

Series solutions to the Tolman-Oppenheimer-Volkoff equations

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We present general series solutions to the Tolman-Oppenheimer-Volkoff equations for compact stellar objects. We develop an algorithm to compute the coefficients of the power series in terms of the equation of state and its derivatives with respect to the thermodynamic variables. Using these results, we establish general properties of analytic solutions and their relation to the regularity of the equation of state. Applying the theory of Padé approximants, we derive series representations for meromorphic functions whose domains of convergence may include isolated poles. These analytic solutions are then used to obtain closed-form expressions to approximate the radius and mass of stellar objects. We apply the formalism to specific models, namely fluids with affine equations of state and polytropic fluids, and compare the results with those obtained from numerical integration. Lastly, we extend the formalism to piecewise equations of state, deriving series solutions that can be matched across transition hypersurfaces.

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I. INTRODUCTION

Compact stellar objects are among the richest environments for probing fundamental physics in regimes that are far beyond those presently accessible to manmade laboratories. Active and upcoming electromagnetic and gravitational-wave detectors are expected to provide accurate and statistically significant data on the macroscopic properties and the evolution of massive compact objects. Relating the macroscopic features of stellar objects to fundamental properties of the matter fields that compose them will enable tighter constraints on theoretical models and deepen our understanding of matter under extreme conditions [1–8].

To a first approximation, compact stellar objects can be characterized by static, spherically symmetric solutions of the theory of general relativity. Under these symmetries and for a perfect fluid source, the Einstein field equations can be cast in a simplified form: the Tolman-Oppenheimer-Volkoff (TOV) equations. In the presence of matter, the system is closed by specifying the fluid source, typically in the form of an equation of state (EOS). Depending on the properties of the matter fields, different solutions exist with markedly distinct behavior.

Deriving an EOS from nuclear physics or QCD that accurately describes matter under the conditions present in compact objects interiors remains an extremely challenging task. Although significant progress has been made in recent decades, a key difficulty lies in the poorly constrained properties of matter at extreme densities, particularly those

expected to be found in neutron star cores. This uncertainty has motivated the development of numerous models based on different assumptions about the state of matter [9–14]. Alternatively, extensive research has gone into constructing phenomenological models that fit the EOS to available observational data (see, e.g., Refs. [15–17] and references therein). Nonetheless, to constrain the EOS, the theoretical predictions of the TOV equations must be compared against experimental data. In that regard, obtaining reliable solutions to the equations is essential.

Given the complexity of the TOV equations, very few exact solutions suitable for stellar objects are known (see, e.g., Refs. [18,19]). For a realistic EOS, numerical methods are often required to find approximate solutions. However, this makes it difficult to extract general relationships between stellar properties and the parameters of the EOS. In contrast, analytic solutions can provide valuable qualitative insight into these functional dependencies, which is crucial for connecting theoretical predictions with observations. Moreover, the TOV equations form a stiff system of differential equations with a singular point. Depending on the EOS, their numerical integration is nontrivial, often requiring working with very high precision or considering reformulations of the original equations to control numerical errors [20].

It is then not surprising that significant attention has been directed to find exact solutions of the TOV equations. In Refs. [21–25], generating theorems were proposed to find solutions by deforming a starting seed exact solution or by imposing a particular form for one of the metric coefficients. In Ref. [26], the TOV equations were cast in an explicit

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covariant form, allowing for the derivation of new exact solutions. In Ref. [27], power series solutions to the TOV equations were developed for linear and polytropic EOSs. In Ref. [28], an analytic solution was derived for polytropic fluids, where Padé approximation theory was used to extend the radius of convergence.

In this article, we derive series solutions to the TOV equations in two general cases: when the energy density is a known analytic function, and when a sufficiently regular barotropic equation of state is specified. The article is organized as follows: in Sec. II, we derive general series solutions to the TOV equations for a given energy density profile. In Sec. III, we consider the TOV equations coupled to a barotropic equation of state, develop an algorithm to compute the power series coefficients of the solution, and establish general properties. The analytic results are applied to particular types of fluids and compared with the numerical approximations in Sec. IV. In Sec. V, we extend the formalism to piecewise equations of state. In Sec. VI, we conclude. In Appendix, we detail the proof of the parity of the power series expansions of the energy density and pressure about the center the star, for sufficiently regular equations of state.

The mathematical formulas in the article are written in the geometrized unit system, where $8\pi G = c = 1$, and assuming the metric signature $(-+++)$.

II. REFORMULATION OF THE TOV EQUATIONS

A. Equivalent differential system

The TOV equations follow from the Einstein field equations considering a static, spherically symmetric spacetime permeated by a perfect fluid with energy density μ and pressure p . The original version of the equations form a system of first-order, nonlinear differential equations with a singular point. In Schwarzschild coordinates: (t, r, θ, φ) , where r represents the circumferential radius, the TOV equations read

$$\frac{dp}{dr} = -\frac{r^2(\mu + p)}{2(r - 2M)} \left(p + \frac{2M}{r^3} - \Lambda \right), \quad (1)$$

$$\frac{dM}{dr} = \frac{1}{2}(\mu + \Lambda)r^2, \quad (2)$$

where Λ represents the cosmological constant and

$$M(r) = \frac{1}{2} \int_0^r (\mu + \Lambda)x^2 dx, \quad (3)$$

is dubbed the mass function. From hereon, we will omit the mass function dependency on the radial coordinate. The system must be completed by providing an equation that relates the energy density with the pressure: an equation of state for the fluid, or an expression that specifies one of the matter variables as a function of the spacetime coordinates.

Solutions of TOV equations aim to model the interior of compact stellar objects. As such, the contributions from the cosmological constant, Λ , are often disregarded. We will consider this simplification and throughout the article set $\Lambda = 0$. Nonetheless, the inclusion of the cosmological constant is straightforward and does not affect the overall discussion below. Moreover, we will be interested in regular solutions of the TOV equations, therefore, in what follows, we will consider that $r > 2M$ and call such solutions *stars*.

Deriving exact solutions to the TOV equations for an equation of state is a challenging task. However, various alternative strategies can be employed to obtain solutions. One simplifying approach is to prescribe the energy density directly as a function of spacetime. Following this approach, various closed-form solutions have been found. To the best of our knowledge, all known exact solutions share the property that the energy density, $\mu(r)$, is real analytic at the center of the star, at $r = 0$. In this section, we then focus on deriving general series solutions to the TOV equations given a real analytic energy density. In the next section, we build on these results to formulate an algorithm for computing power series solutions to the TOV equations coupled with an equation of state.

We start by reformulating the system of equations to an equivalent system where it is simpler to find the general power series solution of the TOV equations for a known energy density function. Let the potentials be $(\phi, \mathcal{A}, \mathcal{E})$, such that

$$\phi = \frac{2}{r} \sqrt{1 - \frac{2M}{r}}, \quad (4)$$

$$\mathcal{A}\phi = p + \frac{2M}{r^3}, \quad (5)$$

$$\mathcal{E} = \frac{1}{3}\mu - \frac{2M}{r^3}. \quad (6)$$

These potentials follow from the $1 + 1 + 2$ semitetrad decomposition of the spacetime manifold [29–31], having, thus, intrinsic physical meaning. In the considered setup, ϕ represents the spatial expansion coefficient of the normalized radial gradient, $\mathcal{A}$ represents the radial component of the four-acceleration of the elements of volume of the fluid, and $\mathcal{E}$ is the fully radial component of the electric part of the Weyl tensor, characterizing tidal forces. We remark that in general, ϕ diverges at $r = 0$, but $r\phi$ is regular. Moreover, regular solutions for p exist only if $\mathcal{A}$ and $\mathcal{E}$ vanish at the center of the star.

Given a line element of the form

$$ds^2 = -g_{tt}dt^2 + g_{rr}dr^2 + r^2d\Omega^2, \quad (7)$$

where $d\Omega^2$ represents the line element of the unit two-sphere, and g_{tt} and g_{rr} are functions of the radial coordinate only, we have the following relations:

$$\phi = \frac{2}{r\sqrt{g_{rr}}},$$

$$\mathcal{A} = \frac{1}{2g_{tt}\sqrt{g_{rr}}} \frac{dg_{tt}}{dr}. \quad (8)$$

Using the TOV equations (1) and (2), or starting directly from the $1 + 1 + 2$ structure equations, we find

$$\frac{d\mathcal{A}}{dr} = \frac{3\mathcal{E}}{r\phi} + \frac{1}{r}\mathcal{A} - \frac{2}{r\phi}\mathcal{A}^2. \quad (9)$$

Equation (9) is a Riccati equation with a regular singular point at $r = 0$. Applying a change of variables, this equation can be written as a second-order linear ordinary differential equation. Let u be such that

$$\frac{2\mathcal{A}}{r\phi} = \frac{1}{u} \frac{du}{dr}. \quad (10)$$

From Eq. (8), the variable $u \propto \sqrt{g_{tt}}$. Using Eq. (10) in Eq. (9), we find that the new variable verifies the differential equation

$$\frac{d^2u}{dr^2} = \frac{6\mathcal{E}}{r^2\phi^2}u + \left(\frac{1}{r} + \frac{4\mu + 6\mathcal{E}}{3r\phi^2}\right) \frac{du}{dr}. \quad (11)$$

To find general power series solutions, we will apply an algorithm described in Refs. [32,33], also considered in Refs. [34–37], to find power series solutions for perturbations of stars. For this purpose, it is useful to cast the second-order differential equation (11) in the form of a system of first-order differential equations. Introducing $s = du/dr$, in matrix form we have

$$\frac{d\mathbb{W}}{dr} = \left(\frac{1}{r} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \Theta\right) \mathbb{W}, \quad (12)$$

where $\mathbb{W} = [su]^T$ and

$$\Theta = \begin{bmatrix} \frac{4\mu+6\mathcal{E}}{3r\phi^2} & \frac{6\mathcal{E}}{r^2\phi^2} \\ 1 & 0 \end{bmatrix}. \quad (13)$$

For μ analytic at $r = 0$, the matrix Θ is analytic at $r = 0$.

Before proceeding, we note that a system for p , formally analogous to Eq. (12), can be obtained straightforwardly by applying the same procedure directly to Eq. (1). Since the pressure is an important quantity in determining the properties of compact stellar objects, it is natural to consider a differential system that explicitly relates p to the energy density μ . However, as will be shown in the following section, to derive series solutions to the TOV equations coupled with an equation of state, it is advantageous to use the variables μ , p , and $\mathcal{A}$. Consequently, the specific choice of variable, between p or $\mathcal{A}$, in the differential system is not essential. Moreover, solutions of Eq. (12) immediately yield

the g_{tt} component of the metric tensor, up to a multiplicative constant.

B. Power series solutions given an analytic energy density

The first-order linear differential system (12) contains a singular point at $r = 0$. To derive the general power series solutions around that point, we will employ the Coddington-Levinson procedure. This approach allow us to find the power series solutions around the singular point, which may or may not be regular at the singular point. Nevertheless, we will show that in the considered setup all analytic solutions are regular at $r = 0$.

