

Holographic complexity of the Klebanov-Strassler background

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We study the complexity of the gravity dual to the confining $SU(N) \times SU(N + M)$ Klebanov-Strassler gauge theory, which is an important test case for holographic complexity in higher-dimensional and nonconformal gauge-gravity dualities. We emphasize the dependence of the complexity on parameters of the gauge theory, finding a common behavior with confinement scale for several complexity functionals. We also analyze how the complexity diverges with the UV cutoff, which is more complicated than in AdS backgrounds because the theory is nonconformal. Our results may provide new perspectives on questions in the holographic complexity program as well as a starting point for further studies of complexity in general gauge-gravity dualities.

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I. INTRODUCTION

In quantum information theory, (circuit) complexity is a measure of the distance of a quantum state to some reference state; holographic complexity for a gauge theory is the dual of complexity on the gravity side of a gauge-gravity duality such as the AdS/CFT correspondence (see [1] for references and a review). However, the definition of complexity in quantum mechanics or quantum field theory is not unique because there are many possible definitions of distance in the space of states [2–8]. Similarly, there is an infinite family of functionals that evaluate holographic complexity. In general, holographic complexity is a codimension-1 or -0 functional evaluated on a surface or region extending from a slice Σ of the conformal boundary space at a cutoff value of the holographic radius (which we

will call the boundary for simplicity). Via duality, this corresponds to the gauge theory's complexity on the given time slice Σ .

There are several properties we may expect from a general complexity functional, including late-time linear growth and the switchback effect in black hole spacetimes [9,10] or its value as the strongest possible divergence in pure AdS spacetime [11]. And the most commonly studied holographic complexity functionals, including the CV “complexity=volume” proposal [12,13], the CV2.0 proposal [14], and the CA “complexity=action” proposal [15,16], satisfy these properties. However, with the exception of [17–19], discussion of holographic complexity has centered on asymptotically AdS spacetimes without the extra dimensions in the fully string theoretic gauge-gravity duality.

Here, we examine the holographic complexity of the Klebanov-Strassler (KS) gauge theory $\mathbb{Z}_2$ symmetric ground state. The KS gauge theory [20] is an $\mathcal{N} = 1$ theory with gauge group $SU(N + M) \times SU(N)$ with four chiral superfields, two of which transform in the bifundamental and two in its conjugate. This theory is holographic, and [20] proposed that the gravity dual is a ten-dimensional warped product of Minkowski spacetime with a deformed conifold along with 3-form and 5-form flux (the KS background in IIB supergravity). Since the KS background

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is static, there is an unambiguous time-independent complexity, and we can choose Σ to be the $t = 0$ slice of the boundary.

The gauge theory has several important properties, one of which is that it is confining in the infrared, which corresponds to the smooth and horizonless nature of the gravity dual, similar to AdS soliton backgrounds [21,22]. As with complexity of the soliton backgrounds [11,23], how the complexity of the KS background behaves as a function of the confinement scale (given by the conifold deformation parameter ϵ) is an important question. We find a universal scaling with ϵ^2 , which is equivalent to the cube of the confinement scale, when we write the UV cutoff in terms of the dimensionless radial coordinate. We also investigate the dependence of complexity on the gauge group rank parameter M (a flux quantum number in the KS background) and gauge coupling as given by g_s .

In addition to confinement in the infrared, the gauge theory undergoes a series of dualities (a “duality cascade”) to an $SU(M)$ gauge group. Conversely, as the renormalization scale moves toward the ultraviolet, the rank of the gauge group continues to increase. That is, it is also nonconformal at high energies, so the holographic complexity has a different divergence structure as a function of cutoff radius for asymptotically KS backgrounds as opposed to asymptotically AdS backgrounds. However, at large radius, the KS background is geometrically similar to AdS with a slowly varying curvature radius. As a result, well-known results on holographic complexity in AdS give us an expectation for the leading divergence of complexity in KS. We find the leading divergences for various holographic complexity proposals as well as a numerical evaluation of the full holographic complexity for comparison. These results should be useful as a reference for future studies of black brane (thermal) solutions in KS [24–26], particularly in terms of defining a version of the complexity of formation [15,16,27] for asymptotically KS spacetimes.

Further, the KS background actually lies on a one-complex-parameter moduli space of the field theory parametrized by the expectation value of baryonic operators (the “baryonic branch”) at a point of $\mathbb{Z}_2$ symmetry; [28,29] described the gravity dual to the baryonic branch (perturbatively and numerically, respectively). We show that the KS background is a local extremum of the complexity functionals that we consider along the baryonic branch.

We begin with a review of the KS background in Sec. II below. Then we discuss the behavior of several holographic complexity functionals in subsequent sections, including a revised CA complexity and the “complexity=flux” proposals of [19]. In each case, we review the definition of the functional along with comments on their use in 10D gauge-gravity dualities. Then we give their scaling with the physical parameters of the gauge theory and determine the divergence structure analytically in the cutoff radius. We also give a numerical evaluation of each functional as a

function of cutoff radius. In Sec. VI, we argue that the $\mathbb{Z}_2$ symmetric KS solution extremizes the complexity along the baryonic branch of the gauge theory. We conclude with a discussion of their implications and connections to other research in Sec. VII.

II. REVIEW OF KS BACKGROUND

Here we give a brief review of the KS background [20], as influenced by the discussion of [30].

The KS background is a solution of the 10D type IIB supergravity with metric

$$ds^2 = h(\tau)^{-1/2} \eta_{\mu\nu} dx^\mu dx^\nu + h(\tau)^{1/2} d\tilde{s}^2, \quad (1)$$

where x^μ are the boundary coordinates $t, \vec{x}$ and

$$\begin{aligned} d\tilde{s}^2 = & \frac{1}{2} \epsilon^{4/3} K(\tau) \left[\frac{1}{3K(\tau)^3} (d\tau^2 + (g_5)^2) \right. \\ & + \sinh^2\left(\frac{\tau}{2}\right) ((g_1)^2 + (g_2)^2) \\ & \left. + \cosh^2\left(\frac{\tau}{2}\right) ((g_3)^2 + (g_4)^2) \right] \end{aligned} \quad (2)$$

is the metric of the deformed conifold for $K(\tau) = (\sinh(2\tau) - 2\tau)^{1/3} / (2^{1/3} \sinh(\tau))$ and deformation modulus ϵ (which sets the confinement scale in the gauge theory). τ is the radial (holographic energy) direction, and the angular basis forms are

$$\begin{aligned} g_1 &= \frac{e_1 - e_3}{\sqrt{2}}, & g_2 &= \frac{e_2 - e_4}{\sqrt{2}}, & g_3 &= \frac{e_1 + e_3}{\sqrt{2}}, \\ g_4 &= \frac{e_2 + e_4}{\sqrt{2}}, & g_5 &= e_5 \end{aligned} \quad (3)$$

with

$$\begin{aligned} e_1 &= -\sin\theta_1 d\phi_1, & e_2 &= d\theta_1, \\ e_3 &= \cos\psi \sin\theta_2 d\phi_2 - \sin\psi d\theta_2, \\ e_4 &= \sin\psi \sin\theta_2 d\phi_2 + \cos\psi d\theta_2, \\ e_5 &= d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2. \end{aligned} \quad (4)$$

Since KS is translation invariant along the boundary, any measure of complexity is formally infinite (proportional to boundary volume), so we will actually measure the density as measured with respect to the coordinates $\vec{x}$. We denote complexity densities as $\mathcal{C}$. Further, the $\mathcal{C}$ functionals we evaluate below will all contain a common integral of $\sin\theta_1 \sin\theta_2$ over the angular coordinates, the angular volume of $T^{1,1}$, which we will denote Ω_T .

While the deformed conifold is Ricci-flat, the warp factor $h(\tau)$ results from supergravity 3-form fluxes

$$\begin{aligned}
F_3 &= \left(\frac{M\alpha'}{2}\right) [Fg_5 \wedge g_1 \wedge g_2 + (1-F)g_5 \wedge g_3 \wedge g_4 \\
&\quad + F'd\tau \wedge (g_1 \wedge g_3 + g_2 \wedge g_4)], \\
H_3 &= \left(\frac{g_s M\alpha'}{2}\right) \left[d\tau \wedge (f'_- g_1 \wedge g_2 + f'_+ g_3 \wedge g_4) \right. \\
&\quad \left. + \frac{1}{2}(f_+ - f_-)^2 g_5 \wedge (g_1 \wedge g_3 + g_2 \wedge g_4) \right], \quad (5)
\end{aligned}$$

where g_s and α' are the string coupling and Regge parameter, respectively, (as standard convention) and M is the flux quantum number, which determines the gauge group in the IR. The 3-forms satisfy the duality relation $g_s F_3 = -\star H_3$. The functions $f_\pm$ and F are given by

$$f_\pm(\tau) = \left(\frac{\tau \coth \tau - 1}{2 \sinh \tau}\right) (\cosh \tau \pm 1), \quad F(\tau) = \frac{\sinh \tau - \tau}{2 \sinh \tau}. \quad (6)$$

