of black hole solutions with primary hair obtained recently in [32], and we examine the influence of the rotation on the innermost stable circular orbit (ISCO) and on the circular light trajectories in the equatorial plane.

Let us emphasize that the study of rotating black holes in modified gravity theories is particularly interesting as we are collecting a lot of data from compact objects in our universe, using both gravitational wave detectors from the LIGO-Virgo-KAGRA collaboration [33,34], and direct imaging by the Event Horizon Telescope [35]. Up to this day, all the data collected from these compact objects are compatible with them being black holes, or neutron stars, in general relativity [36,37], but possible deviations from GR will be further explored with more precise measurements, for instance with future detectors LISA [38] or Einstein Telescope [39].

The paper is organized as follows. The next section is devoted to the derivation of slowly rotating solutions in DHOST theories. We simplify the equation governing the frame-dragging function ω, present a general integral solution, and discuss the origin of the integrability property of this equation. In Sec. III, we apply our results to shift-symmetric theories, for which the expression of ω simplifies. We also study how slowly rotating solutions transform under disformal transformations and examine the impact of rotation on the ISCO and on the circular light trajectories in the equatorial plane. In Sec. IV, we investigate a possible angular dependence of ω and show that regularity conditions at the horizon and at infinity forbid any θ-dependence of ω. We conclude with a brief summary and provide Appendixes containing technical details.

[1]Thanks to David Kubzňák for giving us the reference to the article of Lense and Thirring

II. SLOWLY ROTATING SOLUTIONS IN DHOST THEORIES

After a brief review of DHOST theories, we study their solutions for slowly rotating spacetime following the Hartle-Thorne *Ansatz*. Interestingly, we find an explicit integral formula for the first order term in the slowly rotating metric, for any (quadratic and cubic) DHOST theory.

A. DHOST theories: Brief review and notations

DHOST theories, introduced and classified in [2–4] (see also [6,40] for reviews), are scalar-tensor theories that propagate only three degrees of freedom (two tensor modes and one scalar mode) even though their Lagrangian can involve nontrivial second-order derivatives of the scalar field.[2] The absence of a pathological extra scalar mode is ensured by the degeneracy conditions imposed on the functions appearing in the Lagrangian.

The DHOST action, up to cubic terms in the second derivatives of the scalar field ϕ, takes the form

$$
\begin{aligned}
S[g_{\mu\nu},\phi] = \int \mathrm{d}^4x\sqrt{-g}\Big[&F_0(\phi,X) + F_1(\phi,X)\Box\phi \\
&+ F_2(\phi,X)R + F_3(\phi,X)G^{\mu\nu}\nabla_\mu\nabla_\nu\phi \\
&+ \sum_{i=1}^{5} A_i(\phi,X)L_i^{(2)} + \sum_{i=1}^{10} B_i(\phi,X)L_i^{(3)}\Big], \qquad (1)
\end{aligned}
$$

where we are using the convention[3] $X = -(1/2)\partial_\mu\phi\partial^\mu\phi$ for the canonical kinetic term of the scalar field. The expressions for the elementary quadratic and cubic Lagrangians $L_i^{(2)}$ and $L_i^{(3)}$ can be found in [4] for example. The action is parametrized by the functions F_i, A_i and B_i, which are in general functions of ϕ and X and must satisfy degeneracy conditions, given explicitly in [2,4].

B. The Hartle-Thorne *Ansatz* for the metric

Many static and spherically symmetric solutions (black hole and more generally exotic compact objects) have been discovered (analytically and numerically) in the context of DHOST theories [11,13,41–46]. By contrast, very few rotating (axisymmetric) solutions have been reported in the literature. The first rotating solution found analytically was the stealth Kerr solution [11,13] where the metric is identical to the Kerr metric. From the stealth Kerr solution, new solutions were constructed [15,16] by applying a disformal transformation (see Sec. III A). Numerical

[2]DHOST theories can be complemented by U-DHOST theories [5] which are degenerate in the unitary gauge where the scalar field is used to define the time coordinate.

[3]The alternative convention $X = \partial_\mu\phi\partial^\mu\phi$ is also often encountered in the literature, in particular in [2–4].

solutions have also been found, for the cubic galileon theory [21,22], and scalar-Gauss-Bonnet theories [23–25].

Here, we study solutions that describe a rotating black hole spacetime with a nontrivial scalar field profile at first order in the angular momentum J. Starting from an arbitrary static and spherically symmetric metric solution $g^{(0)}_{\mu\nu}$, we look for a new solution of the form

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + J g^{(1)}_{\mu\nu} + \mathcal{O}(J^2). \tag{2}$$

Following the Hartle-Thorne *Ansatz*, we assume the new metric to be of the form

$$\begin{aligned}\mathrm{d}s^2 = &-h(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2(\mathrm{d}\theta^2 + \sin^2(\theta)\mathrm{d}\varphi^2)\\ &- 2\omega(r)r^2\sin^2(\theta)\mathrm{d}t\mathrm{d}\varphi + \mathcal{O}(J^2),\end{aligned} \tag{3}$$

where the function $\omega(r)$ is implicitly proportional to J, and characterizes the frame dragging effect.

To our knowledge, this *Ansatz* was first introduced by Lense and Thirring in 1918 [29], and was popularized later by Hartle and Thorne to describe the gravitational field of a rotating star within general relativity [30,31]. One may wonder how general this *Ansatz* is, in particular whether any stationary rotating spacetime reduces to this form at linear order in the angular momentum. The expression (3) was derived by Hartle using the specific symmetry of circular spacetimes [47], but it actually also works for the noncircular solution found in [15,16], suggesting that it might be more general. The possible generalization by adding a dependence of the function ω on the angle θ will be discussed in Sec. IV.

At this stage, we follow the usual procedure, which consists in, first, substituting the *Ansatz* (3) into the equations of motion for the metric, namely

$$\mathcal{E}_{\mu\nu} \equiv \frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = 0, \tag{4}$$

and then in expanding these equations up to linear order in ω. Notice that we do not need to take into account the equation of motion for the scalar field since it is a consequence of the metric equations of motion.

In theories invariant under the global shift symmetry $\phi \to \phi + \mathrm{const}$, which will be discussed in Sec. III, the scalar field can acquire a linear time dependence,

$$\phi = qt + \psi(r), \tag{5}$$

without spoiling the stationarity of the metric, because all physical quantities in this case depend on the gradient of ϕ and not ϕ itself (see for instance [43]). In the most general case, where the theory is not shift-symmetric, the parameter q must vanish. Combining (5) with the metric *Ansatz* (3), and expanding at first order in ω gives the following expression for X,

$$X = \frac{q^2}{2h} - \frac{1}{2}f\psi'^2, \tag{6}$$

which happens to be independent of ω.

Let us recall that, in the absence of rotation (when $\omega = 0$), the diagonal equations $\mathcal{E}_{tt}$, $\mathcal{E}_{rr}$ together with the nondiagonal one $\mathcal{E}_{tr}$ (or equivalently the equation of motion of the scalar field) enable us to determine the two metric components $h(r)$ and $f(r)$ and the scalar field $\psi(r)$. All the other equations can be shown to be redundant.

In the presence of a slow rotation, one can easily see that $\mathcal{E}_{tt}$, $\mathcal{E}_{rr}$, and $\mathcal{E}_{tr}$ remain unchanged compared to the case where $\omega = 0$ (they do not involve terms linear in ω), and are thus satisfied by the metric components $h(r)$ and $f(r)$ of the static solution. Furthermore, as scalar quantities can only depend on even powers of the angular momentum J, the scalar field ϕ also remains unchanged in the slow rotation approximation.

The difference with respect to the static case arises from the nondiagonal equation $\mathcal{E}_{t\varphi}$, which is no longer trivial and yields a second order linear differential equation for $\omega(r)$, with coefficients depending on the static solution, i.e. on the functions $h(r)$, $f(r)$ and $\psi(r)$. Hence $\omega(r)$ can be obtained by solving directly $\mathcal{E}_{t\varphi} = 0$, but we will see that it is more convenient to consider a linear combination of $\mathcal{E}_{t\varphi}$ and $\mathcal{E}_{\varphi\varphi}$ which is straightforward to integrate.