The singular part of Eq. (12) is characterized by a constant diagonal matrix with constant eigenvalues: 0 and 1. This property allow us to derive the general formal solution. Using the Coddington-Levinson algorithm, the analytic solutions of the TOV equation are given by

$$\begin{bmatrix} s \\ u \end{bmatrix} = \begin{bmatrix} r & 0 \\ 0 & 1 \end{bmatrix} \mathbb{P}(r) \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \quad (14)$$

$$\mathcal{A} = \frac{r\phi s}{2u}, \quad (15)$$

$$p = \mathcal{E} + \mathcal{A}\phi - \frac{1}{3}\mu, \quad (16)$$

where c_1 and c_2 are integration constants, and the matrix $\mathbb{P}$ has power series about $r = 0$, as follows:

$$\mathbb{P}(r) = \sum_{m=0}^{+\infty} \mathbb{P}_m r^m, \quad (17)$$

with matrix coefficients $\mathbb{P}_m$ given by the recurrence relation

$$\mathbb{P}_0 = \mathbb{I}_2,$$

$$\mathbb{P}_m = \frac{1}{m} \sum_{k=0}^{m-1} \tilde{\Theta}_{m-1-k} \mathbb{P}_k, \quad \text{for } m \geq 1, \quad (18)$$

where $\mathbb{I}_2$ is the identity matrix of size 2 and

$$\tilde{\Theta} = \begin{bmatrix} \frac{r(\mu - \frac{2M}{r})}{2(1 - \frac{2M}{r})} & \frac{\mu - \frac{6M}{r}}{2r(1 - \frac{2M}{r})} \\ r & 0 \end{bmatrix}. \quad (19)$$

In Eq. (18), the terms $\tilde{\Theta}_m$ represent the matrix coefficients of the power series expansion of the matrix $\tilde{\Theta}$ about $r = 0$, that is $\tilde{\Theta}(r) = \sum_{m=0}^{+\infty} \tilde{\Theta}_m r^m$.

Since $\mathbb{P}_0 = \mathbb{I}_2$, the integration constants c_1 and c_2 can be directly related to the boundary conditions at the center of the star. The important physical quantity is the ratio c_1/c_2 and not their separate values, other than being zero. For instance, setting $c_2 = 1$, we find that c_1 is given in

terms of the central energy density, μ_c , and the central pressure, p_c , as

$$c_1 = \frac{1}{6}(3p_c + \mu_c). \quad (20)$$

For a real analytic function μ , the radius of convergence of the power series representation of $\mathbb{W}$ can be related to the radius of convergence of the power series representation of μ and the fixed points of $2M$. We summarize the result as follows:

Proposition 1. Let the energy density be a real analytic function at $r = 0$, with radius of convergence R . For real values of the central density and central pressure, μ_c and p_c , the TOV equations admit real analytic solutions at $r = 0$ for compact stellar objects. Given the nonzero fixed points of $2M(r)$, $\{r_i\}$, with $i = 1, \dots, n$, the radius of convergence of the power series representation of $\mathbb{W}$, $\bar{R}$, is given by the smallest, positive, modulus of the fixed points of $2M(r)$ or R , that is $\bar{R} = \min_{i=1, \dots, n} \{|r_i|, R\}$.

Proof. The formal solution (14) follows from the Coddington-Levinson algorithm [32]. The proofs in Refs. [32,33] show that the resulting power series solution has radius of convergence equal to the radius of convergence of Θ .

Let μ be real analytic at $r = 0$, such that $M \sim \mathcal{O}(r^3)$. In particular, this implies that $M(r)/r = 0$ at $r = 0$. Then, from Eqs. (3)–(6), Θ is real analytic at $r = 0$. The radius of convergence of Θ is either given by the smallest modulus of the nonzero fixed points of $2M(r)$ or the radius of convergence of the power series expansion of μ , depending on which one is smaller. ■

Proposition 1 allows us to determine the disk of convergence of the power series expansion of u about $r = 0$. Naturally, the radius of the disk might be smaller than the radius of the star. However, it is possible to find series solutions for $\mathcal{A}$ and p that may converge in regions beyond that disk.

Given the power series solutions for u , we can use Eqs. (15) and (16) to find series solutions for the potentials $\mathcal{A}$ and p . The power series representation of u can be used to find, for instance, the Maclaurin series of $\mathcal{A}$ and of p . However, we are not limited to use the Maclaurin series. Alternatively, we may consider using Padé approximants centered at $r = 0$ to expand $\mathcal{A}$ and p by rational functions. Since rational functions are meromorphic, Padé approximants may have a radius of convergence greater than the radius of convergence of the Maclaurin series of u .

Moreover, using Eqs. (15) and (16), we may directly express $\mathcal{A}$ and p as the ratio of power series. Truncating the series in the numerator and denominator to a given order does not, in general, yield a Padé approximant of corresponding order since it may not verify the same conditions on the higher-order derivatives. However, the result also does not necessarily have a radius of convergence equal to the Maclaurin series of u .

We will further this discussion in the following sections.

C. Examples

Specifying a real analytic energy density, we can find series solutions for the pressure and for all quantities that characterize the spacetime's geometry, using Eqs. (14)–(20). To exemplify the application of the results in the previous subsection, we will consider the particular cases of the exact Tolman IV solution [38], the Heintzmann IIa solution [39], and a numerical solution of the TOV equations with a rational energy density.

Assuming a line element (7), the Tolman IV solution is characterized by the nontrivial metric coefficients

$$g_{tt} = B^2 \left(1 + \frac{r^2}{A^2} \right), \quad g_{rr} = \frac{1 + 2 \frac{r^2}{A^2}}{\left(1 + \frac{r^2}{A^2} \right) \left(1 - \frac{r^2}{R^2} \right)}, \quad (21)$$

where A , B , and R are assumed to be nonzero, real constants. The energy density and the pressure are given by

$$\mu = \frac{3A^4 + A^2(7r^2 + 3R^2) + 2r^2(3r^2 + R^2)}{R^2(A^2 + 2r^2)^2}, \quad p = \frac{R^2 - A^2 - 3r^2}{R^2(A^2 + 2r^2)}. \quad (22)$$

The interior of a star can be modeled by the Tolman IV solution. Let the radius of the star, r_b , be defined as the value of the circumferential radius of the surface at which the pressure vanishes, that is

$$p(r_b) = 0. \quad (23)$$

For the Tolman IV solution, we find

$$r_b = \sqrt{\frac{R^2 - A^2}{3}}. \quad (24)$$

From Eqs. (21) and (22), we see that the energy density and the pressure have singular points at $r = \pm iA/\sqrt{2}$, and $\sqrt{g_{tt}}$ has singular points at $r = \pm iA$. Therefore, depending on the values of the parameters R and A , the radius of convergence of the Maclaurin series of u and p might be smaller than the radius of the star. However, p is a rational function, such that both the numerator and the denominator are quadratic polynomials. Therefore, it is equal to its Padé approximant of order $[2, 2]$ about $r = 0$. That is, using Eqs. (14)–(20) to find the series expansion to order 4, we can, in this case, find the exact expression for the pressure.

The Tolman IV solution is simple enough that it is possible to find exact expressions for all the thermodynamic variables, radius of the star, and the singular points of the

various quantities. If the pressure can be expressed as a rational function, we can find an exact expression from Eqs. (14)–(20), using Padé approximant theory. In general, this is not the case.

Consider the exact solution first derived by Heintzmann, such that the nontrivial metric components in Schwarzschild coordinates are given by

$$g_{tt} = A^2(ar^2 + 1)^3, \quad g_{rr} = \left(1 - 3ar^2 \frac{C(4ar^2 + 1)^{-\frac{1}{2}} + 1}{2(ar^2 + 1)}\right)^{-1}, \quad (25)$$

with parameters A , a , and C . The energy density and the pressure of the fluid source are given by

$$\mu = \frac{3a[(4a^2r^4 + 13ar^2 + 3)\sqrt{4ar^2 + 1} + C(9ar^2 + 3)]}{2(ar^2 + 1)^2(4ar^2 + 1)^{\frac{3}{2}}}, \quad p = -\frac{3a(7aCr^2 + 3(ar^2 - 1)\sqrt{4ar^2 + 1} + C)}{2(ar^2 + 1)^2\sqrt{4ar^2 + 1}}. \quad (26)$$

Depending on the values of a and C , $\sqrt{g_{tt}}$ and p might be singular at points in the complex plane with modulus smaller than the radius of the star. The singular points of $\sqrt{g_{tt}}$ are solutions to $ar^2 = -1$, and the singular points p are solutions to $ar^2 = -1$ or to $4ar^2 = -1$. The Heintzmann Ila solution provides an example where using Padé approximants allows us to locally approximate the solution to points outside the disk of convergence of the Maclaurin series, and outside the disk of convergence of the series ratio solutions following from Eqs. (14)–(16).

Figure 1 shows the behavior of the pressure profile, its Taylor polynomial at $r = 0$ truncated to a given order, and the truncated series ratio solution following from Eqs. (14)–(20). In that particular realization of the Heintzmann Ila solution, $\sqrt{g_{tt}}$ and p have singular points with a modulus smaller than the radius of the star. Indeed, in that case, both the radius of convergence of the Maclaurin series of the exact solution, and the radius of convergence of the series ratio solutions following from Eqs. (14)–(16) are smaller than the radius of the star. Therefore, neither can be used to accurately describe the pressure for all points in the interior of the star. However, the pressure has only isolated singular points and can be approximated by a rational function. For instance, the diagonal Padé approximant of order [8, 8], for the spacetime

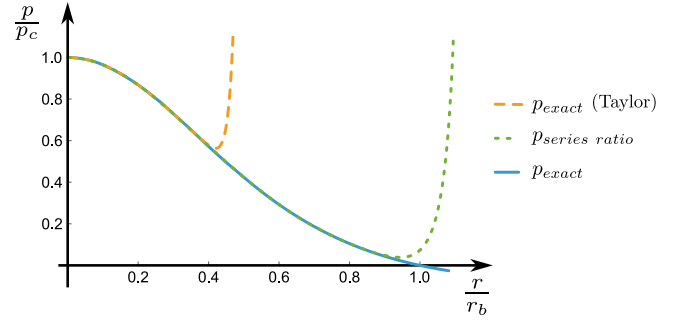


FIG. 1. Pressure profile of the Heintzmann solution (26) assuming the parameters $A = 1$, $a = 16$, and $C = -\frac{1}{3}$, its Taylor polynomial about $r = 0$ truncated to order 30, and the series ratio solution from Eqs. (14)–(20) truncated to order 30.

parameters considered for Fig. 1, differs pointwise from the exact values of the normalized pressure, for any point within the interior of the star, no more than 3.6×10^{-4} .