These source a 5-form flux $\tilde{F}_5 = (1 + \star)\mathcal{F}_5$ with

$$\begin{aligned}
\mathcal{F}_5 &= \left(\frac{g_s M^2 \alpha'^2}{4}\right) l g_1 \wedge g_2 \wedge g_3 \wedge g_4 \wedge g_5, \\
l &= f_+ F + f_- (1 - F) \quad (7)
\end{aligned}$$

and warp factor

$$\begin{aligned}
h(\tau) &= \left(\frac{2^{1/3} g_s M \alpha'}{\epsilon^{4/3}}\right)^2 I(\tau), \\
I(\tau) &= \int_\tau^\infty dx \left(\frac{x \coth x - 1}{\sinh^2 x}\right) (\sinh(2x) - 2x)^{1/3}. \quad (8)
\end{aligned}$$

At large τ , the integral

$$I(\tau) \rightarrow (3/2^{1/3}) \exp(-4\tau/3) (\tau - 1/4). \quad (9)$$

With the coordinate transformation

$$r^2 = (3\epsilon^{4/3}/2^{5/3}) \exp(2\tau/3), \quad (10)$$

the large-radius metric becomes

$$ds^2 = \frac{r^2}{L(r)^2} \eta_{\mu\nu} dx^\mu dx^\nu + \frac{L(r)^2}{r^2} dr^2 + L(r)^2 d\Omega_{T^{1,1}}^2, \quad (11)$$

as first found in [31]. Here, $d\Omega_{T^{1,1}}^2$ is the metric on $T^{1,1}$ (an S^1 bundle over $S^2 \times S^2$) and $L(r)$ is a radially dependent curvature scale given by

$$L(r) = \left(\frac{9g_s M \alpha'}{2\sqrt{2}}\right)^{1/2} \left(\ln\left(\frac{2^{5/6} r}{\sqrt{3}\epsilon^{2/3}}\right)\right)^{1/4}. \quad (12)$$

Since (11) is $\text{AdS}_5 \times T^{1,1}$ with a slowly varying curvature, we expect the leading divergence of complexity for KS to be that of AdS with an additional logarithmic radial dependence.

The KS solution falls in a continuous family of supersymmetric backgrounds dual to a moduli space of the KS gauge theory known as the baryonic branch of the theory. The KS solution has a $\mathbb{Z}_2$ symmetry given by the exchange $(\theta_1, \phi_1) \leftrightarrow (\theta_2, \phi_2)$ in the $T^{1,1}$ angular directions. More general baryonic branch solutions follow an interpolating ansatz given by [32], which breaks the $\mathbb{Z}_2$ symmetry; this geometry is sometimes known as a “resolved warped deformed conifold” [28]. The gravity background for the full baryonic branch (as a solution of the supersymmetry conditions) appears in numerical form in [29]. We will describe perturbations of the KS solution along the baryonic branch (first found in [28]) in Sec. VI.

III. VOLUME COMPLEXITY FUNCTIONALS

We first discuss the two common volume complexity functionals using volumes defined in the ten-dimensional spacetime.

A. CV volume complexity

1. Review

Possibly the simplest proposal for holographic complexity is the CV, or “complexity=volume” proposal [12,13]. The CV complexity is given by $C_V = V/G\ell$, where V is the volume of the slice $\mathcal{B}$ (codimension-1) of the bulk spacetime that maximizes volume subject to $\partial\mathcal{B} = \Sigma$. G is the Newton constant, and ℓ is an arbitrary length scale included to make the complexity dimensionless; in asymptotically AdS backgrounds, ℓ is often chosen equal to the AdS scale. In a ten-dimensional holographic background like KS, there are different ways to define the maximization procedure to find V (see [17–19]). We will choose $\mathcal{B}$ as a nine-dimensional spatial slice of the full ten-dimensional spacetime and G as the ten-dimensional Newton constant. For spacetimes that factorize into a holographic spacetime (such as asymptotically AdS spacetime) and a compact space, this automatically equals the usual CV complexity in the holographic spacetime with the same length ℓ . For example, consider $\text{AdS}_5 \times S^5$ with Poincaré coordinates,

$$ds^2 = \frac{r^2}{L^2} \eta_{\mu\nu} dx^\mu dx^\nu + \frac{L^2}{r^2} dr^2 + L^2 d\Omega_5^2. \quad (13)$$

With a radial cutoff at r_m , this complexity is

$$C_V = \frac{L^5 \Omega_5}{3G_{10} \ell} \frac{r_m^3}{L^2} = \frac{1}{3G_5} \left(\frac{L}{\ell}\right) \left(\frac{r_m}{L}\right)^3, \quad (14)$$

where G_{10} and G_5 are the Newton constants in 10D and 5D, respectively, and Ω_5 is the volume of the unit S^5 . This result will be useful for understanding the divergence structure of complexity in KS. Note that the complexity scales differently with the AdS scale depending on whether we treat G_{10} or G_5 as fixed and that it agrees with the CV complexity of AdS₅ for G_5 fixed (because we have chosen a product spacetime background). In this paper, we will take a definition with G_{10} fixed.

2. Parameter dependence

In the KS background, the CV complexity is simply given by the volume of the $t = 0$ slice. We therefore need the 9D spatial volume element, which is a major ingredient of all complexity functionals. Including the determinant of the deformed conifold and the power $h(\tau)^{3/4}$ of the warp factor, this spatial volume is the integral of

$$\frac{\sqrt{2}(g_s M \alpha')^{3/2} \epsilon^2}{96} [\sin \theta_1 \sin \theta_2] [I(\tau)^{3/4} \sinh^2 \tau]. \quad (15)$$

In total, the CV complexity density is $\mathcal{C}_V = \sqrt{2}(g_s M \alpha')^{3/2} \epsilon^2 \Omega_T J_V / 96 G_{10} \ell$, where J_V is the τ integral of the factor in the last square brackets.

The fact that $\mathcal{C}_V \propto \epsilon^2$ is notable. Since ϵ has dimension of length^{3/2}, this is the same scaling of complexity with confinement scale as found for AdS soliton solutions in [11,23]. Furthermore, we notice that $\mathcal{C}_V \propto M^{3/2}$, in comparison to the AdS₅ $\times$ S^5 case in which $\mathcal{C}_V \propto N^{3/4}$, with N the D3-brane number. This is because the effective D3-brane charge at large distances in KS scales as M^2 ; similar comments apply to the scaling with g_s .

3. Divergence structure

The overall value of the complexity depends on the radial integral

$$J_V(\tau_m) \equiv \int_0^{\tau_m} d\tau I(\tau)^{3/4} \sinh^2 \tau \quad (16)$$

from the origin to a maximal value τ_m , acting as a UV cutoff. While the value for any specific cutoff is not physically interesting, the dependence on the cutoff scale is. It provides a glimpse at the high-energy degrees of freedom of the gauge theory, and it is a necessary component of any definition of complexity of formation in asymptotically KS backgrounds. To evaluate $J_V(\tau_m)$, we first evaluate $I(\tau)$, the functional form of the warp factor, numerically, and then evaluate $J_V(\tau_m)$ numerically. Figure 1 shows $J_V(\tau_m)$; at large τ_m it is nearly exponential.

We would also like to know the form of the leading divergence. Using the asymptotic behavior of $I(\tau)$ from above (11), the divergent behavior is

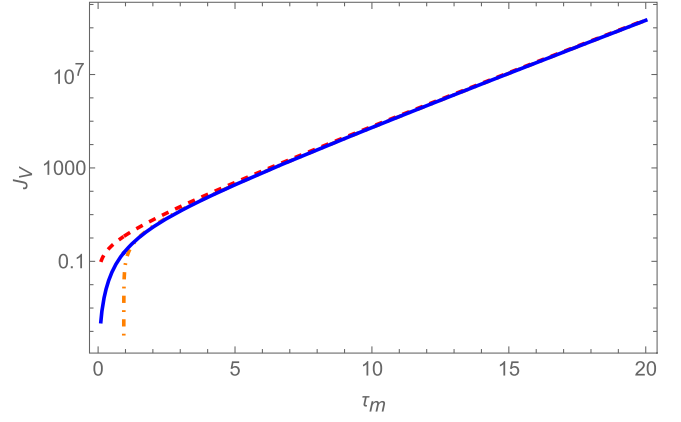


FIG. 1. The dimensionless CV complexity J_V as a function of cutoff τ_m . Compares numerical calculation [solid blue, Eq. (16)], leading divergence [dashed red, (20)], and leading plus next-to-leading divergences [dot-dashed orange, (20)].

$$J_V(\tau_m) \sim \left(\frac{3}{8}\right)^{3/4} \int^{\tau_m} d\tau \left(\tau - \frac{1}{4}\right)^{3/4} e^\tau, \quad (17)$$

with other contributions exponentially suppressed and convergent.¹ Note that the choice of any constant lower limit does not affect the functional form of the divergence. If we choose the lower limit to be $1/4$, then

$$\begin{aligned} J_V(\tau_m) &\sim \left(\frac{3}{8}\right)^{3/4} e^{1/4} \gamma_1\left(\frac{7}{4}, \tau_m - \frac{1}{4}\right) \\ &= \left(\frac{3}{8}\right)^{3/4} e^{1/4} \left(\tau_m - \frac{1}{4}\right)^{7/4} \Gamma\left(\frac{7}{4}\right) \\ &\quad \times \gamma^*\left(\frac{7}{4}, \frac{1}{4} - \tau_m\right), \end{aligned} \quad (18)$$

where

$$\gamma_1(a, x) = \int_0^x dt t^{a-1} e^t, \quad \gamma^*(a, z) = \frac{1}{\Gamma(a)} \int_0^1 dt t^{a-1} e^{-zt} \quad (19)$$

are incomplete gamma functions following Tricomi [33,34] and Böhmer [35] [with $\text{Re}(a) > 0$ for γ_1 while γ^* is entire in both arguments]. The γ^* function has an expression in terms of confluent hypergeometric functions.

