

Pentaquarks on the light front and their mixing to baryons

Nicholas Miesch^{✉,*}, Edward Shuryak^{✉,†} and Ismail Zahed[‡]

*Center for Nuclear Theory, Department of Physics and Astronomy,
Stony Brook University, Stony Brook, New York 11794–3800, USA*



(Received 8 December 2025; accepted 22 January 2026; published 7 May 2026)

In previous papers we developed the light front formulation for Hamiltonians and wave functions (WFs) for mesons and baryons, with both confinement and chiral symmetry breaking. For baryons limited to the lowest Fock component with three quarks, the longitudinal WF is valued in an equilateral triangle with momentum fractions x_i , $i = 1, 2, 3$. In this paper we extend this analysis to light quark pentaquarks ($n = 5$) at angular momenta $L = 0$ and $L = 1$, in which case the longitudinal WF is defined on 4d simplex A_4 with 5 corners. The WF was developed both numerically and using a basis function that diagonalizes the Laplacian with Dirichlet boundary conditions. From the WF follows many predictions, we discuss PDFs of antiquarks specifically. We then focus on mixing between baryons with $27 L = 1$ pentaquark states, via two operators related to adding pionlike and sigmalike $\bar{q}q$ pairs. While similar mechanisms of introduction of nonzero angular momentum and flavor asymmetry of the antiquark sea was discussed before, our pentaquark-based approach is unique in providing WF with full Fermi statistics of 4 quarks.

DOI: [10.1103/3crd-ldcv](https://doi.org/10.1103/3crd-ldcv)

I. INTRODUCTION

Light front wave function was used by [1] in order to relate form factors with large- x PDFs. For further discussion of partonic observables and light-front wave function has been done in detail in [2].

Models of hadronic structure on light front (LF) were developed mostly in the last decade. The so called LF holographic approach [3] was used, simplifying LFWFs to dependence of just one single variable ζ , incorporating transverse coordinates and longitudinal momenta.

The basis light-front quantization (BFLQ) approach (see, e.g., [4,5]) use Hamiltonians and LFWFs partially related to ζ .

Our approach is in the simplicity of the Hamiltonian as suggested by quantum chromodynamics (QCD) confinement. We also always quantize those using minimally needed set of Jacobi variables, thus eliminating fictitious motion of the center-of-mass.

This paper is part of a series devoted to bridging the gap between hadronic spectroscopy and partonic observables. In the original paper [6] we showed how the quark model for hadrons with constituent quarks and confinement, can be

extended to the light front. We then focused on the mesons in [7], and then the baryons in [7,8]. In both descriptions, the Hamiltonian included both the kinetic energy and confinement, quantized in the momentum representation, with the hyperfine interactions treated in perturbation. The light cone WFs depends on the (Bjorken-Feynman) momentum fractions inside the 3D cube $x_1, x_2, x_3 \in [0, 1]^3$, cut by the momentum normalization condition $x_1 + x_2 + x_3 = 1$. As shown in Fig. 1, this leads to quantum mechanics on a 2D equilateral triangle (also known as regular 2-simplex called A_2). For completeness, some of the details of this procedure will be briefly recalled in the section A below. In this paper we generalize this approach to pentaquarks. The forward part of the wave functions is defined at regular 4-simplex called A_4 . We study WFs on it, both by approximate method (Ritz variational) and exact analytic one based on sum of many interfering waves.

Several of our recent papers developed a novel approach to finding the WFs antisymmetric under quark permutations, as required by Fermi statistics. This method was used to analyze baryons in P-shells [9] and D-shells [10], tetraquarks in [11] and even multi-quark states with 6, 9, and 12 quarks. In all those cases the method is based on a *brute force* representation of the permutation groups S_N generators and their diagonalization, first in special “good basis” notations, and then translating to standard monom representation. Since in the pentaquark case discussed the “monom space” is fairly large, $\mathcal{O}(10^6)$, we relied heavily on Wolfram *Mathematica*. Needless to say, that the method was checked to reproduce the standard group representations for few quark systems. The latter becomes

*Contact author: nicholas.miesch@stonybrook.edu

†Contact author: edward.shuryak@stonybrook.edu

‡Contact author: ismail.zahed@stonybrook.edu

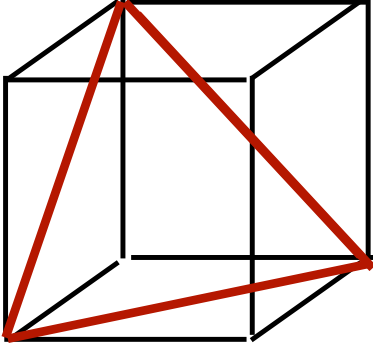


FIG. 1. Equilateral triangle formed by cube $x_1, x_2, x_3 \in [0, 1]^3$ cut by momentum normalization condition $x_1 + x_2 + x_3 = 1$.

increasingly untractable for higher quark systems, as the mixed representations mushroom, hence the need of a more advanced approach.