For the Tolman IV and Heintzmann Ila spacetimes, the energy density is given by an exact expression. However, Eqs. (14)–(20) can be used to approximate solutions to the TOV equations for an analytic energy density that cannot be expressed in exact form and can only be approximated through numerical methods.

The TOV equations form a very stiff system of differential equations. Moreover, Eq. (1) has a singular point at $r = 0$, such that numerical approximations might require high working precision. Nonetheless, a fundamental limitation of numerical solutions is that it might be very difficult to determine general dependencies of the solutions on the parameters. In that regard, considering series solutions, it is possible to establish general analytic expressions to estimate the properties of the solutions, considering only a few terms of the series.

To exemplify this idea, consider a solution of the TOV equations with energy density

$$\mu = a + \frac{b}{1 - cr^2}. \quad (27)$$

Using Eqs. (14)–(20), the Padé approximant of order [2, 2] for the pressure yields a rational function where both numerator and denominator are polynomials of order 2. Computing its positive roots, we find the analytic approximation for the radius of the star is

$$r_b \approx \sqrt{\frac{60p_c}{5(a+b)^2 + 2(a+b)\left[10p_c + 3bc\left(\frac{6}{a+b} - \frac{5}{a+b+p_c} - \frac{1}{a+b+3p_c}\right)\right]}}. \quad (28)$$

Table I presents a comparison between the circumferential radii obtained by numerically integrating the TOV equations and those predicted by the analytic approximation (28), for stars with energy density function (27). In the example, for lower values of the compactness parameter, the analytic approximation closely matches the exact value. For larger compactness

TABLE I. Values of the circumferential radii of stars with energy density function (27) for different values of the central pressure. In all cases, $a = 1$, $b = -\frac{1}{5}$, and $c = \frac{1}{4}$. The second column indicates the value of the compactness parameter of the star; the third column gives the value of r_b found from the numerical integration; the fourth column presents the value of r_b found from the analytic approximation (28); and the last column is the percent error, δ , for the analytic approximation of r_b relative to the numeric approximation.

	$\frac{M}{r_b}$	$r_b(\text{Num})$	$r_b(\text{An approx})$	$\delta(\%)$
$p_c = 0.05$	0.101	0.883	0.881	0.3
$p_c = 0.10$	0.167	1.159	1.148	0.9
$p_c = 0.15$	0.213	1.333	1.309	1.8
$p_c = 0.20$	0.247	1.461	1.418	3.0
$p_c = 0.25$	0.271	1.566	1.497	4.5
$p_c = 0.30$	0.288	1.668	1.556	6.7

values, the analytic approximation significantly deviates from the numeric result and a higher-order analytic approximation should be used. We note, however, that such stars become dynamically unstable at higher values of the compactness parameter. For instance, keeping $a = 1$, $b = -\frac{1}{5}$ and $c = \frac{1}{4}$, and considering $p_c \gtrsim 0.32$, such that $M/r_b \gtrsim 0.29$, the star is unstable under linear, adiabatic radial perturbations.

III. SERIES SOLUTIONS TO THE TOV WITH AN EQUATION OF STATE

The results in the previous section allow us to compute general series solutions to the TOV equations, if the energy density is a known real analytic function at $r = 0$. However, physically, the energy density is not a known function of the spacetime *ab initio*. Solutions for compact stellar objects are found by coupling the TOV equations with a barotropic equation of state for the fluid source, relating the pressure with the energy density. In this section, assuming a sufficiently regular barotropic EOS, we derive series solutions for the spacetime potentials.

Let the energy density and the pressure of the fluid source be related by an EOS of the form

$$f(\mu, p) = 0, \quad (29)$$

where f is an analytic function of the matter variables at the center of the star, and $\partial_\mu f \neq 0$, also at the center of the star. Then, the spacetime is characterized by the TOV equations coupled with the constraint equation (29).

Using the chain rule, Eq. (29) implies

$$f_\mu \partial_r \mu + f_p \partial_r p = 0, \quad (30)$$

where the subscript in f represents the partial derivative with respect to the indicated thermodynamic variable. Using Eqs. (1)–(5), we have

$$\frac{dp}{dr} = -\frac{2(\mu + p)\mathcal{A}}{r\phi}. \quad (31)$$

Combining Eqs. (30) and (31) yields the important relation

$$\frac{d\mu}{dr} = \frac{2(\mu + p)f_p}{r\phi f_\mu} \mathcal{A}. \quad (32)$$

We remark that defining the square of the adiabatic speed of sound as $c_s^2 \equiv (dp/d\mu)_S$, where the derivative is taken at constant entropy, we have $c_s^2 = -f_\mu/f_p$. This implies that the adiabatic index of a perfect fluid, γ , is given by

$$\gamma = -\frac{(\mu + p)f_\mu}{pf_p}. \quad (33)$$

Therefore, Eq. (32) can be equivalently written in terms of the adiabatic speed of sound or the adiabatic index.

Continuing, Eq. (32) relates the derivative of the energy density directly with μ , p , $\mathcal{A}$, and the derivatives of f . Following the results in Appendix, if μ , p , $\mathcal{A}$, and f are analytic at $r = 0$, taking the derivative of order n of Eq. (32) implies that the coefficient of the $(n + 1)$ th-order term of the power series of μ about $r = 0$ depends, at most, on the n th-order derivatives of μ , p , and $\mathcal{A}$ at the center. In turn, from Eq. (14)–(20), for $n > 0$, the coefficients of the n th-order terms of the power series of $\mathcal{A}$ and of p depend at most on the derivative $\mu^{(n-1)}(0)$. Then, provided the central energy density, μ_c , and the central pressure, p_c , we can use Eqs. (14)–(20) together with Eq. (32) and its derivatives to compute the coefficients of the power series of μ , p , and $\mathcal{A}$ recursively.

Using this procedure, the coefficients of the power series of the potentials can be computed symbolically to arbitrary order, for a general barotropic EOS or for a specific family of EOSs. Below, we present the first terms of the power series of μ , p , and $\mathcal{A}$ about $r = 0$, up to order 4. Expectedly, in the general case, the expressions for the higher-order terms become rapidly quite large due to the number of higher-order derivatives of the function f .

$$\begin{aligned}
 \mu(r) &= \mu_c + \frac{(\mu_c + p_c)(\mu_c + 3p_c)f_p}{12f_\mu} r^2 \\
 &\quad + \frac{\mu_c^2 + 3p_c^2 + 4\mu_c p_c}{48f_\mu} \left[\frac{(4\mu_c + 9p_c)(f_p)^2}{15f_\mu} - \frac{(\mu_c + p_c)(\mu_c + 3p_c)f_{pp}}{6} \right. \\
 &\quad \left. + \frac{(\mu_c + p_c)(\mu_c + 3p_c)f_p f_{\mu p}}{3f_\mu} - \frac{(\mu_c + p_c)(\mu_c + 3p_c)(f_p)^2 f_{\mu\mu}}{6(f_\mu)^2} - p_c f_p \right] r^4 + \mathcal{O}(r^6), \\
 p(r) &= p_c - \frac{(\mu_c + p_c)(\mu_c + 3p_c)}{12} r^2 - \frac{(\mu_c + p_c)(\mu_c + 3p_c)[(4\mu_c + 9p_c)f_p - 15p_c f_\mu]}{720f_\mu} r^4 + \mathcal{O}(r^6), \\
 \mathcal{A}(r) &= \frac{\mu_c + 3p_c}{6} r + \frac{(\mu_c + 3p_c)[3(\mu_c + p_c)f_p - 5(\mu_c + 3p_c)f_\mu]}{360f_\mu} r^3 + \mathcal{O}(r^5),
 \end{aligned} \tag{34}$$

where the derivatives of the function f are evaluated at (μ_c, p_c) .

As previously discussed, regularity of solutions of the TOV equations require that $\mathcal{A}$ vanishes at $r = 0$. In turn, this implies that $\partial_r p = 0$ at the center of the star. Moreover, if $f_\mu \neq 0$ at the center of the star, it follows from Eq. (32) that $\partial_r \mu = 0$. Indeed, the expressions in Eq. (34) suggest that the energy density and the pressure are even functions of the radial coordinate, whereas $\mathcal{A}$ is an odd function. The following result establishes that this is a consequence of the regularity of the EOS at the center of the star.

Theorem 1. Consider a perfect fluid with an equation of state (29). If real analytic solutions at $r = 0$ of the TOV equations together with an equation of state exist for compact stellar objects, then the energy density is an even function of the circumferential radius coordinate, such that its power series about $r = 0$ contains only even powers of r .

The proof of Theorem 1 is given in Appendix. In proving this theorem, we have also shown the following:

Corollary 1. In the conditions of Theorem 1, the pressure, p , is an even function of the circumferential radius, and $\mathcal{A}$ is an odd function.

Theorem 1 and its corollary establish a relation between the behavior of the thermodynamic variables in a neighborhood of the center of the star and the nonvanishing of the adiabatic speed of sound. These results allow us to simplify the computation of the coefficients of the power series.

Under the conditions of Theorem 1, the odd-order terms of the power series expansion of $\mathbb{P}$, Eq. (17), are identically zero. Then, the recurrence relation (18) simplifies to

$$\mathbb{P}_m = \begin{cases} \mathbb{I}_2, & \text{if } m = 0 \\ \mathbf{0}, & \text{if } m \text{ odd} \\ \frac{1}{m} \sum_{s=0}^{\frac{m}{2}-1} \tilde{\Theta}_{m-1-2s} \mathbb{P}_{2s}, & \text{if } m \text{ even.} \end{cases} \tag{35}$$

Knowledge of the derivatives of an analytic function at a given point can be used to find the power series representation of the function. However, we can approximate the function using other type of series. As we have discussed in the previous section, for instance, we can use Padé approximants. Using Eq. (34), we present the general Padé approximants of order $[2, 2]$ for the energy density, $\mu_{[2,2]}$, and the pressure, $p_{[2,2]}$, for an EOS characterized by Eq. (30), as follows:

$$\mu_{[2,2]} = \frac{\mu_c + \mathbf{m}_1 r^2}{1 + \mathbf{m}_2 r^2}, \tag{36}$$

$$p_{[2,2]} = \frac{p_c - \frac{1}{60} [5\mu_c(\mu_c + 4p_c) + \frac{p_c(4\mu_c + 9p_c)f_p}{f_\mu}] r^2}{1 + (\frac{p_c}{4} - \frac{(4\mu_c + 9p_c)f_p}{60f_\mu}) r^2}, \tag{37}$$

where

$$\begin{aligned}
 \mathbf{m}_1 &= \frac{1}{4} \mu_c p_c + \frac{[5\mu_c(\mu_c + p_c)(\mu_c + 3p_c)f_{\mu\mu} + 2(\mu_c^2 + 15p_c^2 + 11\mu_c p_c)f_\mu]f_p}{120(f_\mu)^2} \\
 &\quad + \frac{\mu_c(\mu_c + p_c)(\mu_c + 3p_c)f_{pp}}{24f_p} - \frac{\mu_c(\mu_c + p_c)(\mu_c + 3p_c)f_{\mu p}}{12f_\mu}, \\
 \mathbf{m}_2 &= \frac{1}{4} p_c + \frac{[5(\mu_c + p_c)(\mu_c + 3p_c)f_{\mu\mu} - 2(4\mu_c + 9p_c)f_\mu]f_p}{120(f_\mu)^2} \\
 &\quad + \frac{(\mu_c + p_c)(\mu_c + 3p_c)f_{pp}}{24f_p} - \frac{(\mu_c + p_c)(\mu_c + 3p_c)f_{\mu p}}{12f_\mu},
 \end{aligned} \tag{38}$$

and the derivatives of the function f are evaluated at (μ_c, p_c) . We remark that in the light of Theorem 1, the Padé approximants for the energy density and the pressure are even functions. In particular, for each of these functions, the Padé approximant of order $[2n, 2n]$ is equal to the Padé approximant of order $[2n+1, 2n+1]$, for $n \geq 0$, if they exist.