The gamma functions above give the divergence as an exponential of τ_m times a series with decreasing powers of τ_m . To illustrate the behavior of the divergence, consider the leading and next-to-leading terms

¹We take $\sim$ to indicate that the divergences are identical, while $\approx$ will indicate an approximation that improves at large argument.

$$J_V(\tau_m) \approx \left(\frac{3}{8}\right)^{3/4} \int^{\tau_m} d\tau \tau^{3/4} \left(1 - \frac{3}{16\tau}\right) e^\tau$$

$$\approx \left(\frac{3}{8}\right)^{3/4} \tau_m^{3/4} \left(1 - \frac{15}{16\tau_m}\right) e^{\tau_m}. \quad (20)$$

Figure 1 compares the numerically calculated $J_V(\tau_m)$ (shown in solid blue) to the leading divergence (dashed red) and first two divergences (dot-dashed orange). As the expansion (20) suggests, including the leading and next-to-leading terms gives an approximation valid to within a percent for $\tau_m \gtrsim 6$, while the leading term is accurate to within a percent for $\tau_m \gtrsim 90$. In the radial variable r , the leading divergence becomes

$$C_V = \frac{\Omega_T}{(18)^2 G_{10} \ell} L(r_m)^3 r_m^3, \quad (21)$$

which is parametrically identical to the complexity of $\text{AdS}_5 \times S^5$ with AdS length chosen equal to the curvature scale of KS at r_m . Since a common picture of KS at large radius is that of an AdS throat with slowly varying curvature, this result is very intuitive. It serves to demonstrate that the UV degrees of freedom dominate the complexity.

B. CV2.0 volume complexity

1. Review

Another purely geometric holographic complexity functional, known as CV2.0, is the volume of the Wheeler-DeWitt (WDW) patch, the codimension-0 region bounded by future- and past-directed light sheets emanating from Σ on the boundary at the maximal radius r_m . Then the complexity is $C_2 = V_{\text{WDW}}/G\ell^2$, where ℓ is again an arbitrary length scale [14]. In the $\text{AdS}_5 \times S^5$ Poincaré patch (13), the future-directed light sheet is given by $t_+(r) = L^2/r - L^2/r_m$ (and past-directed light sheet by $t_- = -t_+$), and the complexity comes out to

$$C_2 = \frac{L^5 \Omega_5}{6G_{10} \ell^2} \frac{r_m^3}{L} = \frac{1}{6G_5} \left(\frac{L}{\ell}\right)^2 \left(\frac{r_m}{L}\right)^3. \quad (22)$$

The complexity has the same divergence with r_m as the CV complexity, and both have the same scaling with AdS radius if the arbitrary length is chosen as $\ell = L$.

2. Parameter dependence

Since the KS background is static, the CV2.0 complexity is

$$C_2 = \frac{2\Omega_T}{96G_{10}\ell^2} \int_0^{\tau_m} d\tau t_+(\tau, \tau_m) h(\tau)^{1/2} \sinh^2 \tau \quad (23)$$

where the additional factor of $h(\tau)^{-1/4}$ comes from $\sqrt{|g_{tt}|}$ and the factor of $2t_+(\tau)$ follows from the integral along t . Since

$$\frac{dt_+}{d\tau} = -\sqrt{\frac{h(\tau)\epsilon^{4/3}}{6K(\tau)^2}}, \quad (24)$$

we see that $t_+ = 2^{2/3}(g_s M \alpha') \epsilon^{-2/3} j_+(\tau, \tau_m)/\sqrt{6}$, where $j_+(\tau, \tau_m)$ is the dimensionless function defined below. In total, $C_2 = (g_s M \alpha' \epsilon)^2 \Omega_T J_2 / 24 \sqrt{6} G_{10} \ell^2$; J_2 is a dimensionless form of integral over τ from (23). This has the same dependence on the confinement scale, $C_2 \propto \epsilon^2$, as the C_V complexity.

3. Divergence structure

To find the divergence of the CV2.0 complexity, we need the dimensionless form of the light sheet

$$j_+(\tau, \tau_m) = \int_\tau^{\tau_m} d\eta \frac{I(\eta)^{1/2} \sinh \eta}{(\sinh(2\eta) - 2\eta)^{1/3}}. \quad (25)$$

For large values of τ , this is approximately an incomplete gamma function

$$j_+(\tau, \tau_m) \approx \frac{\sqrt{3}}{2^{5/6}} \int_\tau^{\tau_m} d\eta \sqrt{\eta - \frac{1}{4}} e^{-\eta/3}$$

$$= \frac{9}{2^{5/6}} e^{-1/12} \Gamma\left(\frac{3}{2}; \frac{\tau}{3} - \frac{1}{12}, \frac{\tau_m}{3} - \frac{1}{12}\right) \quad (26)$$

including all terms that are not exponentially suppressed. Dropping also subleading terms in powers of τ and τ_m , we have

$$j_+(\tau, \tau_m) \approx \frac{3^{3/2}}{2^{5/6}} (\sqrt{\tau} e^{-\tau/3} - \sqrt{\tau_m} e^{-\tau_m/3}). \quad (27)$$

This is very close to an exponential decay with a rapid cutoff near τ_m and behaves like AdS in the r coordinate. As expected from the form of the power series, we have verified that the relative difference of (27) and the numerically determined value of (25) is approximately $1/\tau$, while (25) and (26) are very close for $\tau \gtrsim 1$; Fig. 2 shows the numerically evaluated and leading forms.

The dimensionless form of the CV2.0 complexity is given by the integral

$$J_2(\tau_m) = \int_0^{\tau_m} d\tau j_+(\tau, \tau_m) I(\tau)^{1/2} \sinh^2 \tau, \quad (28)$$

which we evaluate as a double integral by substituting (25). If we exchange the order of integration, the integral

$$J_2(\tau_m) \sim \int^\tau d\eta \int^\eta d\tau \sqrt{\left(\eta - \frac{1}{4}\right) \left(\tau - \frac{1}{4}\right)} e^{4\tau/3} e^{-\eta/3} \quad (29)$$

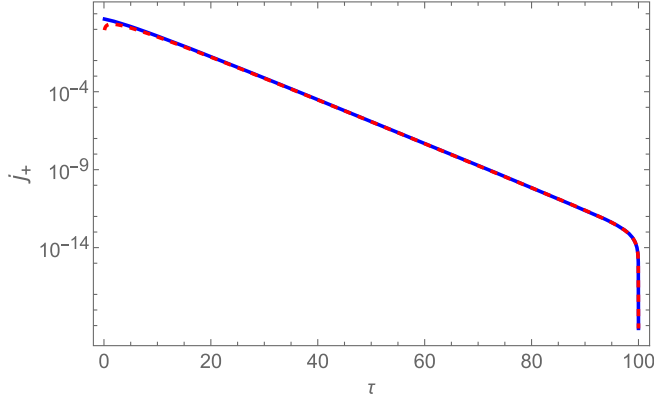


FIG. 2. The dimensionless profile of the WDW patch future boundary, comparing the numerical calculation [solid blue, Eq. (25)] and leading large τ approximation [dashed red, (27)].

captures the divergence structure of the complexity. This integral does not appear to have a closed form in terms of commonly known special functions, though related convergent integrals for $\tau_m \rightarrow \infty$ evaluate to hypergeometric functions.

If we further ignore terms suppressed by powers of $1/\tau_m$, we find the leading divergence

$$J_2(\tau_m) \approx \frac{9}{32} \tau_m e^{\tau_m}. \quad (30)$$

We compare the leading divergence (dashed red) to the numerically evaluated $J_2(\tau_m)$ (solid blue) in Fig. 3. We have confirmed that the relative difference of the leading divergence (30) and the numerical value of (28) scales as $1/\tau_m$ at large τ_m .