Before discussing DHOST theories, let us recall that, for the Kerr metric in general relativity, the function ω is given by

$$\omega_{\mathrm{Kerr}}(r) = \frac{2J}{r^3}, \tag{7}$$

where J is the Arnowitt-Deser-Misner (ADM) angular momentum of the black hole.[4]

In the following, the integration constant appearing in $\omega(r)$ when solving the equations will be expressed in term of J to match the asymptotic behavior of the Kerr solution, whenever possible.

C. Solutions for DHOST theories

We now discuss the specific case of DHOST theories, which includes most known scalar-tensor theories. To write and solve the equation solved by the function $\omega(r)$, it is convenient to consider the linear combination $\mathcal{E}_{t\varphi} + \omega\mathcal{E}_{\varphi\varphi}$,

[4]The expression (7) generalizes to

$$\omega(r) = \frac{2J}{r^3} + \frac{J\Lambda}{3}$$

for Kerr-(A)dS slowly rotating metric, where Λ is the cosmological constant which can be shown to reduce to the very simple form

$$\mathcal{E}_{t\varphi} + \omega\mathcal{E}_{\varphi\varphi} = \frac{1}{2r^2}\sqrt{hf}\sin^2(\theta)\frac{\mathrm{d}}{\mathrm{d}r}[Q(r)\omega'(r)], \quad (8)$$

with $Q(r)$ given by[5]

$$Q = \sqrt{\frac{f}{h}}\left[(F_2 + 2XA_1)r^4 + 2f\psi' XB_2 r^3 - \frac{1}{2}f\psi' X'(F_{3X} + 4XB_6 - 2B_2)r^4\right], \quad (10)$$

where a subscript X denotes a partial derivative with respect to X and a prime a derivative with respect to r. Note that only two quadratic functions, F_2 and A_1, and only three cubic functions, F_3, B_2, and B_6 enter in the above expression, which is also the case for the equations governing axial perturbations in DHOST theories [48,49].

The simplicity of the equation for ω,

$$\frac{\mathrm{d}}{\mathrm{d}r}[Q(r)\omega'(r)] = 0, \quad (11)$$

can be understood in light of some symmetry arguments, as will be explained in the next section (II D) where we will also provide details to compute $Q(r)$. The integration of (8) leads to the expression

$$\omega(r) = k\int\frac{\mathrm{d}r}{Q(r)}, \quad (12)$$

where k is an integration constant, proportional to the angular momentum J.

In Horndeski theories [50], where $A_1 = -F_{2X}$, $2B_2 = F_{3X}$ and $B_6 = 0$, the expression of Q reduces to

$$Q = \sqrt{\frac{f}{h}}[(F_2 - 2XF_{2X})r^4 + f\psi' XF_{3X}r^3]. \quad (13)$$

In general relativity, where the only nonvanishing DHOST function is $F_2 = 1$, the expression of Q simplifies drastically and reduces to $Q = r^4$. Therefore, we recover (7) with $k = -6J$.

[5]The expression of Q, starting from the action (1), is given by

$$Q = \sqrt{\frac{f}{h}}\left[(F_2 + 2XA_1)r^4 + f\psi' X(4B_2 + 3B_3)r^3 - \frac{1}{2}f\psi' X'(F_{3X} + 3B_3 + 4XB_6)r^4 - \frac{X}{h\psi'}(Xh)'(3B_3 + 2B_2)r^4\right], \quad (9)$$

and reduces to (10) when one takes into account one of the degeneracy conditions, namely $3B_3 + 2B_2 = 0$.

Let us remark that slowly rotating black holes are closely related to the study of linear perturbations about static and spherically symmetric solutions, as the Hartle-Thorne metric (3) can be seen as an odd dipolar perturbation to the static black hole. Such perturbations have been studied in [51–53] for quadratic DHOST theories and their results are fully consistent with what we have obtained here.

D. Integrability of the equation from a conservation law

Let us now try to understand why Eq. (11) is so simple. For that purpose, we first notice that under a change of the angular coordinates $\varphi \to \tilde{\varphi} = \varphi + \omega_0 t$ where ω_0 is a constant, the Hartle-Thorne metric (3) becomes,

$$\mathrm{d}s^2 = -[h(r) - r^2(\omega_0^2 + 2\omega_0\omega(r))\sin^2(\theta)]\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2[\mathrm{d}\theta^2 + \sin^2(\theta)\mathrm{d}\tilde{\varphi}^2 - 2(\omega(r) + \omega_0)\sin^2(\theta)\mathrm{d}t\mathrm{d}\tilde{\varphi}]. \quad (14)$$

At first order in $\omega(r)$ and ω_0, the line element above reduces to the original Hartle-Thorne line element where the function ω has been shifted by a constant, i.e. $\omega(r) \to \omega(r) + \omega_0$. As a consequence, from the invariance under diffeomorphisms, this implies that if $\omega(r)$ is a solution of the linearized equations, then $\omega(r) + \omega_0$ is also a solution, because they are related to the same metric written in two different coordinate systems (at linear order in ω). This corresponds to a continuous symmetry in the space of solutions which can be exploited in order to obtain a "conserved" (i.e. a constant) quantity in the theory, which could be useful to integrate some of the equations.

However, at this stage, this symmetry is only true at linear order. Interestingly, we can make it exact by changing the initial Hartle-Thorne *Ansatz* for the metric. Indeed, let us consider instead the following line element

$$\begin{aligned}\mathrm{d}s^2 &= -h(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2[\mathrm{d}\theta^2 + \sin^2(\theta)(\mathrm{d}\varphi - \omega(r)\mathrm{d}t)^2] \\ &= -(h(r) - r^2\sin^2(\theta)\omega(r)^2)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2[\mathrm{d}\theta^2 + \sin^2(\theta)\mathrm{d}\varphi^2 - 2\omega(r)\sin^2(\theta)\mathrm{d}t\mathrm{d}\varphi] \quad (15)\end{aligned}$$

which differs from (3) by terms[6] quadratic in ω. Therefore, the equations of motion are unchanged at the linear level in ω, which means that the two *Ansätze* lead to the same slowly rotating solutions. Furthermore, now, the change of

[6]It must be stressed that the quadratic term here does not have any physical meaning. It has been added only to make the symmetry mentioned above exact. If one expands a rotating black hole metric up to quadratic terms in the angular momentum, one does not get generally a metric of this form. However, since we are only taking into account linear terms in ω at the end, the quadratic term does not affect our physical results.

the angular coordinate $\varphi \to \tilde{\varphi} = \varphi + \omega_0 t$ is exactly equivalent to the shift $\omega(r) \to \omega(r) + \omega_0$. As we now show, the conservation equation (11) is directly linked to this "shift-symmetry" (not to be confused with the shift-symmetry of the scalar field in the DHOST action).

First of all, substituting the *Ansatz* (15) into the action and computing the equation of motion for ω, we get

$$\frac{\delta S}{\delta \omega} = \frac{\delta S}{\delta g^{\mu\nu}} \frac{\delta g^{\mu\nu}}{\delta \omega} = -\frac{2\sqrt{-g}}{h}(\mathcal{E}_{t\varphi} + \omega \mathcal{E}_{\varphi\varphi}), \qquad (16)$$

where we have used the explicit expression of the inverse metric. We recognize here the combination (8) which leads directly to an integral expression for ω.

Then an immediate consequence of the symmetry mentioned above is that the Lagrangian density $\mathcal{L}$ does not depend directly on ω but on its first and second derivatives. Hence, the equation for ω is necessarily proportional to the total derivative of a quantity $\mathcal{J}$,

$$\frac{\delta S}{\delta \omega} = -\frac{\mathrm{d}\mathcal{J}}{\mathrm{d}r}, \qquad \mathcal{J} = \frac{\partial \mathcal{L}}{\partial \omega'} - \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\partial \mathcal{L}}{\partial \omega''}\right). \qquad (17)$$

Finally, a straightforward calculation leads to

$$\mathcal{J} = \sin^3(\theta) Q(r) \omega'(r) + \mathcal{O}(\omega^3), \qquad (18)$$

where we have used the explicit expression of the determinant of the metric and $Q(r)$ stands for the expression given in (10). In summary, the integrability of the equation for ω is directly linked to a conservation law associated with the symmetry identified above.