In Sec. II C, we have introduced the idea of finding an analytic approximation for the radius of the star, r_b , defined by Eq. (23), by considering the Padé approximant of order $[2, 2]$ of the pressure. Evaluating the roots of $p_{[2,2]}$, Eq. (37), yields the following approximation in terms of the EOS:

$$r_b \approx \sqrt{\frac{60p_c}{5\mu_c(\mu_c + 4p_c) + \frac{p_c(4\mu_c + 9p_c)f_p}{f_\mu}}}. \quad (39)$$

Following the same reasoning, we can use Eq. (34) in Eq. (3) and compute the Padé approximant of order $[3, 4]$ to find an approximation for the value of the mass function at a given circumferential radius. We find

$$M \approx \frac{\mu_c r^3}{6 - \frac{3(\mu_c + p_c)(\mu_c + 3p_c)f_p}{10\mu_c f_\mu} r^2 + \frac{(\mu_c + p_c)(\mu_c + 3p_c)M}{2800\mu_c^2 (f_\mu)^3} r^4}, \quad (40)$$

where

$$\begin{aligned} M = & [2(\mu_c^2 + 63p_c^2 + 39\mu_c p_c)f_\mu \\ & + 25\mu_c(\mu_c + p_c)(\mu_c + 3p_c)f_{\mu\mu}](f_p)^2 \\ & - 50\mu_c f_\mu [(\mu_c + p_c)(\mu_c + 3p_c)f_{\mu p} - 3p_c f_\mu] f_p \\ & + 25\mu_c(\mu_c + p_c)(\mu_c + 3p_c)f_{pp}(f_\mu)^2. \end{aligned} \quad (41)$$

Equations (39) and (40) can be used to estimate the compactness parameter: $M(r_b)/r_b$, from the values of μ_c , p_c , and the derivatives up to second order of the function f .

To close this section, we note that it is straightforward to find symbolic expressions for analytic approximations for r_b and M using higher-order approximants. However, the expressions for the coefficients of the polynomials become increasingly larger, so that it is not sensible to present general formulas for higher-order analytic approximations. In the next section, we will consider the higher-order approximations for particular EOSs.

IV. APPLICATIONS TO SELECTED EQUATIONS OF STATE

A. Affine equation of state

Affine EOSs are simple, important models used to characterize states of matter in the interior of stars [40–43]. Generically, affine EOSs can be written as

$$p - \alpha\mu - \beta = 0, \quad (42)$$

where α and β are constants. For an affine EOS, the function f is characterized by $f_\mu = -\alpha$ and $f_p = 1$, and all higher-order derivatives are identically zero.

To exemplify the general analytic results derived in Sec. III, we will consider the particular realization of an affine EOS provided by the MIT bag model. The MIT bag model follows from the assumption that hadrons are composed of noninteracting, massless quarks, behaving as a Fermi gas with energy density μ and pressure p , confined in a region by an external, counterbalancing pressure, B , that maintains the quark gas at finite density and chemical potential. In this model, the fluid's EOS is

$$\mu = 3p + 4B. \quad (43)$$

An application of the MIT EOS to compact stellar objects is to characterize the matter fluid source of strange stars. In Ref. [44], it was shown that three-flavor, (u, d, s) , quark matter is stable for values of B between 57 MeV/fm³ and 94 MeV/fm³, allowing for the possibility that stellar objects composed of strange quark matter might exist in the Universe. Such objects are often referred as strange stars. We will then analyze the properties of these objects by numerically integrating the TOV equations for a fluid characterized by the MIT EOS (43), and comparing the results with those of the analytic approximations introduced in the previous section. To allow easy comparison with the results in the literature on strange stars, we will present the results in terms of the bag constant, B , in MeV/fm³.

Figure 2 shows the behavior of the radius of strange stars as a function of the bag constant, following from numerical integration of the TOV equations together with the EOS (43), the analytic approximation (39), and the respective percent error. Strange stars solutions aim to represent highly compact configurations, where the matter fields are subjected to extreme forces. As expected, in these scenarios, the analytic approximation to the radius fails to fully capture the complexities of the behavior of the fluid's pressure: the approximations becomes increasingly worse, as the value of the bag constant decreases.

The radius estimate is important, since it is used in the analytic estimation of the total mass of the star, $M(r_b)$, using Eq. (40). Given the simplicity of the EOS (43), in this case we can write the expression for the analytic approximation for r_b using the diagonal Padé approximant of order $[4, 4]$ of the pressure in a somewhat compact form. We find

$$r_b \approx \sqrt{\frac{\mathcal{R}_1}{2\mathcal{R}_2} + \varepsilon \frac{\sqrt{\mathcal{R}_1^2 + 4\mathcal{R}_2 p_c}}{2\mathcal{R}_2}}, \quad (44)$$

where $\varepsilon = \pm 1$ and

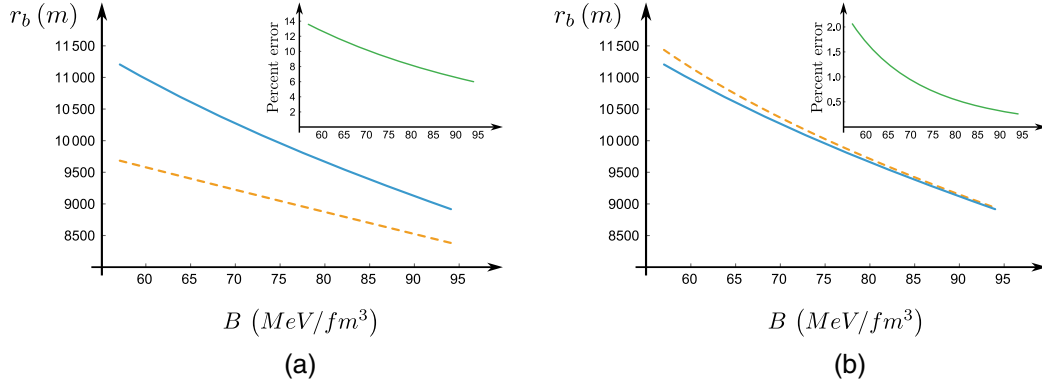


FIG. 2. Radius of strange stars as a function of the bag constant, assuming central energy density $\mu_c = 900 \text{ MeV/fm}^3$. In both plots, the solid blue line follows from numerical integration of the TOV equations. (a) The dashed orange line follows from the analytic approximation (39). (b) The dashed orange line follows from the analytic approximation (44). The insets show the percent error of the respective analytic approximations relative to the numerical result.

$$\begin{aligned} \mathcal{R}_1 &= \frac{-134\mu_c^4 + 9513p_c^4 + 6336\mu_c p_c^3 - 1321\mu_c^2 p_c^2 - 1078\mu_c^3 p_c}{24(67\mu_c^2 + 567p_c^2 + 459\mu_c p_c)}, \\ \mathcal{R}_2 &= \frac{1904\mu_c^5 - 1154979p_c^5 - 1246518\mu_c p_c^4 - 266706\mu_c^2 p_c^3 + 34330\mu_c^3 p_c^2 + 15401\mu_c^4 p_c}{10080(67\mu_c^2 + 567p_c^2 + 459\mu_c p_c)}. \end{aligned} \quad (45)$$

The sign of ε is chosen so that the left-hand side of Eq. (44) is real positive. If the approximation for r_b is real positive for both signs of ε , the one yielding the smallest value is chosen.

As illustrated in Fig. 2, the increased accuracy of the higher-order analytic approximation (44) in comparison with that of Eq. (39) is significant for all values of the bag constant. The fact that the analytic approximation (44) is accurate for such extreme configurations is a consequence of the simplicity of the EOS.

In Fig. 3, we present the compactness parameter, M/r_b , of strange stars as a function of the bag constant following

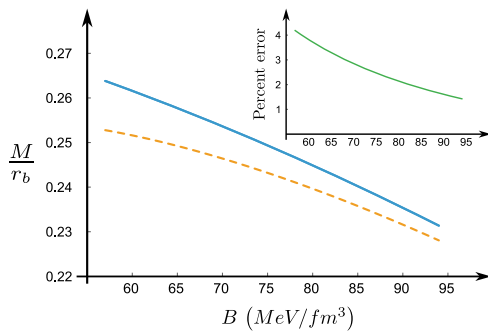


FIG. 3. Compactness parameter of strange stars as a function of the bag constant, assuming central energy density $\mu_c = 900 \text{ MeV/fm}^3$. The solid line follows from numerical integration of the TOV equations. The dashed curve follows from using the analytic approximations (40) and (44). The inset shows the percent error of the analytic approximation relative to the numerical result.

from numerical integrating the TOV equations, and the analytic approximation following from Eqs. (40) and (44). Expectedly, the errors associated with using the analytic expressions become worse for higher values of the compactness parameter, where the behavior of the energy density and the pressure changes significantly, particularly closer to the boundary of the star, and higher-order approximations are required to better fit the model. Nonetheless, the percent error of the analytic approximation relative to the numerical result is surprisingly low, ranging from 1.4% to 4.1%.

B. Relativistic polytropic equation of state

Relativistic polytropic EOSs are a popular family of equations of state to model matter in the interior of compact stellar objects. In these models,

$$p = K\rho^\gamma, \quad (46)$$

where ρ represents the matter rest mass density, γ is the adiabatic index defined in Eq. (33), and the polytropic constant K is determined by the thermodynamic properties of the perfect fluid at a particular point. For instance, considering the values of the rest mass density and the pressure at the center of the star, ρ_c and p_c , respectively, we have

$$K = \frac{p_c}{\rho_c^\gamma}. \quad (47)$$

For a relativistic polytropic EOS, Eq. (29) reads

TABLE II. Approximate values of the radius, in meters, and the compactness parameter found from numerical integration of the TOV equations with a relativistic EOS (48), and computed considering the analytic Padé approximant of order [10, 10], for different values of the adiabatic index γ , with central density $\rho_c = 5 \times 10^{14} \text{g/cm}^3$ and central pressure $p_c = 10^{34} \text{Pa}$. The fourth and last columns list the approximate values of the percent error, δ , of the analytic approximation relative to the numerical approximation for the radius and the compactness, respectively.