In terms of the radial r coordinate, the complexity is

$$C_2 = \frac{\Omega_T}{648 G_{10} \ell^2} L(r_m)^4 r_m^3. \quad (31)$$

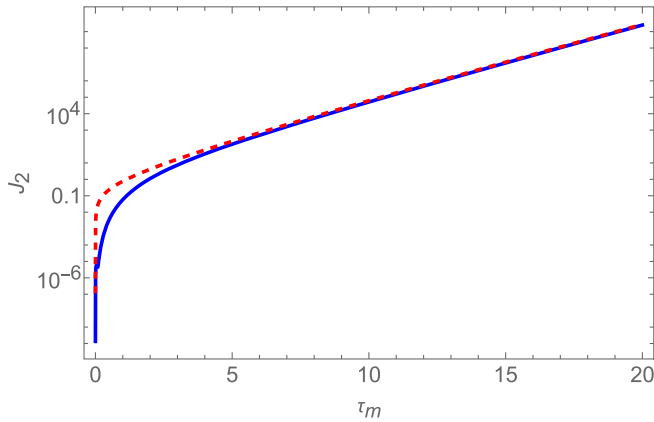


FIG. 3. The dimensionless CV2.0 complexity, evaluated numerically [solid blue, Eq. (28)] and as the leading divergence [dashed red, (30)].

This again agrees with the expectation that the complexity should be the same as in AdS with a slowly varying curvature scale (up to factors of order unity). However, it is interesting to note that the divergence differs from the CV complexity as given in (21), whereas the divergences agree for CV and CV2.0 complexity in AdS. Agreement between the two complexities (up to numerical factors) requires fixing a cutoff and then choosing the length scale ℓ to be the curvature length $L(r_m)$ at the cutoff.

IV. ACTION COMPLEXITY FUNCTIONALS

In this section, we discuss the CA complexity as the ten-dimensional supergravity action, followed by the revised “CA2.0” complexity with additional boundary terms for the supergravity form fields as defined in [19].

A. CA action complexity

1. Review

The CA, also known as “complexity=action,” proposal takes the on-shell action of the background evaluated for the WDW patch as a complexity functional [15,16]. This action includes boundary terms which take the form of integrals on the light sheets that form the future and past boundaries of the WDW patch as well as on the joint at the intersection of those light sheets [36]. In principle, there can be boundary terms for the form fields in a 10D supergravity description of the background; however, agreement between a calculation in $\text{AdS}_5 \times S^5$ and the usual CA complexity of AdS_5 requires only the bulk action, as we will see below. Here, we compare the calculations of the CA complexity for $\text{AdS}_5 \times S^5$ in 5D and 10D since there are some differences.

If we consider AdS_5 only, the complexity is the sum of the bulk action plus a boundary contribution. Since the boundary of the WDW patch consists of two light sheets emitted from Σ on the boundary, the boundary includes a joint at their intersection on Σ , which makes its own contribution. The boundary term includes a logarithmic counterterm necessary for reparametrization invariance which includes an arbitrary length scale ℓ (often chosen as the AdS scale L). The well-known result is $C_A = \ln(3\ell/L)(r_m/L)^3/4\pi^2 G_5$.

In the 10D type IIB supergravity description, the complexity is still $C_A = (S_{\text{bulk}} + S_{\text{bdry}} + S_{\text{joint}})/\pi$, though there are some subtleties. In the bulk action, there is no cosmological constant, which instead follows from the internal curvature and form fluxes. Furthermore, since the 5-form is self-dual, the commonly used manifestly diffeomorphism invariant action is actually a pseudoaction. Instead, we need an action that contains only independent degrees of freedom, i.e., only half the components of $\tilde{F}_5$. We choose to keep the components $\mathcal{F}_5$ with no legs along the time direction. If C_4 are the components of the 4-form potential

also with no legs along t , the bulk action in string frame (and in the absence of free branes) is

$$S_{\text{bulk}} = \frac{1}{2\kappa^2} \int_{\text{WDW}} d^{10}x \sqrt{-g} (e^{-2\phi} R + 4(\partial\phi)^2) - \frac{1}{2\kappa^2} \int_{\text{WDW}} \left(\frac{1}{2} e^{-2\phi} H_3 \wedge \star H_3 + \frac{1}{2} \tilde{F}_1 \wedge \star \tilde{F}_1 + \frac{1}{2} \tilde{F}_3 \wedge \star \tilde{F}_3 + \frac{1}{2} \mathcal{F}_5 \wedge \star \mathcal{F}_5 - \frac{1}{2} C_4 \wedge \tilde{F}_3 \wedge H_3 + \frac{1}{2} \mathcal{F}_5 \wedge C_2 \wedge H_3 - \frac{1}{2} \mathcal{F}_5 \wedge d_t C_4 \right), \quad (32)$$

where $d_t = dt\partial_t$. In $\text{AdS}_5 \times S^5$ and the KS background, the dilaton $\phi = \ln g_s$ is constant, so we take $16\pi G_{10} \equiv 2\kappa^2 g_s^2$. In this case, the boundary and joint terms are the higher-dimensional analogs of the usual terms in AdS :

$$S_{\text{bdry}} = -\frac{1}{8\pi G_{10}} \int_{\partial^+} d\lambda d^8\sigma \sqrt{\gamma} (\kappa_+ + \Theta_+ \ln |\ell \Theta_+|) + \frac{1}{8\pi G_{10}} \int_{\partial^-} d\lambda d^8\sigma \sqrt{\gamma} (\kappa_- + \Theta_- \ln |\ell \Theta_-|), \quad (33)$$

$$S_{\text{joint}} = -\frac{1}{8\pi G_{10}} \int_{\partial^+ \cap \partial^-} d^8\sigma \sqrt{\gamma} \ln |k_+ \cdot k_- / 2|. \quad (34)$$

Here, ∂^+ and ∂^- indicate the future and past boundaries with lightlike parameter λ (of dimension length) increasing

toward the future in both cases. On each light sheet, the vector $k_\pm = dx/d\lambda$ is the lightlike tangent with $\kappa_\pm$ defined by $k_\pm^\mu \nabla_\mu k_\pm^\nu = \kappa_\pm k_\pm^\nu$. The induced metric on each fixed- λ spatial slice is γ_{ab} (along the σ^a coordinates given by $\vec{x}$ and the angular directions), and $\Theta_\pm$ is the logarithmic derivative of $\sqrt{\gamma}$ with respect to λ .

On $\text{AdS}_5 \times S^5$, the boundary and joint terms evaluate in essentially the same fashion as in AdS_5 alone. On the other hand, the bulk term receives no contribution from the Ricci scalar, which cancels between the positively curved sphere and negatively curved AdS , or from any of the flux terms except the integral of $-\mathcal{F}_5 \star \mathcal{F}_5 / 2$, which comes to $-8/L^2$ integrated over the WDW patch. In the end, we find $C_A = \Omega_5 \ln(3\ell/L) L^2 r_m^3 / 4\pi^2 G_{10}$,² which agrees with the pure AdS_5 result under dimensional reduction. Because of this agreement, we do not include boundary terms for $\mathcal{F}_5$ in the action, and we also omit them for H_3 and $\tilde{F}_3$ for consistency.

2. Parameter dependence

We consider the action complexity term by term, starting with the bulk contribution. In the KS background, none of the forms C_4 , $\mathcal{F}_5$, C_2 , $\tilde{F}_3$, or H_3 have any legs along the x^μ directions, so all the Chern-Simons terms [the last line of (32)] vanish. Further, with constant dilaton, the Einstein equation gives $R = (g_s^2 |\tilde{F}_3|^2 + |H_3|^2) / 4$,³ so the bulk action becomes

$$S_{\text{bulk}} = -\frac{1}{16\pi G_{10}} \int_{\text{WDW}} d^{10}x \sqrt{-g} \left(\frac{1}{4} |H_3|^2 + \frac{g_s^2}{4} |\tilde{F}_3|^2 + \frac{g_s^2}{2} |\mathcal{F}_5|^2 \right) = -\frac{\Omega_T}{3(16)^2 \pi G_{10}} \int d^3\vec{x} \int_0^{\tau_m} d\tau \tau_+ (\tau, \tau_m) h(\tau)^{1/2} \sinh^2 \tau \left(\frac{1}{4} |H_3|^2 + \frac{g_s^2}{4} |\tilde{F}_3|^2 + \frac{g_s^2}{2} |\mathcal{F}_5|^2 \right). \quad (35)$$

We can simplify further by noting that the Hodge duality relation between 3-forms implies $|H_3|^2 = g_s^2 |\tilde{F}_3|^2$. Then direct calculation gives us $g_s^2 |\tilde{F}_3|^2 = 3j_3 / (g_s M\alpha')$ and $g_s^2 |\mathcal{F}_5|^2 = 3j_5 / (g_s M\alpha')$, where j_3 and j_5 are dimensionless functions of τ . We end up with

$$S_{\text{bulk}} = -\frac{g_s M\alpha' \epsilon^2 \Omega_T}{2(16)^2 \pi G_{10}} \sqrt{\frac{2}{3}} \left(\int d^3\vec{x} \right) \int_0^{\tau_m} d\tau j_+(\tau, \tau_m) I(\tau)^{1/2} \sinh^2(\tau) (j_3(\tau) + j_5(\tau)). \quad (36)$$

The overall scaling of the boundary and joint terms (33) and (34) comes from the spatial slice metric

$$\sqrt{\gamma} = \frac{g_s M\alpha' \epsilon^2}{16\sqrt{6}} [\sin \theta_1 \sin \theta_2] \sqrt{I(\tau)} \sinh \tau (\sinh(2\tau) - 2\tau)^{1/3}. \quad (37)$$

This also leads to a scaling $C_A \propto (g_s M\alpha') \epsilon^2 \Omega_T$. However, there is also a logarithmic dependence on $(g_s M\alpha')$, which appears in the joint term through

$$k_+ \cdot k_- = -\left(\frac{1}{\alpha_+ \alpha_- \ell^2} \right) \left[h(\tau)^{-1/2} \left(\frac{dt_+}{d\tau} \right)^2 + \frac{h(\tau)^{1/2} \epsilon^{4/3}}{6K(\tau)^2} \right], \quad (38)$$

where we define the lightlike parameter $\lambda = \pm \alpha_\pm \ell (\tau_m - \tau)$ on the future (past) light sheet [note that $k_\pm^\tau = \mp 1/\ell \alpha_\pm$,

²One of us (Frey) originally calculated this result in collaboration with N. Agarwal in separate work [37].