In a recent study [12], we analyzed (light quark) *pentaquarks* in this framework. More specifically, we derived the explicit WFs for their S- and P-shell states, which are needed for the baryon-pentaquark mixing. This is key to solving longstanding puzzles, such as the flavor asymmetry of the antiquark sea, and the strong orbital motion in the nucleons. The present paper follows [12] closely, by extending the description of pentaquarks to the light front (LF).

The paper is organized as follows. We start with II where the main ideas and technical steps encountered are recalled, by setting quark models on the light front. These are then applied to light pentaquarks $qqqq\bar{q}$ in Sec. III. The representations of the permutation group S_4 , and the explicit construction of the orbital-color-spin-flavor wave functions for S-shell and P-shell pentaquarks, are carried in Sec. IV. The P-wave (negative parity) states are important for addressing the $\bar{q}q$ admixture in qqq baryons, as discussed in Sec. V. The relation between the results obtained and the properties of the antiquark sea of the nucleon, are discussed in Sec. VI. The paper is summarized in Sec. VII.

Most of the technical content of the paper is delegated to the Appendices. Appendix A recalls the quantization method for baryons and other hadrons on the LF, in particularly their relation to the well known problem of n fermions on a circle. Appendix B considers a variational calculation of the longitudinal pentaquark wave function. Appendix C derives an exact eigensystem for the Laplacian on the A_4 simplex, as needed for pentaquarks. Appendix D considers the large N limit, in which pertinent simplices can be approximated by spheres.

II. QUARK MODELS ON THE LIGHT FRONT

Quark models, including confining potentials and spin-dependent forces, have for decades provided a reasonably good explanation of hadronic spectroscopy. They are naturally formulated in the center of mass (CM) frame.

One may wonder why people, us included, have put efforts to rederive them using LF kinematics.

One reason is to get a more direct connection to the relativistic form-factors and “partonic observables” (PDF, TMD, GPD etc). Another, is that in the LF formulation one can treat light and heavy quark systems on equal footing, the nonrelativistic approximation is not needed. Finally, what looks like static π , σ “clouds” around hadrons in their rest frame, under a boost they give rise to the $\bar{q}q$ “sea,” or multi-quark components of the wave functions. The complex nature of the boost operator is what forced the development of hadronic models directly on the LF. These models usually posit some Hamiltonian, and then solve it to account for the full spectroscopy, with masses and wave functions.

A. Confined N -quarks in 1 + 1 dimensions

For simplicity, consider first the dimensionally reduced longitudinal LF Hamiltonian for N confined quarks of equal mass,

$$H_{LF,L} = \sum_{i=1}^N \left(\frac{m_Q^2}{x_i} + 2\sigma_T \left| \frac{i\partial}{\partial x_i} \right| \right) \quad (1)$$

The corresponding WFs for N -quarks are solutions to

$$\sum_{i=1}^N \left(\frac{m_Q^2}{x_i} + 2\sigma_T \left| \frac{i\partial}{\partial x_i} \right| \right) \varphi_n[x] = M_n^2 \varphi_n[x] \quad (2)$$

giving mass squared of specific states.

Note that this Hamiltonian already includes two main nonperturbative phenomena in QCD. The first term is the kinetic energy, it includes effective “constituent” quark masses. The second, proportional to the string tension σ_T , describes quark confinement. It will be solved in the momentum representation, so the coordinates are derivative of momenta.

The variables x_i are Bjorken-Feynman momentum fractions for N identical particles moving in a box $0 \leq x_i \leq 1$ and subject to the momentum constraint

$$X = \sum_{i=1}^N x_i = 1. \quad (3)$$

The condition (3) makes the physical support of the problem nontrivial. It is an $\tilde{N}$ -simplex with one dimension lower $\tilde{N} = N - 1$. For baryons $N = 3$, the domain is an equilateral triangle, see Fig. 1, and our solution for quantum mechanics on it was developed earlier. For larger N the domain is the N -dimensional generalization of an equilateral triangle. For $N = 4$ it is a tetrahedron or 3-simplex, and for pentaquarks $N = 5$ it is the 4-dimensional 5-simplex A_4 , a pentachoron with 5 corners, 10 edges and 10 triangular faces also referred by us as a “starfish.”