III. SHIFT-SYMMETRIC THEORIES

This section is devoted to study properties of slowly rotating solutions in shift symmetric DHOST theories. We recall that, in these theories, the functions entering in the action (1) depend on the kinetic density X only. We start with a study of disformal transformations of slowly rotating metrics. Then, we explicitly compute $\omega(r)$ in the case of quadratic DHOST theories.

A. Disformal transformations

In this section, we consider disformal transformations

$$g_{\mu\nu} \longrightarrow \tilde{g}_{\mu\nu} = C(\phi, X) g_{\mu\nu} + D(\phi, X) \partial_\mu \phi \partial_\nu \phi, \quad (19)$$

where C and D are in general functions of ϕ and X, constrained by the requirement that the disformal transformation is invertible and preserves the signature of the metric, which implies that $C > 0$ and $C - 2XD > 0$. For shift-symmetric theories, the functions C and D are supposed to depend on X only.

Disformal transformations induce transformations in the space of DHOST theories as follows:

$$\tilde{S}[\tilde{g}_{\mu\nu}, \phi] = S[g_{\mu\nu}, \phi], \qquad (20)$$

which means that, given a theory $\tilde{S}$ for the metric $\tilde{g}$, one defines a new theory S for the metric g. When the disformal transformation is invertible, the two theories are classically equivalent in vacuum only while they differ when one takes into account (generic) coupling to matter. In particular, their causal structure is different since light geodesics in one metric do not coincide with the light geodesics in the other metric, unless the transformation is purely conformal, i.e. $D = 0$.

Let us now examine what the approximate slow-rotating metric becomes in a disformal transformation. We start with a DHOST action S and we consider a slowly rotating metric characterized by the functions $h(r)$, $f(r)$, and $\omega(r)$. We have shown above that ω satisfies the simple differential equation,

$$\frac{\mathrm{d}\omega}{\mathrm{d}r} = \frac{k}{Q(r)}, \qquad (21)$$

where k is an integration constant and $Q(r)$ has been introduced in (10). Performing a disformal transformation of the metric (3), we find that the new metric $\tilde{g}_{\mu\nu}$ can be written (at linear order in ω) in the form (3)

$$\mathrm{d}\tilde{s}^2 = -\tilde{h}\mathrm{d}\tilde{t}^2 + \frac{\mathrm{d}\tilde{r}^2}{\tilde{f}} + \tilde{r}^2 \mathrm{d}\Omega^2 - 2\tilde{r}^2 \tilde{\omega} \sin^2\theta \mathrm{d}\tilde{t}\mathrm{d}\tilde{\varphi} + \mathcal{O}(\omega^2), \qquad (22)$$

when using new coordinates $(\tilde{t}, \tilde{r}, \tilde{\varphi})$ defined by

$$\begin{aligned} \mathrm{d}\tilde{t} &= \mathrm{d}t - \frac{qD(r)\psi'(r)}{C(r)h(r) - D(r)q^2}\mathrm{d}r, \\ \mathrm{d}\tilde{\varphi} &= \mathrm{d}\varphi - \frac{qD(r)\psi'(r)\omega(r)}{C(r)h(r) - D(r)q^2}\mathrm{d}r, \\ \tilde{r} &= R(r) \equiv r\sqrt{C(r)}. \end{aligned} \qquad (23)$$

The new metric components are related to the old ones via the expressions

$$\tilde{h}(\tilde{r}) = Ch - q^2 D, \qquad \frac{1}{\tilde{f}(\tilde{r})} = \frac{C}{R'^2}\left(\frac{1}{f} + \frac{Dh\psi'^2}{Ch - q^2 D}\right),$$
$$\tilde{\omega}(\tilde{r}) = \omega, \qquad (24)$$

where the functions of the right-hand side (rhs) of the two identities are evaluated on r, viewed as a function of $\tilde{r}$. Let us make two comments: we have used the notation $C(r)$ for $C(X(r))$ and $D(r)$ for $D(X(r))$; as C is supposed to be positive, the coordinate $\tilde{r}$ is well defined. We can formally integrate the two differential equations in (23) so that

$$\tilde{t} = t + \mu(r), \qquad \tilde{\varphi} = \varphi + \nu(r), \qquad (25)$$

where μ and ν are given in terms of integrals. As a result, $\tilde{\omega}$ satisfies an equation similar to (21),

$$\frac{\mathrm{d}\tilde{\omega}}{\mathrm{d}\tilde{r}} = \frac{k}{\tilde{Q}(r)}, \tag{26}$$

where $\tilde{Q}$ is the same function as in (10) but expressed in terms of the new functions $\tilde{F}_i$, $\tilde{A}_i$, and $\tilde{B}_i$ characterizing the action $\tilde{S}$. Of course, the two equations (21) and (26) must be equivalent, which leads to the consistency relation

$$R' \frac{Q}{\tilde{Q}} = 1. \tag{27}$$

This relation can be verified explicitly by using the expressions of $\tilde{F}_i$, $\tilde{A}_i$ and $\tilde{B}_i$ given in terms of F_i, A_i and B_i, as we show in the Appendix A.

In summary, given a DHOST action S that admits a slowly rotating solution associated with the function f, h and ω, then the disformally related action $\tilde{S}$ admits also a slowly rotating solution associated to $\tilde{f}$, $\tilde{h}$ and $\tilde{\omega}$ related to the original functions by (24). In particular, the frame dragging functions are related by

$$\omega(r) = \tilde{\omega}(R(r)), \quad \text{with} \quad R(r) = r\sqrt{C(r)}. \tag{28}$$

Expressing $\tilde{\omega}$ in terms of ω can be done only if the coordinate transformation $\tilde{r} = \sqrt{C(r)}r$ is invertible, or equivalently $\tilde{r}$ is a monotonous function of r.

B. Quadratic theories

In this subsection, we focus on quadratic shift-symmetric DHOST theories (1), which can be related to quadratic Horndeski theories via disformal transformations.

For quadratic Horndeski theories, the expression of Q (13) reduces to the simple form

$$Q = \sqrt{\frac{f}{h}}(F_2 - 2XF_{2X})r^4. \tag{29}$$

This expression can be further simplified by taking into account the equations of motion, as we now show. Due to the shift-symmetry of the theory, the equation of motion for the scalar field is a conservation equation, which means that it can be written as the divergence of a Noether current J^μ [43]

$$\mathcal{E}_\phi = \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta \phi} = 0 \Leftrightarrow \nabla_\mu J^\mu = 0. \tag{30}$$

Furthermore, it was shown in [54] that $\mathcal{E}_t^r = qJ^r$, which necessarily implies $J^r = 0$ when $q \neq 0$. In the case where $q = 0$, an argument using the physical necessity to keep $J_\mu J^\mu$ finite on the horizon also gives $J^r = 0$ (see [9] for instance).

Finally, the combination

$$\begin{aligned} &\mathcal{E}_{rr} + \frac{1}{fh}\mathcal{E}_{tt} - \frac{1}{f}\left(\psi' + \frac{q^2}{fh\psi'}\right)J^r \\ &= -\frac{2}{r}\sqrt{\frac{h}{f}}\frac{\mathrm{d}}{\mathrm{d}r}\left(\sqrt{\frac{f}{h}}(F_2 - 2XF_{2X})\right) = 0, \end{aligned} \tag{31}$$

shows that the coefficient in front of r^4 in the expression (29) of Q is constant. As a consequence, Q is proportional to r^4, as in general relativity, and the function $\omega(r)$ is identical to the one of Kerr (7) for all quadratic shift-symmetric Horndeski theories (even when $f(r)$ and $h(r)$ are different from the Schwarzschild metric coefficients). Hence, we have

$$\omega(r) = \frac{2J}{r^3}. \tag{32}$$

This result can be extrapolated to other quadratic shift-symmetric DHOST theories by resorting to disformal transformations. Indeed, given a slowly rotating metric $\tilde{g}_{\mu\nu}$ defined in a Horndeski theory $\tilde{S}$, the slowly rotating metric $g_{\mu\nu}$ in the DHOST theory S verifies

$$\omega(r) = k\int \frac{\sqrt{h/f}}{(F_2 + 2XA_1)r^4}\mathrm{d}r = \frac{2J}{r^3}C(r)^{-3/2}, \tag{33}$$

according to (12) and (10) for the first equality, and to (28) and (32) for the second equality. Notice that all the functions entering into the integral are viewed as functions of r and the integration constant has been fixed to zero.