	$r_b(\text{Num})$	$r_b(\text{An approx})$	δ (%)	$\frac{M}{r_b}(\text{Num})$	$\frac{M}{r_b}(\text{An approx})$	δ (%)
$\gamma = 2.0$	20 624	19 584	5.0	0.190	0.198	4.4
$\gamma = 2.2$	18 874	18 411	2.5	0.200	0.204	2.3
$\gamma = 2.4$	17 748	17 445	1.7	0.206	0.210	1.6
$\gamma = 2.6$	16 957	16 729	1.3	0.211	0.213	1.3
$\gamma = 2.8$	16 370	16 171	1.2	0.214	0.216	1.1
$\gamma = 3.0$	15 915	15 742	1.1	0.217	0.219	1.0

$$\left(\frac{p}{K}\right)^{\frac{1}{\gamma}} + \frac{p}{\gamma - 1} - \mu = 0. \quad (48)$$

The value of the adiabatic index, γ , and the polytropic constant, K , markedly change the properties of the fluid, such that relativistic polytropic EOSs have been considered to model distinct types of compact stellar objects. Indeed, fluids characterized by relativistic polytropic EOSs have been extensively considered as effective models, instead of microphysics based models of the equation of state (see, e.g., Refs. [45–49] and references therein).

In general, the thermodynamic properties of fluids described by a relativistic polytropic EOS generally have more complex behavior than those of fluids characterized by an affine equation of state. Depending on the values of the parameters, the analytic approximations discussed previously might give very poor results or fail completely. For instance, if the pressure distribution is characterized by a long decreasing tail, the analytic approximation for the radius of the star (44) might yield imaginary results. Nonetheless, we have found that the Padé approximants converge fast to the solutions, yielding accurate results by including a few more terms of the series.

To exemplify the accuracy of analytic series solutions, in Table II, we present the values of the radius and the compactness parameter of a self-gravitating fluid characterized by a relativistic polytropic EOS for different values of the adiabatic index, for fixed values of the central density and pressure. For comparison, we show the results for the pressure and the mass function from integrating the TOV equations numerically and the analytic results using the diagonal Padé approximant of the order [10, 10].

Depending on the values of γ and K , numerical integration of the TOV equations can be challenging. In that regard, we have found that for this type of EOS, it is preferable to use the formulation of the TOV equations introduced by Lindblom [20] to solve the equations numerically.

As we see, for the considered values, the analytic results agree with the numerical approximations, with small percent

errors. For smaller values of γ , the pressure profile has longer decreasing tails toward the boundary, such that the percent error increases with decreasing values of γ .

V. SOLUTIONS FOR PIECEWISE EQUATIONS OF STATE

A. Series solutions about arbitrary points

Properties of the matter fields may vary significantly as we move from the inner stellar core toward the outer crust regions. Determining from microphysical models an EOS that characterizes fluid sources across all regimes within compact object interiors is extremely challenging. Moreover, stellar objects may be structured in stratified layers, where matter exists in distinct states resulting from phase transitions, such that thermodynamic variables may be discontinuous across transition layers. It is therefore common to adopt piecewise EOS models, in which each branch describes the matter fluid in a particular regime [5, 15–17, 50, 51].

In this section, we derive solutions to the TOV-EOS system for piecewise equations of state.

In Eq. (12), we have separated the singular part from the nonsingular part to apply the Coddington-Levinson algorithm to find series solutions about $r = 0$. Assuming Θ is regular, the system does not have other singular points. Then, if we are interested in determining series solutions in a neighborhood centered at any point other than at $r = 0$, we may write the differential system simply as

$$\frac{d\mathbb{W}}{dr} = \mathbb{A}(r)\mathbb{W}, \quad (49)$$

where

$$\mathbb{A}(t) = \begin{bmatrix} \frac{1}{r} + \frac{4\mu+6\mathcal{E}}{3r\phi^2} & \frac{6\mathcal{E}}{r^2\phi^2} \\ 1 & 0 \end{bmatrix}. \quad (50)$$

The general analytic solutions around a given point, say $r = r_J > 0$, are formally given by

$$\mathbb{W} = \mathbb{C}(r) \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \quad (51)$$

where c_1 and c_2 are integration constants, and the matrix $\mathbb{C}$ has power series

$$\mathbb{C}(r) = \sum_{m=0}^{+\infty} \mathbb{C}_m (r - r_J)^m, \quad (52)$$

where the matrix coefficients $\mathbb{C}_m$ are given by the recurrence relation

$$\begin{aligned} \mathbb{C}_0 &= \mathbb{I}_2, \\ \mathbb{C}_m &= \frac{1}{m} \sum_{k=0}^{m-1} \mathbb{A}_{m-1-k} \mathbb{C}_k, \quad \text{for } m \geq 1. \end{aligned} \quad (53)$$

Equations (51)–(53) define the power series solutions to the TOV equations about a particular hypersurface. These can be used to construct solutions, such that the spacetime

is characterized by piecewise potentials. Let $r = r_J$ define the junction hypersurface within the star. From Eqs. (15) and (16), setting, for instance, $c_2 = 1$, we find that c_1 is given in terms of the energy density and the pressure at the junction, μ_J and p_J , respectively, as

$$c_1 = \frac{r_J}{2(1 - \frac{2M_J}{r_J})} \left(p_J + \frac{2M_J}{r_J^3} \right), \quad (54)$$

where

$$M_J = \frac{1}{2} \int_0^{r_J} \mu(x) x^2 dx, \quad (55)$$

represents the value of the mass at the junction hypersurface.

Similarly to the algorithm introduced in Sec. III, using Eqs. (15), (16), and (32), and Eqs. (51)–(54), we can compute the power series coefficients of μ , p , and $\mathcal{A}$ at r_J , for a given EOS of the form (29). We present below the first terms of the power series:

$$\begin{aligned} \mu(r) &= \mu_J + \frac{(\mu_J + p_J)(2M_J + p_J r_J^3) f_p}{2r_J(r_J - 2M_J) f_\mu} (r - r_J) \\ &\quad - \frac{(r_J - 2M_J)(\mu_J + p_J) \mathbf{m}_J}{8r_J^2[(2M_J - r_J) f_\mu]^3} (r - r_J)^2 + \mathcal{O}((r - r_J)^3), \\ p(r) &= p_J - \frac{(\mu_J + p_J)(2M_J + p_J r_J^3)}{2[r_J(r_J - 2M_J)]} (r - r_J) \\ &\quad - \frac{(\mu_J + p_J) \mathbf{p}_J}{8r_J^2(r_J - 2M_J)^2 f_\mu} (r - r_J)^2 + \mathcal{O}((r - r_J)^3), \\ \mathcal{A}(r) &= \frac{2M_J + p_J r_J^3}{2r_J^2 \sqrt{1 - \frac{2M_J}{r_J}}} + \frac{1}{4r_J^4 (1 - \frac{2M_J}{r_J})^{\frac{3}{2}}} [12M_J^2 - 4M_J r_J (2p_J r_J^2 + \mu_J r_J^2 + 2) \\ &\quad + r_J^4 (2\mu_J - p_J^2 r_J^2 + 2p_J)] (r - r_J) + \mathcal{O}((r - r_J)^2), \end{aligned} \quad (56)$$

where the derivatives of f are taken at (μ_J, p_J) and

$$\begin{aligned} \mathbf{m}_J &= f_p(f_\mu)^2 [4M_J^2 - 2M_J r_J (7p_J r_J^2 + \mu_J r_J^2 + 4) + r_J^4 (2\mu_J - 2p_J^2 r_J^2 + \mu_J p_J r_J^2 + 2p_J)] \\ &\quad - (\mu_J + p_J)(2M_J + p_J r_J^3)^2 f_\mu (f_{pp} f_\mu - f_p f_{\mu p}) + (2M_J + p_J r_J^3)^2 f_\mu (f_p)^2 \\ &\quad - (\mu_J + p_J)(2M_J + p_J r_J^3)^2 f_p (f_p f_{\mu\mu} - f_\mu f_{\mu p}), \\ \mathbf{p}_J &= f_\mu (4M_J^2 - 2M_J r_J (7p_J r_J^2 + \mu_J r_J^2 + 4) + r_J^4 (2\mu_J - 2p_J^2 r_J^2 + \mu_J p_J r_J^2 + 2p_J)) \\ &\quad + (2M_J + p_J r_J^3)^2 f_p. \end{aligned} \quad (57)$$

Note that since $\mathcal{A}(r_J)$ is not necessarily zero, the power series solutions for μ and p may contain terms with odd powers of $(r - r_J)$. Furthermore, these power series can be used to construct other series solutions about r_J , such as Padé approximants.

Equations (56) and (57) depend on derivatives of the equation of state with respect to the energy density and the pressure at the junction points: $(\mu_J, p_J) = (\mu(r_J), p(r_J))$. For piecewise EOSs, some derivatives might not exist. Nonetheless, the series solutions from Eqs. (51)–(54) aim to describe a region where $r > r_J$. Therefore, only the one-sided derivatives of f , in the direction $r \rightarrow r_J^+$, are required. As such, the EOS does not have to be smooth, or even continuous across the matching hypersurfaces.

To close this subsection, we remark that the Darmois–Israel junction conditions must be verified at the matching hypersurfaces for the full interior spacetime to be a solution of the Einstein field equations [52,53]. Whether the junction is smooth or requires the presence of thin matter shells at the matching hypersurfaces determines the values of p_J and M_J . Nonetheless, matching of the series solutions is well defined.

B. Example: Piecewise polytrope

To exemplify the analytic results of the previous subsection and see how they compare with those found from numerical integration, we will consider a simple toy model of an EOS defined as a piecewise polytrope characterizing three regions within a stellar object. The three subregions are determined by the density of the fluid, as follows:

$$p(\rho) = \begin{cases} K_1 \rho^{\gamma_1}, & \rho_c \leq \rho < \rho_1 \\ K_2 \rho^{\gamma_2}, & \rho_1 \leq \rho < \rho_2 \\ K_3 \rho^{\gamma_3}, & \rho \geq \rho_2 \end{cases} \quad (58)$$

where ρ represents the matter rest mass density and γ_i , with $i \in \{1, 2, 3\}$, are the adiabatic indexes for each region. We assume the values of the adiabatic index for each subdomain are known values. To determine the values of the polytropic constants, we consider that the pressure is continuous throughout the star, but not differentiable. Imposing these conditions, we find

$$p(\rho) = \begin{cases} \frac{p_c}{\rho_c^{\gamma_1}} \rho^{\gamma_1}, & \rho_c \leq \rho < \rho_1 \\ \frac{\rho_1^{\gamma_1 - \gamma_2} p_c}{\rho_c^{\gamma_1}} \rho^{\gamma_2}, & \rho_1 \leq \rho < \rho_2 \\ \frac{\rho_1^{\gamma_1 - \gamma_2} \rho_2^{\gamma_2 - \gamma_3} p_c}{\rho_c^{\gamma_1}} \rho^{\gamma_3}, & \rho \geq \rho_2 \end{cases} \quad (59)$$

In Table III, we present a comparison between values found from numerically integrating the TOV equations and from the analytic series solutions, for particular values of the model parameters. Namely, we show approximate values of the total masses within the transition hypersurfaces and the respective radii, and the total mass and radius of the star.