³We take the convention that $|F_p|^2 = F_{M_1 \dots M_p} F^{M_1 \dots M_p} / p!$.

$k_{\pm}^t = k_{\pm}^{\tau} dt_{\pm}/d\tau = -(1/\ell\alpha_{\pm})dt_{\pm}/d\tau]$. We have introduced the length scale ℓ to give λ the correct dimensionality; $\alpha_{\pm}$ are additional arbitrary constants. From the scaling of the warp factor and $t_{\pm}$, we find $k_{+} \cdot k_{-} = -(g_s M\alpha'/\ell^2) \times (j_{\text{joint}}/\alpha_{+}\alpha_{-})$, where j_{joint} is a dimensionless function of τ_m . Once we convert the light sheet actions to integrals over τ , $\alpha_{\pm}$ will cancel between the $\ln|\Theta|$ and joint terms. This leaves $\ln(g_s M\alpha'/\ell^2)$ as a further parameter dependence.

3. Divergence structure

As is typical for the action complexity, finding the divergence structure J_A is more complicated than for volume complexities.

We begin with the bulk action term. After some simplification, the dimensionless flux functions are

$$j_3(\tau) = \frac{1}{4I(\tau)^{3/2} \sinh^6(\tau)} [\cosh(4\tau) - 24\tau \sinh(2\tau) + 8(1 + \tau^2) \cosh(2\tau) + 16\tau^2 - 9], \quad (39)$$

$$j_5(\tau) = \frac{1}{I(\tau)^{5/2} \sinh^6(\tau)} (\tau \coth \tau - 1)^2 (\sinh(2\tau) - 2\tau)^{4/3}, \quad (40)$$

and we define the dimensionless bulk action integral

$$J_{\text{bulk}}(\tau_m) = \int_0^{\tau_m} d\tau j_{+}(\tau, \tau_m) \sqrt{I(\tau)} \sinh^2 \tau (j_3(\tau) + j_5(\tau)). \quad (41)$$

To evaluate this numerically, we substitute (25) for j_{+} and calculate J_{bulk} as a double integral. We can also write the

divergent part as

$$\begin{aligned} J_{\text{bulk}}(\tau_m) &\sim 6\sqrt{\frac{2}{3}} \int_0^{\tau_m} d\tau \frac{2\tau^2/3 - \tau/3 + 5/12}{(\tau - 1/4)^2} \\ &\times e^{4\tau/3} (\sqrt{\tau} e^{-\tau/3} - \sqrt{\tau_m} e^{-\tau_m/3}) \\ &\approx \sqrt{\frac{2}{3}} \sqrt{\tau_m} e^{\tau_m}. \end{aligned} \quad (42)$$

It is possible to rewrite this form in terms of γ_1 or γ^* functions as in (19)⁴; however, we keep only the leading divergence as given because the light sheet action divergence does not have a general closed form.

In the action on the light sheets $\partial^{\pm}$, we can define $\kappa_{\pm} \equiv k_{\pm}^{\tau} \bar{\kappa}$ and $\Theta_{\pm} \equiv k_{\pm}^{\tau} \bar{\Theta}$. Then

$$\begin{aligned} S_{\text{bdry}} &= \frac{1}{4\pi G_{10}} \int d^8\sigma \int_0^{\tau_m} d\tau \sqrt{\gamma} (\bar{\kappa} + \bar{\Theta} \ln |\bar{\Theta}|) \\ &+ \frac{1}{2} \bar{\Theta} \ln |k_{+}^{\tau} k_{-}^{\tau} \ell^2|. \end{aligned} \quad (43)$$

Since $k_{\pm}^{\tau}$ are constant and $\sqrt{\gamma} \bar{\Theta} = \partial_{\tau} \sqrt{\gamma}$, the last term is supported only on the joint. Therefore,

$$\begin{aligned} S_{\text{bdry}} + S_{\text{joint}} &= \frac{1}{4\pi G_{10}} \int d^8\sigma \int_0^{\tau_m} d\tau \sqrt{\gamma} (\bar{\kappa} + \bar{\Theta} \ln |\bar{\Theta}|) \\ &- \frac{1}{8\pi G_{10}} \int d^8\sigma \sqrt{\gamma} \ln \left(\frac{g_s M\alpha'}{\ell^2} |j_{\text{joint}}(\tau_m)| \right), \end{aligned} \quad (44)$$

where

$$\begin{aligned} \bar{\kappa} &= -\frac{1}{K(\tau)} \frac{dK(\tau)}{d\tau} = \frac{\cosh(2\tau) - 6\tau \coth(\tau) + 5}{3(\sinh(2\tau) - 2\tau)}, \\ \bar{\Theta} &= \frac{2I(\tau)(3 \coth \tau (\sinh(2\tau) - 2\tau) + 2(\cosh(2\tau) - 1)) - 3 \text{csch} \tau (\tau \coth \tau - 1)(\sinh(2\tau) - 2\tau)^{4/3}}{6I(\tau)(\sinh(2\tau) - 2\tau)}, \\ j_{\text{joint}} &= \frac{1}{2g_s M\alpha'} \left[h(\tau_m)^{-1/2} \left(\frac{dt_{+}}{d\tau} \right)^2 + \frac{h(\tau_m)^{1/2} \epsilon^{4/3}}{6K(\tau_m)^2} \right] = \frac{\sqrt{I(\tau_m)} \sinh^2 \tau_m}{3(\sinh(2\tau_m) - 2\tau_m)^{2/3}}. \end{aligned} \quad (45)$$

The dimensionless actions are given by $J_{\text{bdry}}(\tau_m) + J_{\text{joint}}(\tau_m)$, where

$$J_{\text{bdry}}(\tau_m) = \int_0^{\tau_m} d\tau \sqrt{I(\tau)} \sinh \tau (\sinh(2\tau) - 2\tau)^{1/3} (\bar{\kappa} + \bar{\Theta} \ln \bar{\Theta}) \quad (46)$$

and

$$J_{\text{joint}}(\tau_m) = \sqrt{I(\tau_m)} \sinh \tau_m (\sinh(2\tau_m) - 2\tau_m)^{1/3} \ln \left[\left(\frac{g_s M\alpha'}{\ell^2} \right) \frac{\sqrt{I(\tau_m)} \sinh^2(\tau_m)}{3(\sinh(2\tau_m) - 2\tau_m)^{2/3}} \right]. \quad (47)$$

⁴Or rather lower incomplete versions of those due to divergences at the lower limit in some cases.

The divergences are

$$J_{\text{bdry}}(\tau_m) \sim \frac{1}{2} \sqrt{\frac{3}{2}} \int^{\tau_m} d\tau \sqrt{\tau - \frac{1}{4}} e^{\tau} \left[\frac{1}{3} + \left(\frac{4\tau + 1}{4\tau - 1} \right) \ln \left(\frac{4\tau + 1}{4\tau - 1} \right) \right] \approx \frac{1}{2\sqrt{6}} \sqrt{\tau_m} e^{\tau_m} \quad (48)$$

and

$$J_{\text{joint}}(\tau_m) \sim \frac{1}{2} \sqrt{\frac{3}{2}} \sqrt{\tau_m - \frac{1}{4}} e^{\tau_m} \ln \left[\frac{1}{2\sqrt{6}} \left(\frac{g_s M \alpha'}{\ell^2} \right) \sqrt{\tau_m - \frac{1}{4}} \right] \approx \frac{1}{2} \sqrt{\frac{3}{2}} \sqrt{\tau_m} e^{\tau_m} \ln \left[\frac{1}{2\sqrt{6}} \left(\frac{g_s M \alpha'}{\ell^2} \right) \sqrt{\tau_m} \right]. \quad (49)$$

We again keep only the leading divergence. Note that the expansion terms $(\bar{\Theta} \ln \bar{\Theta})$ give only subleading divergences.