In particular, the expression for ω is the same as in general relativity for shift-symmetric quadratic beyond Horndeski theories [55] which are related to Horndeski theories by a pure disformal transformation (i.e. with $C = 1$). This is worth noting because many static solutions were found in this framework, see for instance [11,13,32,42,43,46].

Otherwise, the function ω is different from the Kerr expression. Nonetheless, if we want the disformal transformation to preserve the asymptotic flatness of the metric, we must impose that $C = 1 + o(1)$ at infinity (when $r \to \infty$). In this case, the asymptotic behavior of $\omega(r)$ remains the same as in Kerr.

C. Circular timelike and lightlike geodesics

As an application of the general formalism developed above, we consider the exact static black holes obtained recently [13], in the context of specific shift-symmetric quadratic DHOST theories defined by the nontrivial functions

$$F_0 = -\frac{2\alpha}{\lambda^2}X^p, \quad F_2 = 1 - 2XA_1, \quad A_1 = -A_2 = \frac{\alpha}{2}X^{p-1},$$
$$A_3 = -A_4 = \frac{\alpha}{2}(2p-1)X^{p-2}, \tag{34}$$

and $A_5 = 0$, where α is a coupling constant and p some constant, conveniently chosen to be an integer or half-integer. Notice that these theories belong to the quadratic Beyond Horndeski theories.

The black holes obtained in these theories are interesting as they are the first solutions in DHOST theories characterized by a primary hair, i.e. a scalar charge that can be fixed independently of the mass of the black hole. This scalar parameter q is associated with the linear time dependence of the scalar field, namely

$$\phi(r,t) = qt + \psi(r), \tag{35}$$

where the radial dependent part $\psi(r)$ satisfies

$$\psi'(r)^2 = \frac{q^2}{h(r)^2}\left[1 - \frac{h(r)}{1+(r/\lambda)^2}\right], \tag{36}$$

so that the scalar kinetic term is given by[7]

$$X = \frac{q^2/2}{1+(r/\lambda)^2}, \tag{37}$$

which is regular and decays when going away from the black hole.

The explicit black hole metric depends on the value of p that defines the DHOST theory (34). For $p = 1/2$, the metric is simply Schwarzschild, thus defining a stealth solution with a nontrivial scalar profile. An interesting value is $p = 2$, for which the metric is defined by

$$h(r) = f(r) = 1 - \frac{2M}{r} + \xi_2\left(\frac{\pi/2 - \arctan(r/\lambda)}{r/\lambda} + \frac{1}{1+(r/\lambda)^2}\right),$$
$$\xi_2 \equiv \frac{3}{4}\alpha q^4. \tag{38}$$

From our discussion in the previous subsection, we know that slow-rotating version of these solutions, i.e. of the form (3), are such that $\omega(r)$ is the same as in general relativity.

It is then instructive to study timelike and lightlike geodesics in the equatorial plane $\theta = \pi/2$. As a consequence of the time and angular invariance of the metric, one can introduce as usual the conserved energy and angular momentum

$$E = h\dot{t} + \omega r^2\dot{\varphi}, \qquad L = r^2\dot{\varphi} - \omega r^2\dot{t}, \tag{39}$$

where a dot denotes a derivative with respect to the geodesic parameter (the proper time τ for a timelike geodesic and an affine parameter λ for a lightlike geodesic). Inverting the above system to obtain $\dot{t}$ and $\dot{\varphi}$ in terms of E and L, and substituting in the expression $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -\kappa$ (with $\kappa = 1$ for timelike geodesics and $\kappa = 0$ for lightlike ones), one gets, at first order in ω, the radial equation

$$\frac{1}{2}\dot{r}^2 + V_{\rm eff}(r) = 0, \tag{40}$$

with the effective potential

$$V_{\rm eff}(r) = \frac{L^2 f}{2r^2} - \frac{E^2 f}{2h} + \frac{ELf}{h}\omega + \frac{\kappa}{2}f. \tag{41}$$

For massive particles, the innermost stable circular orbit is defined by the three conditions

$$\text{ISCO}: \; V_{\rm eff}(r) = 0, \qquad V'_{\rm eff}(r) = 0,$$
$$V''_{\rm eff}(r) = 0 \quad (\text{with } \kappa = 1). \tag{42}$$

The first two conditions define a circular orbit whereas the last one characterizes the onset of instability, i.e. the transition from stable circular orbits with $V''_{\rm eff}(r) > 0$ to unstable ones with $V''_{\rm eff}(r) < 0$.

As a first step, one can determine the ISCO as a function of ξ_2, for the nonrotating metric (i.e. $J = 0$) (38). The result is plotted on Fig. 1, together with the position of the BH horizon, which also depends on the parameter ξ_2 (for the choice of parameter $\lambda = M$). We then compute the ISCO when a slow rotation, parametrized by $a = J/M$, is introduced. In the limit $\xi_2 \to 0$, one recovers the well-known results for Schwarzschild $r_{\rm ISCO} = 6M$ and for Kerr in the slow rotation limit

$$\frac{\partial r^-_{\rm ISCO}}{\partial a}(\xi_2 = 0) = 4\sqrt{\frac{2}{3}}, \tag{43}$$

where $r^-_{\rm ISCO}$ denotes here the ISCO for retrograde orbits, i.e. orbiting the BH in the sense opposite to the BH rotation. The analogous expression for prograde orbits has the opposite sign. In other words, the ISCO radius increases with the BH rotation for retrograde orbits but decreases for

[7]The solution is asymptotically characterized by a null scalar gradient with $X \to 0$ in a Minkowski spacetime. Hence, perturbations of this solution may be subtle with possible strong-coupling issues for scalar fluctuations. This point deserves further investigation that we hope to pursue in the future. We thank the referee for the observation.

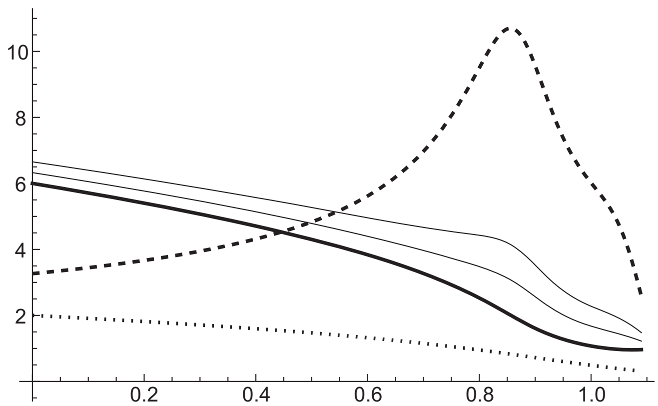

FIG. 1. Plot of the nonrotating ISCO (thick continuous line) as a function of the parameter ξ_2 compared with the BH horizon (dotted line). Both radii are expressed in units of the BH mass M. Above the nonrotating ISCO, two thin lines indicate the modified ISCO for $a = 0.1$ and $a = 0.2$ (in the linear approximation) for retrograde orbits. More generally, the derivative $\partial r^-_{\rm ISCO}/\partial a$ is plotted with a dashed line. For all plots, we take $\lambda = M$.

prograde orbits. One can notice that the position of the ISCO is most sensitive to the rotation when $\xi_2 \simeq 0.9$, for the particular choice $\lambda = M$, as illustrated on Fig. 1.