The analytic series solution is defined piecewise. For the inner core region, where $\rho < \rho_1$, we considered the Padé approximant about $r = 0$ of order [10, 10], following from Eq. (14), both for the pressure and the energy density. For the regions where $\rho_1 \leq \rho \leq \rho_2$ and $\rho \geq \rho_2$, we considered the Padé approximants of order [3, 3], respectively, about the transition radii $(r_J)_1$ and $(r_J)_2$, following from Eqs. (51)–(55).

Comparing the results, we see that the numerical integration and the analytic approximations are in close agreement. We remark that while analytic expressions can be evaluated rapidly, numerical integration can be particularly slow for certain values of the adiabatic index.

TABLE III. Approximate values of the radii, in meters, the masses within the transition hypersurfaces, and the total mass and radius of a star characterized by a piecewise EOS (59), with central density $\rho_c = 5 \times 10^{14} \text{g/cm}^3$, central pressure $p_c = 10^{34} \text{Pa}$, transition rest mass densities: $\rho_1 = 2 \times 10^{14} \text{g/cm}^3$, and $\rho_2 = 10^{12} \text{g/cm}^3$, and the polytropic indexes $(\gamma_1, \gamma_2, \gamma_3) = (3, 2, \frac{4}{3})$. The first row shows the values obtained from numerical integration. The second row shows the values obtained from the analytic series solutions.

	$(r_J)_1$	$\frac{(M_J)_1}{M_\odot}$	$(r_J)_2$	$\frac{(M_J)_2}{M_\odot}$	r_b	$\frac{M}{M_\odot}$
Numeric	13 929	2.043	16 657	2.341	16 701	2.343
Analytic	13 929	1.974	16 681	2.344	16 682	2.344

VI. CONCLUSION

Assuming sufficient regularity of the matter fluid, we have shown that the TOV equations can be solved in terms of analytic series. The series solutions developed in this work allowed us to obtain closed-form approximations to macroscopic properties of compact stellar objects, either in terms of the spacetime parameters or in terms of the underlying equation of state of the fluid.

The considered examples demonstrate that analytic solutions exist for widely used, nontrivial EOSs, and general approximation formulas for the stellar radius and mass can be derived. The accuracy of the analytic approximations depends on the behavior of the thermodynamic variables. In cases where the energy density or the pressure are characterized by long decreasing tails, more terms of the series are required. Nevertheless, using Padé approximants, the series converge sufficiently fast, such that closed-form expressions based on only a few terms of the series yielded accurate results in the cases examined.

Realistic stellar objects are composed of matter in distinct physical regimes. To model the stratification of stellar interiors, piecewise equations of state are often adopted. In such cases, the thermodynamic potentials may fail to be differentiable, or even continuous, across junction hypersurfaces. To deal with such configurations, we have constructed general series expansions about arbitrary points, such that analytic solutions can be obtained for piecewise EOSs and different regions are characterized by different series, matched across the transition hypersurfaces. To test this approach, we have considered a piecewise relativistic polytropic EOS as a toy model. The analytic results closely matched the numerical integration, having the advantage of being much faster to evaluate.

For piecewise solutions, we have not derived approximate formulas for the radius and mass functions. For piecewise EOSs, analytic approximations depend explicitly on the number of transition hypersurfaces. Nevertheless, for a fixed number of transitions, analytic approximation formulas can be constructed following the same reasoning as in the single-expression EOS case.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the author.

APPENDIX: PROOF OF THE PARITY PROPERTY OF POWER SERIES SOLUTIONS

Here we will prove Theorem 1, stating that if solutions to the TOV equations coupled with an equation of state exist, such that μ is real analytic at $r = 0$, its power series contains only even powers of the circumferential radius.

We will prove a number of intermediate results which we summarize in a set of lemmas. To avoid unnecessary repetition, the matrix $\tilde{\Theta}$ is given by Eq. (19) and has power series expansion $\tilde{\Theta}(r) = \sum_{m=0}^{+\infty} \tilde{\Theta}_m r^m$; it is assumed a barotropic EOS of the form (29), analytic at the center of the star and such that $f_\mu(\mu_c, p_c) \neq 0$, where $\mu_c \equiv \mu(0)$ and $p_c \equiv p(0)$ represent the values of the central energy density and central pressure, respectively, and $f_\mu \equiv \partial_\mu f$ and $f_p \equiv \partial_p f$.

Lemma 1. Let $f(\mu, p) = 0$ represent the equation of state of the fluid source, defining p implicitly as a function of μ . For $n \geq 1$ odd, if $\frac{f_p}{f_\mu} \in \mathcal{C}^n(\mu(0))$ and $\mu \in \mathcal{C}^n(0)$, and for all odd $k \in \{1, \dots, n\}$, $\mu^{(k)}(0)$ vanish, then

$$\frac{d^n}{dr^n} \left(\frac{f_p}{f_\mu} \right) (0) = 0. \quad (\text{A1})$$

Proof. A barotropic EOS $f(\mu, p) = 0$ defines $p = p(\mu)$ implicitly as a function of μ , such that $\frac{f_p}{f_\mu}$ is effectively a function of μ . In turn, μ is a function of the radial coordinate r . If $\frac{f_p}{f_\mu} \in \mathcal{C}^n(\mu(0))$ and $\mu \in \mathcal{C}^n(0)$, we can use directly Faà di Bruno's formula for the n th derivative of the composition, yielding

$$\begin{aligned} & \frac{d^n}{dr^n} \left(\frac{f_p}{f_\mu} \right) (0) \\ &= \sum \frac{n!}{m_1! m_2! \cdots m_n!} \left(\frac{f_p}{f_\mu} \right)^{(m_1+m_2+\cdots+m_n)} (\mu(0)) \\ & \quad \times \prod_{i=1}^n \left(\frac{1}{i!} \mu^{(i)}(0) \right)^{m_i}, \end{aligned} \quad (\text{A2})$$

where the sum is taken over all non-negative integers $m_1, m_2, \dots, m_n$ that verify the constraint $m_1 + 2m_2 + 3m_3 + \cdots + nm_n = n$, and the derivatives of $\frac{f_p}{f_\mu}$ in the sum are with respect to μ .

For n odd, that is, for an odd-order derivative, n cannot be the sum of even numbers. Therefore, there is at least one nontrivial odd-indexed m_i in the equation $m_1 + 2m_2 + 3m_3 + \cdots + nm_n = n$. Then, for n odd, in Eq. (A2), we will always have in the product at least one term with an odd-order derivative of μ . Assuming that for all odd $k \leq n$, $\mu^{(k)}(0)$ vanish, Eq. (A2) implies Eq. (A1). ■

Lemma 2. Let $f(\mu, p) = 0$ represent the equation of state of the fluid source, defining p implicitly as a function of μ . For $n \geq 1$ odd, if $p \in \mathcal{C}^n(\mu(0))$, $\mu \in \mathcal{C}^n(0)$ and for all odd $k \in \{1, \dots, n\}$, $\mu^{(k)}(0)$ vanish, then

$$\frac{d^n p}{dr^n} (0) = 0. \quad (\text{A3})$$

The proof of Lemma 2 follows the same reasoning as the proof of Lemma 1.

Lemma 3. Let $n \geq 0$ and the matrix $\mathbb{P}$ verify Eqs. (17) and (18). If $\tilde{\Theta}_{2k} = 0$ for all $k \in \{0, 1, \dots, n\}$, then $\mathbb{P}_{2n+1} = 0$.

Proof. The proof follows directly from the recurrence relation (18). For $m = 2n + 1$, where $n \geq 0$, we have

$$\mathbb{P}_{2n+1} = \frac{1}{2n+1} \sum_{i=0}^{2n} \tilde{\Theta}_{2n-i} \mathbb{P}_i. \quad (\text{A4})$$

If all $\tilde{\Theta}_{2k} = 0$ vanish for $0 \leq k \leq n$, the sum in Eq. (A4) contains only terms with odd index i , that is, $\mathbb{P}_{2n+1}$ depends only on the coefficients $\mathbb{P}_i$ with index i odd smaller than $2n$. If $\tilde{\Theta}_0 = 0$, $\mathbb{P}_1 = 0$. In turn, $\mathbb{P}_1 = 0$ implies $\mathbb{P}_3 = 0$, which implies $\mathbb{P}_5 = 0$ and so on.

Lemma 3 follows from the assumption that for a given value $2n + 1$, all terms of the power series of $\tilde{\Theta}$ of even order smaller than or equal to $2n$ vanish. The following result provides sufficient conditions for this to be verified.

Lemma 4. Let $n \geq 0$ and μ be real analytic at $r = 0$, such that $\mu^{(k)}(0)$ vanish, for all odd $k \in \{1, \dots, 2n + 1\}$. Then, $\tilde{\Theta}_{2n} = 0$.

Proof. To prove the lemma, we will explicitly find the coefficients $\tilde{\Theta}_m$ of the power series of $\tilde{\Theta}$ at $r = 0$, in terms of the derivatives of μ . Since $(\tilde{\Theta})_{21} = r$ and $(\tilde{\Theta})_{22} = 0$, their power series are trivial. Then, we will focus specifically on the power series of the (1, 1) and (1, 2) entries.

Assuming μ is real analytic at $r = 0$ with power series $\mu(r) = \sum_{m=0}^{\infty} \frac{1}{m!} \mu^{(m)}(0) r^m$, from Eq. (3) we have

$$2 \left(1 - \frac{2M}{r} \right) = 2 - 2 \sum_{m=0}^{\infty} \frac{\mu^{(m)}(0)}{m!(m+3)} r^{m+2}. \quad (\text{A5})$$

Therefore,

$$2\left(1 - \frac{2M}{r}\right) = \sum_{m=0}^{\infty} b_m r^m, \quad (\text{A6})$$

where

$$\begin{aligned} b_0 &= 2 \\ b_1 &= 0 \\ b_m &= -\frac{2}{(m+1)(m-2)!} \mu^{(m-2)}(0), \quad m \geq 2. \end{aligned} \quad (\text{A7})$$

In particular, for $m \geq 2$ even, the coefficients b_m depend only on the even-order derivatives of μ at zero.