With these definitions, the complexity is

$$\begin{aligned} C_A &= \frac{1}{8\pi^2 G_{10}} \frac{g_s M \alpha' \epsilon^2 \Omega_T}{16\sqrt{6}} \left(-\frac{1}{2} J_{\text{bulk}}(\tau_m) \right. \\ &\quad \left. + 2J_{\text{bdry}}(\tau_m) - J_{\text{joint}}(\tau_m) \right) \\ &\equiv \frac{1}{8\pi^2 G_{10}} \frac{g_s M \alpha' \epsilon^2 \Omega_T}{16\sqrt{6}} J_A(\tau_m). \end{aligned} \quad (50)$$

Taking the leading divergence, the contributions from the bulk and boundary cancel to give

$$J_A(\tau_m) \approx -\frac{1}{2} \sqrt{\frac{3}{2}} \sqrt{\tau_m} e^{\tau_m} \ln \left[\frac{1}{2\sqrt{6}} \left(\frac{g_s M \alpha'}{\ell^2} \right) \sqrt{\tau_m} \right]. \quad (51)$$

We show J_A and its leading divergence in Fig. 4. Note that the sign of J_A changes as τ_m increases for any fixed value of the arbitrary length ℓ . At large τ_m , the relative difference scales as $1/\tau_m$ as expected.

In terms of the standard radial coordinate r , the leading divergence is

$$C_A = \frac{1}{4\pi^2 G_{10}} \frac{\Omega_T}{108} L(r_m)^2 r_m^3 \ln \left(\frac{3\ell}{L(r_m)} \right), \quad (52)$$

which is again the same functional form as for $\text{AdS}_5 \times S^5$ but with a slowly varying curvature scale. The unusual feature is that the complexity changes sign with changing cutoff r_m if the arbitrary scale ℓ is fixed. However, from the point of view of evaluating complexity of excited states of KS (for example, black holes), we should work with a fixed cutoff, and we are free to choose ℓ based on that cutoff [such as $\ell = L(r_m)$].

B. CA2.0 action complexity

1. Review

As discussed in [19], the gauge transformation of the Chern-Simons terms in the last line of the bulk action (32) leads to nontrivial terms on the boundary of the WDW

patch. While these vanish on the KS background, the CA complexity is therefore not gauge invariant on a general background. As a result, [19] proposed including additional boundary terms for the form fields that cancel this gauge variation on shell. There is an infinite family of boundary terms that gives a gauge-invariant complexity functional; [19] chose a set of boundary terms that respect the $SL(2, Z)$ duality and named the resulting complexity CA2.0, which we denote $C_{\mathbb{A}}$. On shell, it is possible to write those boundary terms as a contribution to the bulk integrand, so $C_{\mathbb{A}} = (S'_{\text{bulk}} + S_{\text{bdry}} + S_{\text{joint}})/\pi$, where S_{bdry} and S_{joint} are

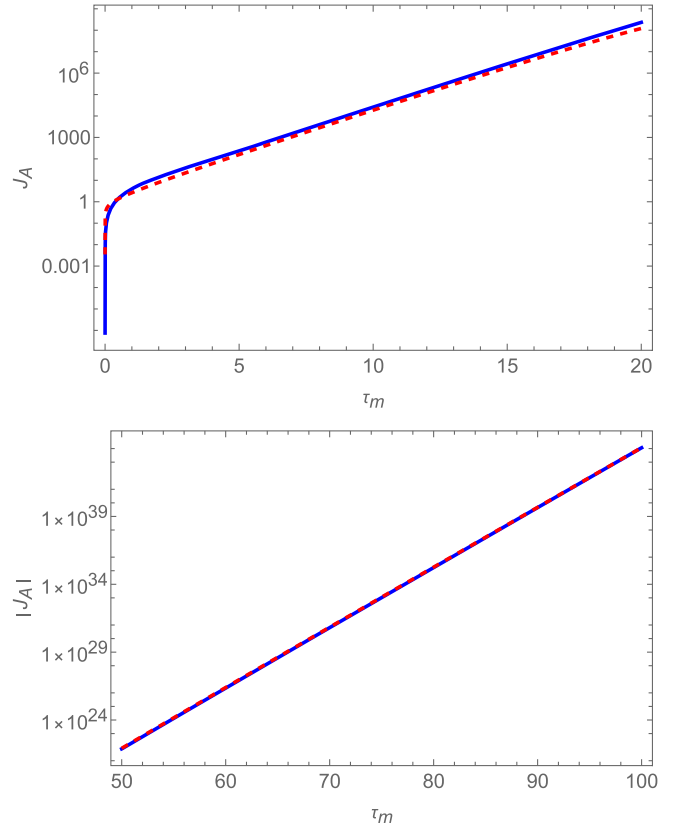


FIG. 4. The dimensionless CA complexity, evaluated numerically [solid blue, from Eqs. (41), (44)–(47), and (50)] and as the leading divergence [dashed red, (51)]. To plot, we choose $\ell^2 = g_s M \alpha'$. Note that the complexity becomes negative at large cutoff, so we show the absolute value.

given by (33) and (34), respectively, and

$$S'_{\text{bulk}} = \frac{1}{2\kappa^2} \int_{\text{WDW}} d^{10}x \sqrt{-g} (e^{-2\phi} R + 4(\partial\phi)^2) - \frac{1}{2\kappa^2} \int_{\text{WDW}} \left(\frac{1}{4} e^{-2\phi} H_3 \wedge \star H_3 + \frac{1}{2} \tilde{F}_1 \wedge \star \tilde{F}_1 + \frac{1}{4} \tilde{F}_3 \wedge \star \tilde{F}_3 \right). \quad (53)$$

For $\text{AdS}_5 \times S^5$ and the KS background, the curvature and modified form field strength kinetic terms cancel, so $S'_{\text{bulk}} = 0$. As a result, $C_{\mathbb{A}}$ is the sum of only the light sheet and joint terms. For $\text{AdS}_5 \times S^5$, we have $C_{\mathbb{A}} = \Omega_5 [\ln(3\ell/L) + 1/3] L^2 r_m^3 / 4\pi^2 G_{10}$.

In the sense that S'_{bulk} vanishes, the CA2.0 complexity presented here is an extreme, and other gauge-invariant choices for the form boundary terms should be intermediate between CA2.0 and the standard CA complexity.

2. Parameter dependence

Because the CA2.0 complexity is composed of a subset of the terms from the CA complexity, the parametric dependence is the same. That is, $C_{\mathbb{A}} \propto (g_s M \alpha') \epsilon^2 \Omega_T$ with an additional logarithmic dependence on $g_s M \alpha' / \ell^2$.

3. Divergence structure

Following the discussion above, the CA2.0 complexity is

$$C_{\mathbb{A}} = \frac{1}{8\pi^2 G_{10}} \frac{g_s M \alpha' \epsilon^2 \Omega_T}{16\sqrt{6}} J_{\mathbb{A}}(\tau_m), \quad J_{\mathbb{A}}(\tau_m) \equiv 2J_{\text{bdry}}(\tau_m) - J_{\text{joint}}(\tau_m), \quad (54)$$

where J_{bdry} and J_{joint} are defined in (46) and (47). The leading divergences are therefore

$$J_{\mathbb{A}}(\tau_m) \approx \frac{1}{\sqrt{6}} \sqrt{\tau_m} e^{\tau_m} \left\{ 1 - \frac{3}{2} \ln \left[\frac{1}{2\sqrt{6}} \left(\frac{g_s M \alpha'}{\ell^2} \right) \sqrt{\tau_m} \right] \right\} \approx J_A(\tau_m). \quad (55)$$

These terms have the same exponential and power law divergence, though the logarithmic term dominates at very large τ_m , so the leading divergence is the same as for the CA complexity. At large enough τ_m , the CA2.0 complexity follows the right panel of Fig. 4.

For fixed ℓ , there is however an important difference, which is the value τ_m^* of the cutoff for which the complexity changes sign. Because of the additional constant term in the brackets of (55), τ_m^* is considerably larger for CA2.0 than for CA complexity. In fact, τ_m^* is exponentially sensitive to terms outside the logarithm, so even the subleading divergences (suppressed above) shift τ_m^* noticeably.

In terms of the radial coordinate r , (55) gives

$$C_{\mathbb{A}} = \frac{1}{4\pi^2 G_{10}} \frac{\Omega_T}{108} L(r_m)^2 r_m^3 \left[\frac{1}{3} + \ln \left(\frac{3\ell}{L(r_m)} \right) \right] \quad (56)$$

for the leading divergence.

V. FLUX COMPLEXITY FUNCTIONALS

In ten-dimensional supergravity, the flux provides a (local) length scale due to its dimensionality, much like the curvature. As a result, [19] defined two “flux complexity” functionals CF and CF2.0 in parallel to the two volume complexities using the flux to replace the arbitrary length scale.