Let us now turn to massless particles and their circular geodesics in the equatorial plane (which are unstable). The radial position of these orbits is obtained by solving

$$V_{\rm eff}(r) = 0, \qquad V'_{\rm eff}(r) = 0, \quad (\text{with } \kappa = 0). \qquad (44)$$

The analysis of the position of the circular light orbits is very similar to that of the ISCO and we have plotted the results on Fig. 2. As expected, we see that the radius of the

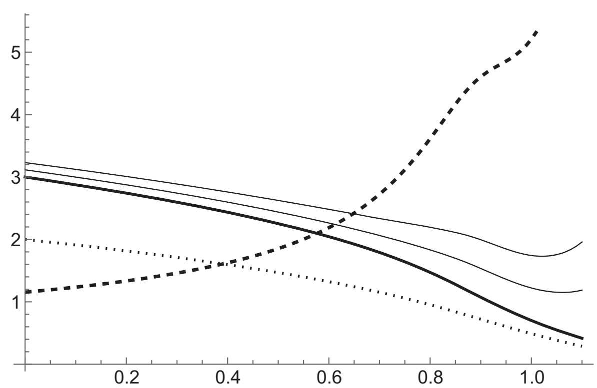

FIG. 2. Plot of the light circular orbit radius $r_{\rm LCO}$ in the nonrotating solution (thick continuous line), as a function of the parameter ξ_2 compared with the BH horizon (dotted line). Both radii are expressed in units of the BH mass M. Above the nonrotating case, two thin lines indicate the modified light radius for $a = 0.1$ and $a = 0.2$ (in the linear approximation) for retrograde orbits. More generally, the derivative $\partial r^-_{\rm LCO}/\partial a$ is plotted with a dashed line. For all plots, we take $\lambda = M$.

photon orbits varies with the deformation parameter ξ_2 and with the rotation (as in GR). We also illustrate the variation of the retrograde orbits $r^-_{\rm LCO}$ which increases with the black hole angular momentum.

IV. ON THE ANGULAR DEPENDENCE OF THE FRAME DRAGGING FUNCTION

So far, we have made the assumption that the frame dragging function ω depends only on the radial coordinate r. In the context of general relativity, such an assumption is well justified because of the no-hair theorem [7] which ensures that the slowly rotating solution must correspond to the small angular momentum limit of the Kerr solution. Thus, there is no reason to expect ω to depend on the angular coordinate θ.

This argument is however no longer valid in modified theories of gravity, which admit hairy black hole solutions. Therefore, we study in this section the possibility that ω depends on θ. We start by computing the equation of motion for ω. Then, we revisit the case of general relativity and recall that regularity arguments prevent ω to depend on θ. Finally, we adapt the analysis to the specific case of DHOST theories and argue that ω should not depend on θ.

A. Equation of motion

Assuming a θ-dependence for ω leads to a partial differential equation, instead of the ordinary differential equation (8). In order to compute this equation, we proceed as in Sec. II D since the shift-symmetry argument remains valid and finally obtain (see Appendix B for more details)

$$\frac{\partial}{\partial r}\left[Q(r)\sin^3(\theta)\frac{\partial\omega}{\partial r}\right] + \frac{\partial}{\partial\theta}\left[\hat{Q}(r)\sin^3(\theta)\frac{\partial\omega}{\partial\theta}\right] = 0, \qquad (45)$$

where Q is the same function as in (10) while $\hat{Q}$ is defined by

$$\hat{Q}(r) = \frac{r^2}{\sqrt{hf}}\left[\left(F_2 + \frac{q^2}{h}A_1\right) + \frac{1}{2}f\psi' X'\left(F_{3X} - 2\frac{q^2}{h}B_6\right) - \frac{q^2}{h}B_2\frac{(hX)'}{h\psi'}\right], \qquad (46)$$

for DHOST theories.[8]

[8]Similarly to Q, the expression (46) for $\hat{Q}$ is obtained from

$$\hat{Q}(r) = \frac{1}{\sqrt{hf}}\left[\left(F_2 + \frac{q^2}{h}A_1\right)r^2 + f\psi'\frac{q^2}{h}(3B_3 + 2B_2)r + \frac{1}{2}f\psi' X'\left(F_{3X} - 2\frac{q^2}{h}B_6\right)r^2 - \frac{q^2}{h}B_2\frac{(hX)'}{h\psi'}r^2\right],$$

after imposing the degeneracy condition $3B_3 + 2B_2 = 0$.

To solve (45), we proceed as in [30] and decompose $\omega(r,\theta)$ on a basis of Legendre polynomials,[9]

$$\omega(r,\theta)\sin^2(\theta) = -\sum_{l=1}^{+\infty} \omega_l(r)\sin(\theta)\frac{\partial P_l}{\partial\theta}, \qquad (47)$$

where P_l is the Legendre polynomial of order l evaluated at $\cos(\theta)$. Substituting (47) into (45) leads to the following ordinary differential equation for each mode ω_l:

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(Q\frac{\mathrm{d}\omega_l}{\mathrm{d}r}\right) + (2 - l(l+1))\hat{Q}\omega_l = 0. \qquad (48)$$

For $l = 1$, the second term in (48) vanishes and we recover the solution (12) discussed in the previous sections. For $l \geq 2$, the analysis depends on the form of the functions Q and $\hat{Q}$. In the next three subsections, we discuss in turn three specific cases: general relativity, quadratic DHOST theories of the form (34) and 4D-Einstein-Gauss-Bonnet (which is another particular case of DHOST theories).

B. The case of general relativity revisited

As we mentioned above, the no-hair theorem [7] in general relativity ensures that ω does not depend on θ. Interestingly, one can recover this result from the differential equation (48), as discussed in [56].

In general relativity, the functions Q and $\hat{Q}$ in (48) reduce to

$$Q(r) = r^4, \qquad \hat{Q} = \frac{r^2}{f(r)}, \qquad f(r) = 1 - \frac{2M}{r}. \qquad (49)$$

Expressing, for simplicity, the radial coordinate r in units of the Schwarzschild radius, which is equivalent to taking $2M = 1$, equation (48) reduces to

$$\left(1 - \frac{1}{r}\right)\frac{\mathrm{d}}{\mathrm{d}r}(r^4\omega_l') + (2 - l(l+1))r^2\omega_l = 0. \qquad (50)$$

[9]The coefficient of $\mathrm{d}t\mathrm{d}\varphi$ in the line element, which involves ω, can be seen as an axial (or odd-parity) perturbation of the static and spherically symmetric background. Hence, it can be decomposed into the following basis of vectors on the 2-sphere,

$$V^a_{(l,m)} = \left(\frac{1}{\sin(\theta)}\frac{\partial Y_l^m}{\partial\varphi}; -\sin(\theta)\frac{\partial Y_l^m}{\partial\theta}\right).$$

Since we are considering an axisymmetric metric (whose coefficients do not depend on φ), only the modes $m = 0$ enter in the decomposition, hence

$$\omega(r,\theta)\sin^2(\theta) = \sum_{l=1}^{+\infty}\omega_l(r)V^{\varphi}_{(l,0)}(\theta) = -\sum_{l=1}^{+\infty}\omega_l(r)\sin(\theta)\frac{\partial P_l}{\partial\theta}.$$

For $l = 1$, we obtain the solution corresponding to the Kerr metric (7) as expected. For $l \geq 2$, the solution is more involved but can be written in terms of special functions [56].

For the present discussion, we simply need to study the asymptotic behavior, at spatial infinity and near the horizon, of $\omega_l(r)$ or, equivalently, of the function $y_l(r) = r^2\omega_l(r)$, which satisfies the simpler differential equation

$$r(r-1)y_l'' + \left(\frac{2}{r} - l(l+1)\right)y_l = 0. \qquad (51)$$

The general solution of (51) can be written as

$$y_l(r) = c_{(1)}y_l^{(1)}(r) + c_{(2)}y_l^{(2)}(r), \qquad (52)$$

where $(y_l^{(1)}, y_l^{(2)})$ is some basis of solutions while $c_{(1)}$ and $c_{(2)}$ are constants (to be fixed with boundary conditions). For $l = 2$, the general solution is given explicitly by

$$y_2(r) = c_{(1)}r^2(r-1) + 5c_{(2)}\left(4r^2 - 2r - \frac{2}{3} - \frac{1}{3r} + 4r^2(r-1)\ln\left(\frac{r-1}{r}\right)\right). \qquad (53)$$

At spatial infinity, the leading order of the two independent solutions $y_l^{(1)}$ and $y_l^{(2)}$ can be deduced from the equation

$$r^2 y_l'' - l(l+1)y_l = 0, \qquad (54)$$

where the coefficients in (51) have been replaced by their leading order expression at large r (see e.g. Chap. 4.2 of [57] for details). Since the general solution of (54) is simply given by a linear combination of r^{l+1} and r^{-l}, we can choose a basis $(y_l^{(1)}, y_l^{(2)})$ such that

$$y_l^{(1)}(r) = r^{l+1}(1 + o(1)), \qquad y_l^{(2)}(r) = r^{-l}(1 + o(1)). \qquad (55)$$

As the solution should not diverge at spatial infinity in order to recover an asymptotically Minkowski metric, we have to fix $c_{(1)} = 0$ for every l in (52).