Continuing,

$$r\left(\mu - \frac{2M}{r^3}\right) = \sum_{m=0}^{\infty} \frac{m+2}{(m+3)m!} \mu^{(m)}(0) r^{m+1}. \quad (\text{A8})$$

Therefore,

$$r\left(\mu - \frac{2M}{r^3}\right) = \sum_{m=0}^{\infty} a_m r^m, \quad (\text{A9})$$

where

$$\begin{aligned} a_0 &= 0 \\ a_m &= \frac{m+1}{(m+2)(m-1)!} \mu^{(m-1)}(0), \quad m \geq 1. \end{aligned} \quad (\text{A10})$$

Lastly,

$$\frac{1}{r} \left(\mu - \frac{6M}{r^3} \right) = \sum_{m=1}^{\infty} \frac{m}{(m+3)m!} \mu^{(m)}(0) r^{m-1},$$

which can be written as

$$\frac{1}{r} \left(\mu - \frac{6M}{r^3} \right) = \sum_{m=0}^{\infty} \bar{a}_m r^m, \quad (\text{A11})$$

with

$$\begin{aligned} \bar{a}_0 &= \frac{1}{4} \mu^{(1)}(0) \\ \bar{a}_m &= \frac{m+1}{(m+4)(m+1)!} \mu^{(m+1)}(0), \quad m \geq 1. \end{aligned} \quad (\text{A12})$$

From Eqs. (A10) and (A12), for m even, the coefficients a_m and $\bar{a}_m$ depend only on odd-order derivatives of μ at zero.

The division of power series can be expressed compactly using determinants. Namely, for the (1, 1) entry we have

$$(\tilde{\Theta})_{11} = \frac{r(\mu - \frac{2M}{r^3})}{2(1 - \frac{2M}{r})} = \frac{\sum_{m=0}^{\infty} a_m r^m}{\sum_{m=0}^{\infty} b_m r^m} = \sum_{m=0}^{\infty} d_m r^m, \quad (\text{A13})$$

where

$$d_0 = \frac{a_0}{b_0},$$

$$d_m = \frac{1}{b_0^{m+1}} \begin{vmatrix} a_m & b_1 & b_2 & \cdots & b_m \\ a_{m-1} & b_0 & b_1 & \cdots & b_{m-1} \\ a_{m-2} & 0 & b_0 & \cdots & b_{m-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_0 & 0 & 0 & \cdots & b_0 \end{vmatrix}, \quad \text{for } m \geq 1. \quad (\text{A14})$$

For (1, 2) entry,

$$(\tilde{\Theta})_{12} = \frac{\mu - \frac{6M}{r^3}}{2r(1 - \frac{2M}{r})} = \frac{\sum_{m=0}^{\infty} \bar{a}_m r^m}{\sum_{m=0}^{\infty} b_m r^m} = \sum_{m=0}^{\infty} \bar{d}_m r^m, \quad (\text{A15})$$

where

$$\bar{d}_0 = \frac{\bar{a}_0}{b_0}$$

$$\bar{d}_m = \frac{1}{b_0^{m+1}} \begin{vmatrix} \bar{a}_m & b_1 & b_2 & \cdots & b_m \\ \bar{a}_{m-1} & b_0 & b_1 & \cdots & b_{m-1} \\ \bar{a}_{m-2} & 0 & b_0 & \cdots & b_{m-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \bar{a}_0 & 0 & 0 & \cdots & b_0 \end{vmatrix}, \quad \text{for } m \geq 1. \quad (\text{A16})$$

From Eqs. (A14) and (A16), $d_0 = 0$, and $\bar{d}_0 = \frac{1}{8} \mu^{(1)}(0)$. Then, $\tilde{\Theta}_0 = 0$, if $\mu^{(1)}(0) = 0$. For $m \geq 2$ even, we will show that if all derivatives $\mu^{(k)}(0)$, with odd $k \in \{1, \dots, m-1\}$, vanish, then $(\tilde{\Theta}_m)_{11}$ is zero. The proof for $(\tilde{\Theta}_m)_{12}$ follows similarly, assuming that all derivatives $\mu^{(k)}(0)$, with odd $k \in \{1, \dots, m+1\}$, vanish.

We will prove by recurrence that for $m \geq 2$ even, $d_m = 0$, using Laplace cofactor expansion for the determinant. To not have to worry about the signs that follow from applying Laplace expansion, we will consider the absolute value, $|d_m|$.

For an even $m \geq 2$, $(\tilde{\Theta}_m)_{11}$ is given by

$$d_m = \frac{1}{2^{m+1}} \det \begin{bmatrix} a_m & b_1 & b_2 & \cdots & b_m \\ a_{m-1} & b_0 & b_1 & \cdots & b_{m-1} \\ a_{m-2} & 0 & b_0 & \cdots & b_{m-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_0 & 0 & 0 & \cdots & b_0 \end{bmatrix}, \quad (\text{A17})$$

where we have used $b_0 = 2$. If all derivatives $\mu^{(k)}(0)$, with odd $k \leq m-1$, vanish, all b_k , and a_{k+1} , are zero. Given the structure of the matrix in Eq. (A17), using Laplace expansion, the determinant is associated with the determinant of the submatrix with the last line and last column removed from the original matrix, such that

$$\begin{aligned} |d_m| &= \frac{1}{2^{m+1}} \det \begin{bmatrix} a_m & b_1 & b_2 & \cdots & b_m \\ a_{m-1} & b_0 & b_1 & \cdots & b_{m-1} \\ a_{m-2} & 0 & b_0 & \cdots & b_{m-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_0 & 0 & 0 & \cdots & b_0 \end{bmatrix} \\ &= \frac{1}{2^m} \det \begin{bmatrix} a_m & b_1 & b_2 & \cdots & b_{m-1} \\ a_{m-1} & b_0 & b_1 & \cdots & b_{m-2} \\ a_{m-2} & 0 & b_0 & \cdots & b_{m-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & 0 & 0 & \cdots & b_0 \end{bmatrix}. \end{aligned} \quad (\text{A18})$$

In turn, since $b_1 = 0$, the second column only has one nontrivial entry, hence

$$|d_m| = \frac{1}{2^{m-1}} \det \begin{bmatrix} a_m & b_2 & b_3 & \cdots & b_{m-1} \\ a_{m-2} & b_0 & b_1 & \cdots & b_{m-3} \\ a_{m-3} & 0 & b_0 & \cdots & b_{m-4} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & 0 & 0 & \cdots & b_0 \end{bmatrix}. \quad (\text{A19})$$

Since $b_1 = b_3 = 0$, again the second column only has one nontrivial entry, therefore

$$|d_m| = \frac{1}{2^{m-2}} \det \begin{bmatrix} a_m & b_2 & b_4 & \cdots & b_{m-1} \\ a_{m-2} & b_0 & b_2 & \cdots & b_{m-3} \\ a_{m-4} & 0 & b_0 & \cdots & b_{m-5} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & 0 & 0 & \cdots & b_0 \end{bmatrix}. \quad (\text{A20})$$

Following this procedure, since all even indexed coefficients a_i vanish, we will end up with a matrix of size $\frac{m}{2} \times \frac{m}{2}$ whose first column is composed of only zeros, hence the determinant is zero. Curiously, the procedure starts and ends by considering a submatrix with the last line and last column removed from a previous matrix.

We have shown that, for m even, $d_m = 0$. The proof that $\bar{d}_m = 0$, for m even, follows similarly. Then, for m even, $\tilde{\Theta}_m = 0$. ■

The last intermediate result that we will need concerns the structure of the power series of $\frac{\mathcal{A}}{r\phi}$.

Lemma 5. Let s and u be a solution of Eqs. (14)–(19), then

$$\left(\frac{\mathcal{A}}{r\phi}\right)(0) = 0. \quad (\text{A21})$$

If μ is a real analytic function at $r = 0$, such that for an even $m \geq 2$, $\mu^{(k)}(0)$ vanish, for all odd $k \in \{1, \dots, m-1\}$, then,

$$\left(\frac{\mathcal{A}}{r\phi}\right)^{(k+1)}(0) = 0. \quad (\text{A22})$$

Proof. Equation (15) implies $\frac{2\mathcal{A}}{r\phi} = \frac{s}{u}$. Then, it suffices to show that truncating the power series of u at some even order, it contains only terms with even powers of r .

From Eqs. (14) and (17), we have

$$\begin{bmatrix} s \\ u \end{bmatrix} = \begin{bmatrix} r & 0 \\ 0 & 1 \end{bmatrix} (\mathbb{P}_0 + \mathbb{P}_1 r + \mathbb{P}_2 r^2 + \cdots) \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \quad (\text{A23})$$

hence

$$\begin{aligned} s &= \sum_{i=0}^{\infty} [c_1(\mathbb{P}_i)_{11} + c_2(\mathbb{P}_i)_{12}] r^{i+1} = \sum_{i=0}^{\infty} s_i r^{i+1}, \\ u &= \sum_{i=0}^{\infty} [c_1(\mathbb{P}_i)_{21} + c_2(\mathbb{P}_i)_{22}] r^i = \sum_{i=0}^{\infty} u_i r^i. \end{aligned} \quad (\text{A24})$$

In particular, $s_0 = 0$ and, for $i \geq 1$, s_i depends on $\mathbb{P}_{i-1}$, and u_i depends on $\mathbb{P}_i$. The quotient s/u can be written as a power series in terms of the power series of s and u , such that

$$\frac{s}{u} = \sum_{i=0}^{\infty} d_i r^i, \quad (\text{A25})$$

where

$$d_0 = 0,$$

$$d_m = \frac{1}{u_0^{m+1}} \begin{vmatrix} s_m & u_1 & u_2 & \cdots & u_m \\ s_{m-1} & u_0 & u_1 & \cdots & u_{m-1} \\ s_{m-2} & 0 & u_0 & \cdots & u_{m-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_0 & 0 & 0 & \cdots & u_0 \end{vmatrix}, \quad \text{for } m \geq 1. \quad (\text{A26})$$

In particular, since the zeroth-order coefficient d_0 is identically zero, it implies that $(\frac{2A}{r\phi})(0) = 0$. Moreover, for a given $m \geq 1$, the coefficient d_m depends on the matrix coefficients $\mathbb{P}_i$, with $i = \{0, 1, \dots, m\}$.

For an even $m \geq 2$, if the derivatives $\mu^{(k)}(0)$ vanish for all odd $k \in \{1, \dots, m-1\}$, Lemmas 3 and 4 imply that all $\mathbb{P}_k$ are identically zero. Therefore, all u_k vanish and $\sum_{i=0}^m u_i r^i$ defines an even function of the radial coordinate. Naturally, this implies $\sum_{i=0}^m s_i r^i$ defines an odd function, since $s = du/dr$. Therefore,

$$\frac{\sum_{i=0}^m s_i r^i}{\sum_{i=0}^m u_i r^i} \quad (\text{A27})$$

is an odd function in the radial coordinate, in particular, all even-order terms of its power series expansion about $r = 0$ are identically zero. ■

Lemmas 1–5 follow from assuming that all odd-order derivatives of μ of order smaller than or equal to a particular

n vanish at $r = 0$. The last step in the proof of Theorem 1 is to show that analytic solutions of Eqs. (14)–(19) together with Eq. (32) have exactly this property.