A. CF flux complexity

1. Review

The CF complexity (denoted C_F) [19] is the integral

$$C_F \equiv \frac{1}{G_{10}} \int_{\mathcal{B}} d^9x \sqrt{|g|} \sqrt{\mathcal{G}}, \quad (57)$$

where $\mathcal{B}$ is the spatial slice that optimizes the integral and

$$\mathcal{G} \equiv \frac{g_s^2}{16} |\mathcal{F}_5|^2 + \frac{a}{16} (|H_3|^2 + g_s^2 |\tilde{F}_3|^2). \quad (58)$$

In $\text{AdS}_5 \times S^5$, $g_s^2 |\mathcal{F}_5|^2 = 16/L^2$ is constant, so $C_F = C_V$ for $\text{AdS}_5 \times S^5$ with the given normalization (with $\ell = L$ for CV), and a is an arbitrary constant.

Compared to [19], we have set an additional term in $\mathcal{G}$ (which is constant in KS and $\text{AdS}_5 \times S^5$) to zero, and we have taken into account a simplification due to the fact that these flux components have no legs in the time direction.

2. Parameter dependence

In the notation of Sec. IVA 2, the flux complexity density is

$$C_F = \frac{\sqrt{6} g_s M \alpha' \epsilon^2 \Omega_T}{384 G_{10}} \int_0^{\tau_m} d\tau I(\tau)^{3/4} \sinh^2(\tau) \times (j_5(\tau) + 2a j_3(\tau))^{1/2} \equiv \frac{\sqrt{6} g_s M \alpha' \epsilon^2 \Omega_T}{384 G_{10}} J_F(\tau_m). \quad (59)$$

For a fixed τ_m , this shows a different scaling with $g_s M \alpha'$ than the CV complexity but is instead similar to the action complexities.

3. Divergence structure

The dimensionless complexity has the divergence

$$J_F(\tau_m) \sim \int^{\tau_m} d\tau \sqrt{a + \frac{2(\tau-1)^2}{3\tau-1/4}} e^\tau \approx \sqrt{\frac{2}{3}} \sqrt{\tau_m} e^{\tau_m}. \quad (60)$$

Notice that the leading divergence, given after the $\approx$ sign, is independent of the free parameter a because $j_5/j_3 \approx 2\tau/3$. Therefore, the leading divergence is given entirely by the 5-form flux rather than the 3-forms; however, the 3-forms contribute to subleading divergences.

For illustrative purposes, we have calculated $J_F(\tau_m)$ for both $a = 0$ and $a = 5$, which we show in Fig. 5 along with the leading divergence. We have verified that the relative difference of the three curves drops as $1/\tau_m$ at large values of the cutoff.

While CF equals the CV complexity in $\text{AdS}_5 \times S^5$ when the arbitrary scale $\ell = L$, the AdS radius, the leading divergence (60) is softer than the corresponding divergence for CV complexity in KS. Specifically, if the cutoff is written in terms of the radial coordinate r , the divergence is

$$C_F = \frac{\Omega_T}{(18)^2 G_{10}} L(r_m)^2 r_m^3. \quad (61)$$

This matches the CV result (21) precisely when the arbitrary length is chosen to match the curvature scale at the cutoff $\ell = L(r_m)$. However, the subleading divergences and finite parts are presumably different.

B. CF2.0 flux complexity

1. Review

From [19], the CF2.0 complexity (indicated with $\mathbb{F}$) is given by the integral

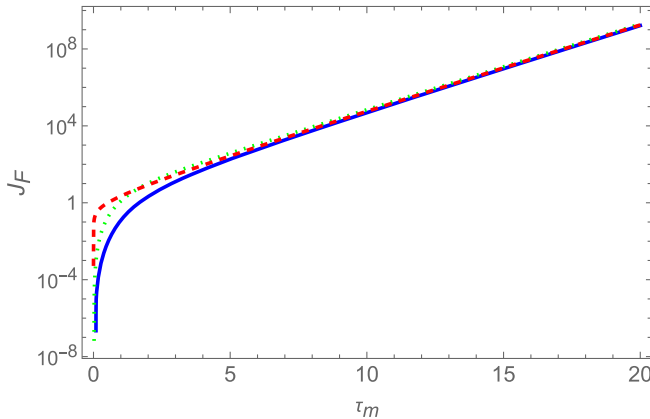


FIG. 5. The dimensionless CF complexity J_F as a function of cutoff τ_m . Compares the numerical calculation for $a = 0$ (solid blue) and $a = 5$ (dotted green) [both from (59)] to the leading divergence [dashed red, (60)].

$$C_{\mathbb{F}} \equiv \frac{1}{G_{10}} \int_{\text{WDW}} d^{10}x \sqrt{-g} \mathcal{G} \quad (62)$$

over the WDW patch with $\mathcal{G}$ as in (58). Like the CF complexity equals the CV complexity with the arbitrary length scale ℓ assigned to the AdS scale L , this CF2.0 complexity is the same as the CV2.0 complexity with $\ell = L$. Specifically, the complexity density is $C_{\mathbb{F}} = \Omega_5 L^2 r_m^3 / 6G_{10}$.

2. Parameter dependence

Following our previous discussions, the CF2.0 complexity is

$$C_{\mathbb{F}} = \frac{g_s M \alpha' \epsilon^2 \Omega_T}{128 \sqrt{6} G_{10}} J_{\mathbb{F}}(\tau_m),$$

$$J_{\mathbb{F}}(\tau_m) = \int_0^{\tau_m} d\tau j_+(\tau, \tau_m) \sqrt{I(\tau)} \sinh^2(\tau) (j_5(\tau) + 2a j_3(\tau)). \quad (63)$$

Like the CF complexity, the scaling with $g_s M \alpha'$ is the same as in the action complexities as opposed to the volume complexities, even though the flux complexities mimic the volume complexities. The reason for the difference is the absence of an arbitrary length scale in the overall prefactor in the flux complexities. Even though setting the overall normalization of the flux complexity is akin to specifying the length scale of the volume complexity, it is dimensional analysis that sets the scaling with $g_s M \alpha'$.

3. Divergence structure

The dimensionless complexity functional $J_{\mathbb{F}}(\tau_m)$ reduces to the functional $J_{\text{bulk}}(\tau_m)$ from CA complexity when $a = 1/2$. Like that integral, we evaluate it numerically by writing $j_+(\tau, \tau_m)$ using (25) and calculating $J_{\mathbb{F}}(\tau_m)$ as a double integral.

Analytically, the divergence part of the complexity is

$$J_{\mathbb{F}}(\tau_m) \sim 4 \sqrt{\frac{3}{2}} \int^{\tau_m} d\tau (\sqrt{\tau} e^{-\tau/3} - \sqrt{\tau_m} e^{-\tau_m/3}) e^{4\tau/3} \times \left[\frac{a + 2(\tau-1)^2/3(\tau-1/4)}{\tau-1/4} \right] \quad (64)$$

with leading term

$$J_{\mathbb{F}}(\tau_m) \approx \frac{2}{\sqrt{6}} \sqrt{\tau_m} e^{\tau_m}. \quad (65)$$

We show the leading CF2.0 divergence in Fig. 6 along with numerical calculations of $J_{\mathbb{F}}$ for $a = 0$ and $a = 5$. For large cutoff values τ_m , the difference among these curves decreases as $1/\tau_m$ as expected.

In terms of the radial coordinate r , the leading divergence is the same as for the CF complexity, up to the numerical coefficient:

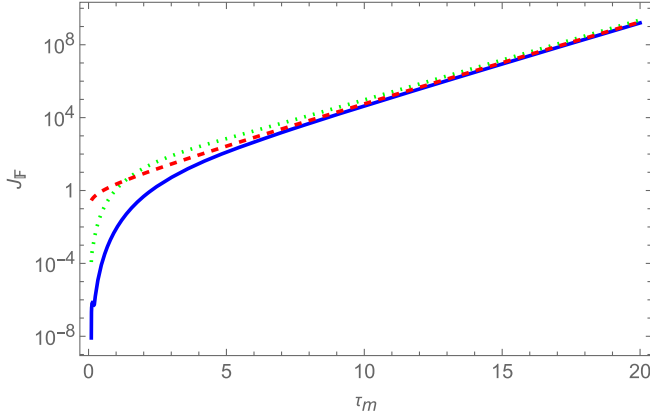


FIG. 6. The dimensionless CF2.0 complexity $J_{\mathbb{F}}$ as a function of cutoff τ_m . Compares the numerical calculation for $a = 0$ (solid blue) and $a = 5$ (dotted green) [both Eq. (63)] to the leading divergence [dashed red, (65)].

$$C_{\mathbb{F}} = \frac{\Omega_T}{648G_{10}} L(r_m)^2 r_m^3. \quad (66)$$

The numerical prefactor is the same as for C_2 , but the divergence is considerably softer for a fixed value of the CV2.0 length scale ℓ . The idea that the leading divergence should match based on intuition from $\text{AdS}_5 \times X^5$ backgrounds suggests choosing the arbitrary scale to be the curvature length at the cutoff after fixing the cutoff. We note, however, that subleading divergences and finite parts will differ between the CF2.0 and CV2.0 complexities.

VI. VARIATION ON THE BARYONIC BRANCH

In this section, we consider the variation of holographic complexity around the KS background along the baryonic branch. We will work with linear perturbations and demonstrate that the KS background is an extremum of all of the complexity functionals considered here (as well as a broad class of others).