Let us now turn to the asymptotic behavior at the horizon. In the case $l = 2$, one sees that the solution $y_l^{(2)}$ is pathological at the horizon ($r \to 1$) because of the logarithm term: the solution is continuous but not differentiable. As a consequence, $c_{(2)}$ must also vanish if one imposes that the metric components is both continuous and differentiable. The same conclusion applies for $l > 2$, due to the presence of a similar logarithmic term leading to a pathological behavior at the horizon (see [56] for instance). Hence, all modes $l \geq 2$ must vanish and ω cannot depend on θ, as expected from the no-hair theorem.

C. Generalization to a class of quadratic DHOST theories

We now extend the analysis of the previous subsection to the family of quadratic shift-symmetric DHOST theories considered in Sec. III C, which admit interesting hairy black hole solutions [13] where the scalar field is of the form $\phi = qt + \psi(r)$ while the metric is static and homogeneous, i.e. $f(r) = h(r)$.

We can easily compute the two functions entering the equation for ω and we find $Q(r) = r^4$, as in general relativity, while $\hat{Q}$ is given by

$$\hat{Q} = \frac{r^2}{f}\left(F_2 + \frac{q^2}{f}A_1\right) = \frac{r^2}{f}(1 + f\psi'^2 A_1). \qquad (56)$$

The equation for the function $y_l(r) = r^2\omega_l(r)$ now reads

$$\alpha_2(r)y_l'' + \alpha_0(r)y_l = 0, \qquad \alpha_2 = r^2 f,$$
$$\alpha_0 = (2 - l(l+1))(1 + f\psi'^2 A_1) - 2f. \qquad (57)$$

where the coefficients behave, at spatial infinity, as

$$\alpha_2(r) = r^2(1 + o(1)), \qquad \alpha_0(r) = -l(l+1) + o(1), \qquad (58)$$

assuming[10] $p > 1$. At spatial infinity, we thus recover the same asymptotic behavior as in general relativity, which implies that one of the two branches of solutions has to be eliminated, say $c_{(1)} = 0$.

At the horizon, by contrast, the behavior of the solution differs from that in general relativity. Indeed, near the horizon radius r_H (defined by $f(r_H) = 0$), the two functions α_0 and α_2 behave as

$$\alpha_2 = \alpha_{2H}\varepsilon(1 + o(1)), \qquad \alpha_0 = \frac{\alpha_{0H}}{\varepsilon}(1 + o(1)), \qquad (59)$$

where $\varepsilon = r - r_H$, $X_H = X(r_H)$ and

$$\alpha_{2H} = r_H^2 f'(r_H), \qquad \alpha_{0H} = \alpha q^2 \frac{X_H^{p-1}}{2f'(r_H)}(2 - l(l+1)). \qquad (60)$$

Hence, one can choose a basis of solutions (52) of (57) such that

$$y_l^{(1)} = \varepsilon^{n_+}(1 + o(1)), \qquad y_l^{(2)} = \varepsilon^{n_-}(1 + o(1)), \qquad (61)$$

where the powers $n_\pm$ are given by

$$n_\pm = \frac{1}{2}\left[1 \pm \left(1 - 4\frac{\alpha_{0H}}{\alpha_{2H}}\right)^{1/2}\right]. \qquad (62)$$

When $l = 1$, $n_+ = 1$ and $n_- = 0$, thus the two branches are regular (continuous and differentiable) at the horizon. Hence, as in general relativity, the mode ω_1 should be included. For the modes $l \geq 2$, we must examine the near-horizon behavior depending on the sign of α.

If $\alpha > 0$, we see that $n_- < 0$, giving a diverging branch at the horizon, which must be eliminated. Provided that this branch is not the same as the one already eliminated at spatial infinity, we conclude that $y_l = 0$ for all $l \geq 2$. Moreover, except for some specific values of α, n_+ is not an integer and so the remaining branch at the horizon is not $\mathcal{C}^\infty$. If we demand the solution to verify this property, we can eliminate both branches at the horizon.

If $\alpha < 0$, $n_\pm$ are either real or complex. When they are real, it is easy to see that $0 < n_- < n_+ < 1$, so both branches are continuous but not differentiable at the horizon, and must therefore be eliminated. When $n_\pm$ are complex, the solutions are of the form,

$$y_l^{(1)} = \sqrt{\varepsilon}e^{i\Im(n_+)\ln(\varepsilon)}(1 + o(1)),$$
$$y_l^{(2)} = \sqrt{\varepsilon}e^{-i\Im(n_+)\ln(\varepsilon)}(1 + o(1)), \qquad (63)$$

where $\Im(n_\pm)$ denotes the imaginary part. Both branches are continuous but not differentiable at the horizon, so we can eliminate both again if we demand the solution to be differentiable.

In conclusion, ω cannot depend on the angle θ.

D. Four dimensional Einstein-Gauss-Bonnet

In this section, we extend our analysis to the case of a four dimensional Einstein-Gauss-Bonnet theory,[11] for which a static black hole solution is known [45]. A slowly rotating solution has already been found in [26], and was studied in [27]. This theory belongs to the class of Horndeski theories, and is defined by the following nontrivial functions,

$$F_0 = 8\alpha X^2, \qquad F_1 = 8\alpha X,$$
$$F_2 = 1 + 4\alpha X, \qquad A_1 = -A_2 = -4\alpha,$$
$$F_3 = -4\alpha\ln|X|, \qquad 6B_1 = -2B_2 = 3B_3 = \frac{4\alpha}{X}, \qquad (64)$$

[10] The case $p = 1$ corresponds to a nonasymptotically flat spacetime.

[11] In this work, we adopt the prescription of [45], in which the limiting procedure introduces an additional scalar degree of freedom. Other prescriptions exist, for instance those that construct a consistent $D \to 4$ Einstein-Gauss-Bonnet theory (in an unique way) with only two dynamical degrees of freedom by breaking temporal diffeomorphism invariance [58].

with α being a coupling constant of dimension length squared. As this theory contains cubic terms in the second derivatives of the scalar field, it does not fall under the class of theories we have studied in the previous section III B.

The static black hole solution is given by

$$f(r) = h(r) = 1 + \frac{r^2}{2\alpha}\left(1 - \sqrt{1 + \frac{8\alpha M}{r^3}}\right), \qquad (65)$$

while the scalar field can be obtained from

$$\psi'(r) = \frac{\sqrt{f(r)} - 1}{r\sqrt{f(r)}}, \qquad (66)$$

where, for simplicity, we restrict ourselves to the case of $q = 0$.

The slowly rotating function $\omega(r)$ still satisfies the Eq. (45) where the two functions Q and $\hat{Q}$ are now given by

$$Q = r^4\sqrt{1 + \frac{8\alpha M}{r^3}}, \qquad \hat{Q} = \frac{r^2}{f}\left[1 + \frac{2\alpha}{r^2}(f - 1 - rf')\right]. \qquad (67)$$

We start by considering the $l = 1$ mode, and an immediate calculation leads to the same solution as the one obtained in [26],

$$\omega_1(r) = \frac{J}{2\alpha M}\left(\sqrt{1 + \frac{8\alpha M}{r^3}} - 1\right). \qquad (68)$$

Notice that the two integration constants that show up when we integrate ω_1 are chosen such that ω_1 tends to the Kerr solution at spatial infinity.