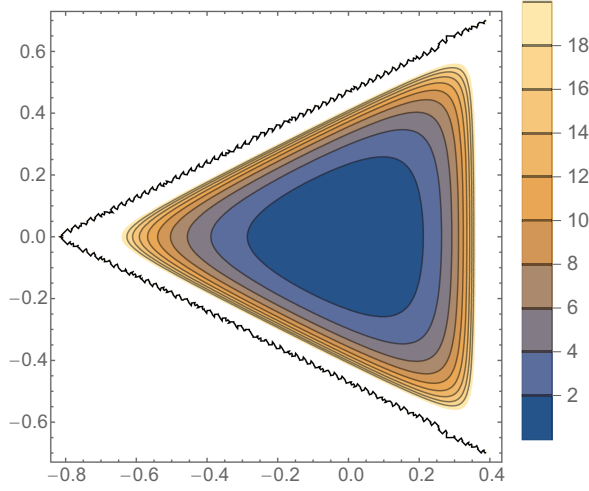


FIG. 2. Effective potential for equilateral triangular domain for $N = 3$, as defined in [8].

The singular kinetic energy term forces the wave function to vanish at these boundaries (Dirichlet boundary condition). The shape of the effective “cup potential” is shown in Fig. 2 for the baryons.

To solve this problem, we proceed in three steps. First, we unwind the square roots by using the einbein trick

$$\begin{aligned} \sum_{i=1}^N \left| \frac{i\partial}{\partial x_i} \right| &= \sum_{i=1}^N \frac{1}{2} \left(\frac{1}{e_{iL}} + e_{iL} \left(\frac{i\partial}{\partial x_i} \right)^2 \right) \\ &\rightarrow \frac{1}{2} \left(\frac{3}{e_L} + e_L \sum_{i=1}^N \left(\frac{i\partial}{\partial x_i} \right)^2 \right) \end{aligned} \quad (4)$$

and assume equal $e_{iL} = e_L$ at the extrema, in the steepest descent approximation. Second, we diagonalize the free Laplacian after isolating the center of mass coordinate, using normal mode coordinates (analog of Jacobi coordinates) see below. Third, we minimize the energies with respect to e_L .

The Hamiltonian and wave functions use the momentum representation and Jacobi coordinates (so, unlike the work carried by Vary’s group, we do not have any issue with the center-of-CM motion, which is set to zero from the onset.).

There are two methods to do quantum mechanics for such a Hamiltonian:

- (i) One, derived earlier for a triangle, is to work out the complete set of eigenfunctions for the Laplacian (the second term), and then represents the second term as a matrix. In some truncated form it can be diagonalized.
- (ii) For a triangle, one can numerically solve the Schrodinger equation, with the Laplacian and a cup potential included. For the reader convenience, the main points in this approach can be found in Appendix A. Recently, for pentaquarks we were able to numerically diagonalize this Hamiltonian by using a set of determinantal wave functions [13].

Here we will develop a variational approximation, as well as a simplex-motivated derivation.

III. PENTAQUARKS ON THE LIGHT FRONT

The first step in conventional spectroscopy (in the CM frame) is the selection of appropriate coordinates, excluding the fictitious motion of the center of mass. In the momentum representation and for momentum fractions, this is captured by the longitudinal momentum constraint (3). In both cases the problem was solved for us long ago, by choosing Jacobi coordinates. For baryons ($N = 3$) they are traditionally called ρ, λ

$$\begin{aligned} x_1 &= (\sqrt{6}\lambda + 3\sqrt{2}\rho + 2X)/6 \\ x_2 &= (\sqrt{6}\lambda - 3\sqrt{2}\rho + 2X)/6 \\ x_3 &= (-\sqrt{6}\lambda + X)/3 \end{aligned} \quad (5)$$

Note that Laplacian in Jacobi coordinates is

$$\sum_{i=1}^3 \left(\frac{i\partial}{\partial x_i} \right)^2 = \left(\frac{i\partial}{\partial \lambda} \right)^2 + \left(\frac{i\partial}{\partial \rho} \right)^2 + 3 \left(\frac{i\partial}{\partial X} \right)^2 \quad (6)$$

and because the total momentum does not change, only the two first terms remain. So, without a cup potential, the problem thereby is reduced to two free particles confined to the equilateral triangle box. Our initial study addressed this problem.