We begin by describing the baryonic branch linearized around the KS background as given by [28] following the discussion in [29]. At the linear level in deformation parameter q , the warp factor $h(\tau)$ is unchanged, as are $\tilde{F}_3$ and $\mathcal{F}_5$. In terms of the basis forms g_i of (3), the linear deformation of H_3 is

$$\delta H_3 = q\chi'(\tau)d\tau \wedge (g_2 \wedge g_3 - g_1 \wedge g_4). \quad (67)$$

Also writing the metric in terms of the g_i basis,

$$\begin{aligned} \delta\tilde{g}_{13} &= \delta\tilde{g}_{24} = q \cosh(\tau) K(\tau) Z(\tau), \\ \delta\tilde{g}^{13} &= \delta\tilde{g}^{24} = -q \cosh(\tau) \coth^2(\tau) K(\tau) Z(\tau). \end{aligned} \quad (68)$$

In the above, $\chi'(\tau)$ and $Z(\tau)$ can be written in terms of hypertrigonometric functions, but their precise form is not

important to us. Finally, the axiodilaton is unperturbed (and remains constant). We note that [29] constructed the baryonic branch backgrounds only along a one-parameter curve in the moduli space because other solutions are related by S duality; see more discussion below.

We can now see that the constituents of the complexity functionals discussed here have vanishing first-order variation along the baryonic branch.⁵ First, because $\delta\tilde{g}_{mn}$ is traceless and the warp factor is unchanged at linear order, neither the nine-dimensional volume measure of a constant t slice nor the ten-dimensional volume measure (within the WDW patch) changes. Next, it is straightforward to see by checking indices that

$$\delta(|\tilde{F}_3|^2) = h^{-3/2}(\tau) \frac{1}{2} \tilde{F}_{mnp} \tilde{F}_{qrs} \tilde{g}^{mq} \tilde{g}^{nr} \delta\tilde{g}^{ps} = 0. \quad (69)$$

Similarly, $\delta(|\mathcal{F}_5|^2) = 0$. The square of the Neveu-Schwarz-Neveu-Schwarz 3-form flux is slightly more complicated:

$$\begin{aligned} \delta(|H_3|^2) &= h^{-3/2}(\tau) \frac{1}{2} H_{mnp} H_{qrs} \tilde{g}^{mq} \tilde{g}^{nr} \delta\tilde{g}^{ps} \\ &\quad + h^{-3/2}(\tau) \frac{1}{3} H_{mnp} \delta H_{qrs} \tilde{g}^{mq} \tilde{g}^{nr} \tilde{g}^{ps} = 0 \end{aligned} \quad (70)$$

also by comparing the components of H_3 and δH_3 . Additionally, we can see that $\tilde{F}_{mnp} H^{mnp}$ has zero variation. Finally, since the Ricci scalar is given by a sum of $|\tilde{F}_3|^2$ and $|H_3|^2$, it is also unchanged. (Further, the Ricci tensor is diagonal, so powers of it are also unperturbed.)

Taken together, these results mean that the first variation of all the complexity functionals discussed in this paper vanishes when evaluated on the KS background. A large family of more general complexity functionals are also extremized on the KS background for the same reason. Complexity can be defined in quantum mechanics as the distance of the state in question from some reference state along an optimal path in state space [2–4]; following the first law of complexity [38,39], we see that the first variation along the baryonic branch is orthogonal to that optimal path. We also note that our calculation agrees with the first law of complexity in that the variation of complexity depends only on the variation of the target state (namely the supergravity background in our case).

It is worth considering why complexity might be extremized at KS solution along the baryonic branch; we speculate that there is a rotational symmetry of the moduli space with a fixed point at the KS solution. In that case, smoothness (specifically absence of a kink) of complexity as a function of baryonic modulus requires its derivative to

⁵We have also verified these results by evaluating the ansatz of [29] on the linearized baryonic branch around the KS background.

vanish at the KS point. This rotational symmetry is not immediately obvious in the usual description of the baryonic branch in the gauge theory, which is parametrized by a complex variable ξ with $\xi = 1$ at the $\mathbb{Z}_2$ symmetric point [28]. However, as discussed in [29], the supersymmetry of the supergravity dual can be parametrized by a real function α and a complex function β with the KS background at $\beta = 0$; [29] (following [28]) found the baryonic branch solution for imaginary β . Real and imaginary β are related by S duality, so derivatives of the complexity in those directions must vanish at $\beta = 0$ to be compatible with S duality and smoothness.⁶ Smoothness then implies that the variation of the complexity must vanish along any direction of the baryonic branch at the KS background. If this reasoning holds, the extremization of complexity at the KS background is a consequence of duality.

VII. DISCUSSION

We have evaluated the holographic complexity of the Klebanov-Strassler background with several complexity functionals, including the CA2.0, CF, and CF2.0 functionals proposed recently in [19] based on considerations from the 10D supergravity. The KS background is an interesting test case for holographic complexity in both 10D and nonconformal gravity backgrounds, and we compare the complexity in KS to that of $\text{AdS}_5 \times S^5$ (as evaluated in 10D variables). At large radius, KS is approximately $\text{AdS}_5 \times T^{1,1}$ with a logarithmically varying curvature length $L(r)$; substituting this value $L(r_m)$ at the cutoff for the AdS scale transforms the complexity of $\text{AdS}_5 \times T^{1,1}$ into the leading divergence of the complexity of KS (up to numerical factors). Specifically, this leading divergence is $f(L(r_m))r_m^3$, where $f(L)$ is a power or a power times a logarithm, depending on the complexity functional.

In fact, if we insist that the AdS complexity determines the leading divergence, rewriting that divergence in terms of the dimensionless radial coordinate τ controls the dependence on the gauge theory parameters g_s , M , and ϵ . Most importantly, the universal r_m^3 factor is $\epsilon^2 \exp(\tau_m)$ (times numerical factors of order unity), and ϵ^2 is the cube of the confinement length. This is the same scaling as the complexity of formation for AdS solitons, which may suggest a universal role for confinement in complexity. Meanwhile, the dependence on g_s and M follows from the functional dependence on $L(r_m)$. An interesting point is that the dependences on ϵ and with $g_s M$ both rely on how we fix parameters: is the cutoff radius a fixed τ_m or r_m , and do we fix the 10D Newton constant or the 5D one in an effective theory?

⁶We speculate that the supergravity $SL(2, \mathbb{R})$ symmetry is related to rotations in the β plane, although we do not know the detailed mapping.

However, the complexity of KS is not only the leading term, which diverges faster than the AdS complexity due to the growth of $L(r_m)$. There is also a series of divergent terms of the form $\exp(\tau_m) \sum_p \tau_m^{n-p}$ (times $\ln \tau_m$ for some complexities), as well as a finite contribution. As a result, the dependence on g_s , M , and ϵ is simplest if we define the cutoff radius as a fixed τ_m . Furthermore, even the leading divergence differs among complexity functionals if arbitrary lengths ℓ in the functional definitions are held fixed. For example, if the CV, CV2.0, and CA functionals are to have parametrically similar values, we should choose ℓ for all three as the UV length scale $L(r_m)$ (times a constant). We can ask if this is a hint toward associating the arbitrary scale ℓ with a UV scale.

The full complexity of the KS ground state, which we determine numerically for several functionals, is necessary to define the complexity of formation for excited backgrounds like thermal (black brane) states (the complexity of formation is the difference between the excited and ground state complexities). Given that there may be no closed form even for all the divergences in terms of commonly known special functions, it may be useful to make a modified definition for complexity of formation in asymptotically KS backgrounds. On the other hand, it may be possible to define a complexity of formation by subtracting the integrands when writing the complexities as integrals over τ . An interesting important question is whether exciting the KS background can change the subleading divergences, which would seem to make it impossible to define a finite complexity of formation in the standard way.

We make a point of comparison to other work. Recently, [40] proposed a complexity functional for 10D backgrounds involving an overall conformal factor in the metric. In the KS solution, the complexity density of [40] is⁷

$$\mathcal{C} \propto \epsilon^4 \int d\tau h(\tau) \sinh^2(\tau), \quad (71)$$

which has a different scaling with ϵ and a leading divergence of a different form [i.e., not $f(L(r_m))r_m^3$]. These differences highlight the need to understand the relation between different complexity functionals.

Somewhat more speculatively, our work may be helpful in understanding the holographic complexity of de Sitter (dS) spacetime [41] in the context of string theory. Specifically, an important class of dS compactifications [42] includes KS or similar warped throats in the compactification geometry.

Finally, as a separate point, because the KS gauge theory has a moduli space (the baryonic branch), the variation of the complexity along that moduli space is a physically meaningful quantity. We have shown for a broad family of complexity functionals, including all six functionals

⁷We thank C. Nuñez for providing this formula.

considered here, that the holographic complexity is extremized on the KS background. That is, the first variation of the complexity along the baryonic branch, including all divergences and finite contributions, vanishes on the KS background.

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DATA AVAILABILITY

The data that support the findings of this article are openly available [43].

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