In order to study the modes for $l > 1$, we proceed as in the previous section. We first perform the change of variable $y_l = \sqrt{Q}\omega_l$ so that the equation for y_l does not involve a first derivative term,

$$y_l'' + \left[\frac{Q'^2 - 2QQ''}{4Q^2} + (2 - l(l+1))\frac{\hat{Q}}{Q}\right]y_l = 0. \qquad (69)$$

At spatial infinity, the coefficient of y_l behaves as follows:

$$\frac{Q'^2 - 2QQ''}{4Q^2} + (2 - l(l+1))\frac{\hat{Q}}{Q} = -\frac{l(l+1)}{r^2} + o\left(\frac{1}{r^2}\right), \qquad (70)$$

hence, the asymptotic behavior is the same as in general relativity (54) and we can eliminate the branch that diverges like like r^{l+1}.

At the horizon, located at radius $r_H = M + \sqrt{M^2 - \alpha}$, the function $\hat{Q}$ diverges (due to the presence of $f(r)$ in the denominator) while Q and its derivatives are finite. Thus, the coefficient of y_l behaves as follows:

$$(2 - l(l+1))\frac{\alpha_H}{\varepsilon}, \qquad \alpha_H = \frac{r_H^4 - 4\alpha M r_H}{2r_H^2(r_H - M)(r_H^2 + 2\alpha)}, \qquad (71)$$

where $\varepsilon = r - r_H$. Hence, the solution behaves as in general relativity at the horizon with a logarithmic term $(r - r_H)\ln(r - r_H)$ appearing in one of the two branches of solutions (for $l \neq 1$), making it nondifferentiable. Hence, by requiring the solution to be differentiable, we can also eliminate a branch of solution.

We have no proof that the branch eliminated at infinity is not the same as the one eliminated at the horizon, in which case a regular branch of solution would remain. However, given that the theory (64) reduces to GR continuously in the limit $\alpha \to 0$, one can expect that the two branches eliminated at spatial infinity and at the horizon, respectively, are distinct, as in GR, at least for sufficiently small values of α. This would confirm that ω does not depend on the angle θ.

V. CONCLUSION

We studied slowly rotating black-hole solutions within the framework of DHOST theories. Starting from any static, spherically symmetric background of a DHOST model, we derived a general integral expression for the frame-dragging function ω. The possibility to obtain such a general result relies on an interesting "hidden" symmetry of the equations of motion, which leads to their integrability. We then showed that this expression is consistent with disformal transformations and derived how ω transforms under such transformations. We illustrated our results with explicit examples and proved (using regularity conditions at infinity and at the horizon) that no solution for ω with angular dependence can exist, as in general relativity. In the context of the recently discovered black holes with primary hair, we analyzed the timelike and lightlike circular geodesics and quantified how they deviate from the predictions of the Kerr solution, in the slow-rotation limit.

It would be interesting to investigate whether the method for computing slowly rotating solutions, along with the integrability property of the equations of motion, remains valid in other modified theories of gravity. We also aim to generalize our results to the case of slowly rotating stars.

ACKNOWLEDGMENTS

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DATA AVAILABILITY

The data that support the findings of this article are available within this article.

APPENDIX A: DISFORMAL TRANSFORMATIONS

In this Appendix, we provide technical details on the effect of disformal transformations on slowly rotating spacetimes in DHOST theories.

We recall that a disformal transformation of the metric is defined by

$$g_{\mu\nu} \longrightarrow \tilde{g}_{\mu\nu} = C(\phi, X) g_{\mu\nu} + D(\phi, X)\partial_\mu\phi\partial_\nu\phi, \tag{A1}$$

where C and D are functions of ϕ and X, constrained by the requirement that the disformal transformation is invertible. Moreover, we assume that the conformal factor C is a positive function to keep the signature of the metric unchanged. As we restrict ourselves to shift-symmetric theories, the functions C and D are also supposed to depend only on X.

Disformal transformations induce transformations in the space of DHOST theories as follows:

$$\tilde{S}[\tilde{g}_{\mu\nu}, \phi] = S[g_{\mu\nu}, \phi], \tag{A2}$$

which means that, given a theory $\tilde{S}$ for the metric $\tilde{g}$, one defines a new theory S for the metric g. When the disformal transformation is invertible, the two theories are classically equivalent in vacuum only while they differ when one takes into account (generic) coupling to matter. Here, we are constructing slowly rotating solutions for scalar-tensor theories in vacuum and then, considering the action S or any disformally related action $\tilde{S}$ must lead to equivalent metrics. Let us check that this is indeed the case.

We start with a DHOST action S and we consider a slowly rotating metric, characterized by the functions $h(r)$, $f(r)$ and $\omega(r)$. We have shown that ω satisfies the simple differential equation

$$\frac{\mathrm{d}\omega}{\mathrm{d}r} = \frac{k}{Q(r)}, \tag{A3}$$

where k is an integration constant and $Q(r)$ has been introduced in (10) and can be decomposed as follows:

$$Q = (Q_2 + Q_3)r^4 + Q_B r^3, \tag{A4}$$

where the different components are defined by

$$Q_2 = \sqrt{\frac{f}{h}}(F_2 + 2XA_1), \tag{A5}$$

$$Q_3 = -\frac{1}{2} f \sqrt{\frac{f}{h}} \frac{\partial\phi}{\partial r} \frac{dX}{dr} (F_{3X} + 4XB_6 - 2B_2), \tag{A6}$$

$$Q_B = 2f \sqrt{\frac{f}{h}} \frac{\partial\phi}{\partial r} XB_2. \tag{A7}$$

Notice that we have replaced ψ' by the more explicit expression $\partial\phi/\partial r$ to recall that we are taking the derivative of ϕ at constant t (since this will become important later when we change coordinates).

Then, we perform a disformal transformation of the metric and, after a direct calculation, we can show that the metric $\tilde{g}_{\mu\nu}$ still takes the form (3)

$$\mathrm{d}\tilde{s}^2 = -\tilde{h}\mathrm{d}\tilde{t}^2 + \frac{\mathrm{d}\tilde{r}^2}{\tilde{f}} + \tilde{r}^2\mathrm{d}\Omega^2 - 2\tilde{r}^2\tilde{\omega}\sin^2\theta\mathrm{d}\tilde{t}\mathrm{d}\tilde{\varphi}, \tag{A8}$$

where the new coordinates $(\tilde{t}, \tilde{r}, \tilde{\varphi})$ are defined by

$$\begin{aligned}\mathrm{d}\tilde{t} &= \mathrm{d}t - \frac{qD(r)\psi'(r)}{C(r)h(r) - D(r)q^2}\mathrm{d}r,\\ \mathrm{d}\tilde{\varphi} &= \mathrm{d}\varphi - \frac{qD(r)\psi'(r)\omega(r)}{C(r)h(r) - D(r)q^2}\mathrm{d}r, \qquad \tilde{r} = R(r),\end{aligned} \tag{A9}$$

with $R(r) = \sqrt{C(r)}r$, while the new functions are given by the relations

$$\tilde{h}(\tilde{r}) = Ch - q^2 D, \qquad \frac{1}{\tilde{f}}(\tilde{r}) = \frac{C}{R'^2}\left(\frac{1}{f} + \frac{Dh\psi'^2}{Ch - q^2 D}\right),\qquad \tilde{\omega}(\tilde{r}) = \omega, \tag{A10}$$

where the functions of the rhs of the two identities are evaluated on r, viewed as a function of $\tilde{r}$. Let us make two comments: we have used the notation $C(r)$ for $C(X(r))$ and $D(r)$ for $D(X(r))$; as C is supposed to be positive, the coordinate $\tilde{r}$ is well defined. We can formally integrate the two differential equations in (A9) so that

$$\tilde{t} = t + \mu(r), \qquad \tilde{\varphi} = \varphi + \nu(r), \tag{A11}$$

where μ and ν are given in terms of integrals. As a result, $\tilde{\omega}$ satisfies an equation similar to (A3),

$$\frac{\mathrm{d}\tilde{\omega}}{\mathrm{d}\tilde{r}} = \frac{\tilde{k}}{\tilde{Q}(r)}, \tag{A12}$$

where $\tilde{k}$ is still a constant and $\tilde{Q}$ is the same function as in (10) with the new functions $\tilde{F}_i$, $\tilde{A}_i$ and $\tilde{B}_i$ given in terms of F_i, A_i and B_i from disformal transformations (see the Appendix of [59] for instance). Of course, the two equations (A3) and (A12) must be equivalent, which leads to the consistency relation

$$R'\frac{Q}{\tilde{Q}} = \frac{k}{\tilde{k}} = \mathrm{cst}, \tag{A13}$$

which can be verified explicitly.