Taking the derivative of order n of Eq. (32) yields

$$\begin{aligned} (\mu)^{(n+1)}(0) &= 2 \sum_{k=0}^n \sum_{i=0}^k \binom{n}{k} \binom{k}{i} \left(\frac{f_p}{f_\mu} \right)^{(n-k)} \\ &\quad \times (\mu + p)^{(k-i)} \left(\frac{A}{r\phi} \right)^{(i)}, \end{aligned} \quad (\text{A28})$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \quad (\text{A29})$$

and all derivatives are taken with respect to r and evaluated at $r = 0$.

For $n = 0$, Lemma 5 states that $(\frac{A}{r\phi})(0) = 0$. Then, from Eq. (A28), $\mu^{(1)}(0) = 0$.

For $n = 2$, since $\mu^{(1)}(0) = 0$, from Lemma 5, the derivatives $(\frac{A}{r\phi})^{(i)}$, with $i \in \{0, 2\}$, vanish, such that in the sums in Eq. (A28), only the terms with i odd, $i = 1$, may be nontrivial. However, if k is even, then $k - i$ is odd. Using Lemma 2, the term $(\mu + p)^{(k-i)}$ vanishes. If k is odd, $n - k$ is odd. From Lemma 1, the term $(\frac{f_p}{f_\mu})^{(n-k)}$ is zero.

Therefore, $\mu^{(3)}(0) = 0$.

The cases for $n = \{5, 7, \dots\}$ follow similarly.

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- [1] R. Huxford, R. Kashyap, S. Borhanian, A. Dhani, I. Gupta, and B. S. Sathyaprakash, Accuracy of neutron star radius measurement with the next generation of terrestrial gravitational-wave observatories, *Phys. Rev. D* **109**, 103035 (2024).
 - [2] J. M. Lattimer and M. Prakash, Neutron star structure and the equation of state, *Astrophys. J.* **550**, 426 (2001).
 - [3] N. A. Webb and D. Barret, Constraining the equation of state of supranuclear dense matter from XMM-Newton observations of neutron stars in globular clusters, *Astrophys. J.* **671**, 727 (2007).
 - [4] F. Özel and D. Psaltis, Reconstructing the neutron-star equation of state from astrophysical measurements, *Phys. Rev. D* **80**, 103003 (2009).
 - [5] L. Lindblom, Spectral representations of neutron-star equations of state, *Phys. Rev. D* **82**, 103011 (2010).
 - [6] M. Oertel, M. Hempel, T. Klähn, and S. Typel, Equations of state for supernovae and compact stars, *Rev. Mod. Phys.* **89**, 015007 (2017).
 - [7] G. Baym, T. Hatsuda, T. Kojo, P. D. Powell, Y. Song, and T. Takatsuka, From hadrons to quarks in neutron stars: A review, *Rep. Prog. Phys.* **81**, 056902 (2018).
 - [8] T. Kojo, G. Baym, and T. Hatsuda, Implications of NICER for neutron star matter: The QHC21 equation of state, *Astrophys. J.* **934**, 46 (2022).
 - [9] A. Akmal, V. R. Pandharipande, and D. G. Ravenhall, Equation of state of nucleon matter and neutron star structure, *Phys. Rev. C* **58**, 1804 (1998).
 - [10] J. Morales Jr., V. R. Pandharipande, and D. G. Ravenhall, Improved variational calculations of nucleon matter, *Phys. Rev. C* **66**, 054308 (2002).
 - [11] A. Mukherjee, Variational theory of hot nucleon matter. II. Spin-isospin correlations and equation of state of nuclear and neutron matter, *Phys. Rev. C* **79**, 045811 (2009).
 - [12] T. Krüger, I. Tews, K. Hebeler, and A. Schwenk, Neutron matter from chiral effective field theory interactions, *Phys. Rev. C* **88**, 025802 (2013).

- [13] K. Otto, M. Oertel, and B. J. Schaefer, Hybrid and quark star matter based on a nonperturbative equation of state, *Phys. Rev. D* **101**, 103021 (2020).
- [14] A. R. Raduta, M. Oertel, and A. Sedrakian, Proto-neutron stars with heavy baryons and universal relations, *Mon. Not. R. Astron. Soc.* **499**, 914 (2020).
- [15] J. S. Read, B. D. Lackey, B. J. Owen, and J. L. Friedman, Constraints on a phenomenologically parametrized neutron-star equation of state, *Phys. Rev. D* **79**, 124032 (2009).
- [16] M. F. O’Boyle, C. Markakis, N. Stergioulas, and J. S. Read, A parametrized equation of state for neutron star matter with continuous sound speed, *Phys. Rev. D* **102**, 083027 (2020).
- [17] L. Suleiman, M. Fortin, J. L. Zdunik, and C. Providência, Polytropic fits of modern and unified equations of state, *Phys. Rev. C* **106**, 035805 (2022).
- [18] M. S. R. Delgaty and K. Lake, Physical acceptability of isolated, static, spherically symmetric, perfect fluid solutions of Einstein’s equations, *Comput. Phys. Commun.* **115**, 395 (1998).
- [19] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, and E. Herlt, *Exact Solutions of Einstein’s Field Equations* (Cambridge University Press, Cambridge, England, 2009).
- [20] L. Lindblom, Phase transitions and the mass-radius curves of relativistic stars, *Phys. Rev. D* **58**, 024008 (1998).
- [21] S. Rahman and M. Visser, Spacetime geometry of static fluid spheres, *Classical Quantum Gravity* **19**, 935 (2002).
- [22] K. Lake, All static spherically symmetric perfect fluid solutions of Einstein’s equations, *Phys. Rev. D* **67**, 104015 (2003).
- [23] D. Martin and M. Visser, Algorithmic construction of static perfect fluid spheres, *Phys. Rev. D* **69**, 104028 (2004).
- [24] P. Boonserm, M. Visser, and S. Weinfurtner, Generating perfect fluid spheres in general relativity, *Phys. Rev. D* **71**, 124037 (2005).
- [25] P. Boonserm, M. Visser, and S. Weinfurtner, Solution generating theorems for the TOV equation, *Phys. Rev. D* **76**, 044024 (2007).
- [26] S. Carloni and D. Vernieri, Covariant Tolman-Oppenheimer-Volkoff equations. I. The isotropic case, *Phys. Rev. D* **97**, 124056 (2018).
- [27] T. Harko and M. K. Mak, Exact power series solutions of the structure equations of the general relativistic isotropic fluid stars with linear barotropic and polytropic equations of state, *Astrophys. Space Sci.* **361**, 283 (2016).
- [28] A. S. Saad, M. I. Nouh, A. A. Shaker, and T. M. Kamel, Approximate analytical solutions to the relativistic isothermal gas spheres, *Rev. Mex. Astron. Astrofis.* **53**, 247 (2017).
- [29] C. A. Clarkson and R. K. Barrett, Covariant perturbations of Schwarzschild black holes, *Classical Quantum Gravity* **20**, 3855 (2003).
- [30] G. Betschart and C. A. Clarkson, Scalar field and electromagnetic perturbations on locally rotationally symmetric spacetimes, *Classical Quantum Gravity* **21**, 5587 (2004).
- [31] C. A. Clarkson, A covariant approach for perturbations of rotationally symmetric spacetimes, *Phys. Rev. D* **76**, 104034 (2007).
- [32] E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations* (McGraw-Hill, New York, 1955).
- [33] A. Coddington and R. Carlson, *Linear Ordinary Differential Equations* (Society for Industrial and Applied Mathematics, Philadelphia, 1997).
- [34] P. Luz and S. Carloni, Gauge invariant perturbations of static spatially compact LRS II spacetimes, *Classical Quantum Gravity* **41**, 235012 (2024).
- [35] P. Luz and S. Carloni, Adiabatic radial perturbations of relativistic stars: Analytic solutions to an old problem, *Phys. Rev. D* **110**, 084054 (2024).
- [36] P. Luz and S. Carloni, Noncomoving description of adiabatic radial perturbations of relativistic stars, *Phys. Rev. D* **110**, 084055 (2024).
- [37] P. Luz and S. Carloni, Radial adiabatic perturbations of stellar compact objects, [arXiv:2604.20960](https://arxiv.org/abs/2604.20960).
- [38] R. C. Tolman, Static solutions of Einstein’s field equations for spheres of fluid, *Phys. Rev.* **55**, 364 (1939).
- [39] H. Heintzmann, New exact static solutions of Einstein’s field equations, *Z. Phys.* **228**, 489 (1969).
- [40] E. Witten, Cosmic separation of phases, *Phys. Rev. D* **30**, 272 (1984).
- [41] D. Christodoulou, Self-gravitating relativistic fluids: A two-phase model, *Arch. Ration. Mech. Anal.* **130**, 343 (1995).
- [42] A. Urbano and H. Veermäe, On gravitational echoes from ultracompact exotic stars, *J. Cosmol. Astropart. Phys.* **04** (2019) 011.
- [43] R. Sharma and S. D. Maharaj, A class of relativistic stars with a linear equation of state, *Mon. Not. R. Astron. Soc.* **375**, 1265 (2007).
- [44] E. Farhi and R. L. Jaffe, Strange matter, *Phys. Rev. D* **30**, 2379 (1984).
- [45] R. F. Tooper, Adiabatic fluid spheres in general relativity, *Astrophys. J.* **142**, 1541 (1965).
- [46] U. S. Nilsson and C. Uggla, General relativistic stars: Polytropic equations of state, *Ann. Phys. (N.Y.)* **286**, 292 (2000).
- [47] J. M. Heinzle, N. Rohr, and C. Uggla, Dynamical systems approach to relativistic spherically symmetric static perfect fluid models, *Classical Quantum Gravity* **20**, 4567 (2003).
- [48] L. Herrera and W. Barreto, General relativistic polytropes for anisotropic matter: The general formalism and applications, *Phys. Rev. D* **88**, 084022 (2013).
- [49] L. Herrera, A. Di Prisco, W. Barreto, and J. Ospino, Conformally flat polytropes for anisotropic matter, *Gen. Relativ. Gravit.* **46**, 1827 (2014).
- [50] L. Lindblom, Determining the nuclear equation of state from neutron-star masses and radii, *Astrophys. J.* **398**, 569 (1992).
- [51] C. Vuille, Evidence for a hard equation of state in the cores of neutron stars, [arXiv:0812.3828](https://arxiv.org/abs/0812.3828).
- [52] G. Darmon, Les équations de la gravitation einsteinienne, in *Mémoires des sciences mathématiques, fascicule* (Gauthier-Villars, Paris, 1927), Vol. 25.
- [53] W. Israel, Singular hypersurfaces and thin shells in general relativity, *Il Nuovo Cimento* **44**, 10 (1966).