To show that this is indeed the case, we have to express $\tilde{Q}$ in terms of the coefficients of the metric $g_{\mu\nu}$ and the radial coordinate r. This can be done using the relation

$$\frac{\tilde{h}}{\tilde{f}} = \frac{C(C-2DX)}{R'^2}\frac{h}{f}, \tag{A14}$$

and we also need to transform the derivative of ϕ with respect to $\tilde{r}$ in terms of a derivative with respect to r,

$$\frac{\partial\phi}{\partial\tilde{r}} = \frac{\partial}{\partial\tilde{r}}[q\tilde{t} + q\mu(r) + \psi(r)] = \frac{1}{R'}(q\mu' + \psi') = C\frac{h}{\tilde{h}}\frac{\psi'}{R'}. \tag{A15}$$

After a direct calculation, we can see that $\tilde{Q}$ takes the same form as (A4),

$$\tilde{Q} = (\tilde{Q}_2 + \tilde{Q}_3)r^4 + \tilde{Q}_B r^3, \tag{A16}$$

where the three components are now given by

$$\tilde{Q}_2 = R'\sqrt{\frac{f}{h}}\frac{C^{3/2}}{(C-2DX)^{1/2}}(\tilde{F}_2 + 2\tilde{X}\tilde{A}_1), \tag{A17}$$

$$\tilde{Q}_3 = -\frac{R'}{2}f\sqrt{\frac{f}{h}}\psi' X'\frac{C^{3/2}}{(C-2DX)^{3/2}}\frac{\partial\tilde{X}}{\partial X} \times\left[\tilde{F}_{3\tilde{X}} + 4\tilde{X}\tilde{B}_6 - 2\left(1+\frac{\tilde{X}C_{\tilde{X}}}{C}\right)\tilde{B}_2\right], \tag{A18}$$

$$\tilde{Q}_B = 2R'\sqrt{\frac{f}{h}}f\psi'\frac{C^{3/2}}{(C-2DX)^{3/2}}\tilde{X}\tilde{B}_2, \tag{A19}$$

with $\tilde{X} = X/(C-2XD)$. As a consequence, we have to show that

$$R'\frac{Q_2}{\tilde{Q}_2} = R'\frac{Q_3}{\tilde{Q}_3} = R'\frac{Q_B}{\tilde{Q}_B} = \text{cst}. \tag{A20}$$

In the case of quadratic theories, only Q_2 is nontrivial, and verifying the relation between Q_2 and $\tilde{Q}_2$ is easy and can be done from a direct calculation using disformal transformations formulas [59],

$$\begin{aligned} F_2 &= (C(C-2DX))^{1/2}\tilde{F}_2, \\ A_1 &= \frac{DC^{1/2}}{(C-2DX)^{1/2}}\tilde{F}_2 + \frac{C^{3/2}}{(C-2DX)^{3/2}}\tilde{A}_1, \end{aligned} \tag{A21}$$

which leads immediately to the relation

$$F_2 + 2XA_1 = \frac{C^{3/2}}{(C-2DX)^{1/2}}(\tilde{F}_2 + 2\tilde{X}\tilde{A}_1). \tag{A22}$$

Furthermore, we see that the constant entering in (A20) is equal to 1, which implies that the two integration constants k and $\tilde{k}$ (A12) are identical.

This result can be extended to cubic theories. In that case, we assume that $\tilde{S}$ is the Horndeski action so that we can use the disformal transformations formulas given in Sec. V of [4]. In that case, $\tilde{F}_3$ is an arbitrary function, and

$$\tilde{B}_2 = \frac{1}{2}\tilde{F}_{3\tilde{X}}, \qquad \tilde{B}_6 = 0, \tag{A23}$$

so that, after a few calculations, we have,

$$\tilde{Q}_3 = -\frac{R'}{2}f\sqrt{\frac{f}{h}}\psi' X'\frac{XC_X C^{1/2}}{(C-2DX)^{5/2}}\frac{\partial X}{\partial\tilde{X}}\tilde{F}_{3X}. \tag{A24}$$

The functions entering in the disformed action S are obtained from disformal transformations relations which imply here

$$\begin{aligned} F_{3X} &= \frac{C^{1/2}}{(C-2DX)^{1/2}}\tilde{F}_{3X}, \\ B_2 &= \frac{C^{3/2}\tilde{F}_{3X}}{2(C-2XD)^{1/2}(C-XC_X+2X^2D_X)}, \\ B_6 &= -\frac{C^{1/2}XD_X\tilde{F}_{3X}}{2(C-2XD)^{1/2}(C-XC_X+2X^2D_X)}. \end{aligned} \tag{A25}$$

Thus, we can easily compute Q_3 and Q_B, and check that the remaining two relations (A20) are satisfied with the constant being 1 as well.

There is an immediate consequence of this consistency relations. Given a DHOST action S which admits a slowly rotating solution associated with the function f, h and ω, then the disformally related action $\tilde{S}$ admits also a slowly rotating solution associated to $\tilde{f}$, $\tilde{h}$ and $\tilde{\omega}$ related to the original ones by (A10). In particular, the frame dragging functions are related by

$$\omega(r) = \tilde{\omega}(R(r)), \quad \text{with} \quad R(r) = r\sqrt{C(r)}. \tag{A26}$$

Expressing $\tilde{\omega}$ in terms of ω can be done only if the coordinate transformation $\tilde{r} = \sqrt{C(r)}r$ is invertible, or equivalently $\tilde{r}$ is a monotonous function of r.

APPENDIX B: MORE DETAILS ON THE EQUATION FOR AN ANGULAR DEPENDENT FRAME DRAGGING FUNCTION

In this Appendix, we give more details on the Eq. (45) with (46) for a θ-dependent ω. Following the method described in Sec. II D, the Eq. (17) now becomes

$$\frac{\delta S}{\delta\omega} = -\frac{\partial}{\partial r}\left(\frac{\delta S}{\delta\partial_r\omega}\right) - \frac{\partial}{\partial\theta}\left(\frac{\delta S}{\delta\partial_\theta\omega}\right), \tag{B1}$$

where the two derivatives can be computed directly from the DHOST Lagrangian,

$$\frac{\delta S}{\delta \partial_r \omega} = Q(r)\sin^3(\theta)\frac{\partial \omega}{\partial r} + \frac{1}{2}\frac{\partial}{\partial \theta}\left(\sqrt{\frac{f}{h}}r^2\psi' F_3 \sin^3(\theta)\frac{\partial \omega}{\partial \theta}\right), \tag{B2}$$

$$\frac{\delta S}{\delta \partial_\theta \omega} = \frac{1}{\sqrt{hf}}\left[\left(F_2 + \frac{q^2}{h}A_1\right)r^2 + f\psi'\frac{q^2}{h}(3B_3 + 2B_2)r + \frac{1}{2}f\psi' X'\left(F_{3X} - 2\frac{q^2}{h}B_6\right)r^2 - \frac{q^2}{h}B_2\frac{(hX)'}{h\psi'}r^2\right]\sin^3(\theta)\frac{\partial \omega}{\partial \theta} - \frac{1}{2}\frac{\partial}{\partial r}\left(\sqrt{\frac{f}{h}}r^2\psi' F_3 \sin^3(\theta)\frac{\partial \omega}{\partial \theta}\right). \tag{B3}$$

Notice that $Q(r)$ is the same function as the one introduced in (9).

Remarkably, the last term in (B2) cancels with the last term in (B3) when combined in (B1), yielding an equation of the form of (45). Finally, simplifying the expression of (B3) using the degeneracy condition $3B_3 + 2B_2 = 0$ gives (46).

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