We follow up on this work by considering five partons, subject to the longitudinal constraint $\sum_{i=1}^5 x_i = 1$. Using the (modified) Jacobi coordinates $\alpha, \beta, \gamma, \delta$ amounts to

$$\begin{aligned} x_1 &= (6 + 15\sqrt{2}\alpha + 5\sqrt{6}\beta + 5\sqrt{3}\gamma + 3\sqrt{5}\delta)/30 \\ x_2 &= (6 - 15\sqrt{2}\alpha + 5\sqrt{6}\beta + 5\sqrt{3}\gamma + 3\sqrt{5}\delta)/30, \\ x_3 &= (6 - 10\sqrt{6}\beta + 5\sqrt{3}\gamma + 3\sqrt{5}\delta)/30, \\ x_4 &= 1/10(2 - 5\sqrt{3}\gamma + \sqrt{5}\delta), \\ x_5 &= 1/5 - (2\delta)/\sqrt{5} \end{aligned} \quad (7)$$

The corresponding 4-dimensional domain (similar to Fig. 1 but for obvious reason not plotted) is a simplex with five pointed corners called A_4 . Five corners, as always, correspond to one of the momentum fraction carried by a single parton $x_i = 1$, with all others null. Their explicit locations are

$$\begin{aligned} a_1 &= \{1/\sqrt{2}, 1/\sqrt{6}, 1/(2\sqrt{3}), 1/(2\sqrt{5})\} \\ a_2 &= \{-(1/\sqrt{2}), 1/\sqrt{6}, 1/(2\sqrt{3}), 1/(2\sqrt{5})\} \\ a_3 &= \{0, -\sqrt{2}/3, 1/(2\sqrt{3}), 1/(2\sqrt{5})\} \\ a_4 &= \{0, 0, -(\sqrt{3}/2), 1/(2\sqrt{5})\} \\ a_5 &= \{0, 0, 0, -(2/\sqrt{5})\}. \end{aligned} \quad (8)$$

All have norms $4/5$ and all 10 mutual products are equal to $-1/5$ that angles between them $\theta_{i>j}$ are all the same, $\cos(\theta_{i>j}) = -1/4$.

There are 10 segments connected two corners $a_i, a_j, i \neq j$, called “edges.” For example, the previous formulas show that a_4 to a_5 belong to $\alpha = \beta = 0$ plane. Across them there are “faces” with three corners $a_i, a_j, a_k, i \neq j \neq k$, (e.g. 1, 2, 3). Those are 2-dimensional equilateral triangles. “Across” means that centers of edges $(a_i + a_j)/2$ and centers of complimentary faces $(a_i + a_j + a_k)/3$ are connected by lines going though the global center $\alpha = \beta = \gamma = \delta = 0$.

Our task now is to do quantum mechanics on this simplex. We recall that by a Hamiltonian on the LF, we mean $M^2 = 2P^+P^-$ where LF variables are $P^\pm = P^0 \pm P^z$. Using the same einbein trick, one can get rid of the square roots in the confining terms, so that the linear confining part of the Hamiltonian is traded for a quadratic form in coordinates. In momentum representation it is a Laplacian. The transverse coordinates can also be replaced by derivatives $r_{i\perp}^2 = -\partial/\partial k_{i\perp}^2$, and so our LF Hamiltonian is

$$H_{LF} = \sum_{i=1}^5 \frac{k_{i\perp}^2 + m_Q^2}{x_i} + \sigma_T \left(5a + \frac{1}{a} \sum_{i=1}^5 ((i\partial/\partial x_i)^2 - M^2(\partial/\partial k_{i\perp})^2) \right) \quad (9)$$

If minimization in auxiliary parameter a be performed at this stage, the Hamiltonian would return to its version with linear potential. Instead we solve it as given here, minimizing energies of states afterward.

Note that we neither have attractive Coulomb forces nor a (negative) constant, which are normally included in the Cornell-type potentials. Therefore we would not compare the absolute eigenvalues of this Hamiltonian, but rather its differences and the wave functions.

The first term (former kinetic energy in LF) is rewritten by adding and subtracting mean terms twice

$$\sum_{i=1}^5 \frac{k_{i\perp}^2 - \langle k_{i\perp}^2 \rangle}{x_i} + \langle k_{i\perp}^2 \rangle + m_{Qi}^2 \left(\sum_{i=1}^5 \frac{1}{x_i} - 25 \right) + 25(\langle k_{i\perp}^2 \rangle + m_Q^2) \quad (10)$$

The first term is referred to as the “nonfactorization potential,” the second is “the cup potential” to be referred to as V_{cup} below, and the last is just a constant. The cup potential is a function of only momentum fractions x_i on the simplex. It is near zero in the middle of the simplex, $\alpha \approx \beta \approx \gamma \approx \delta \approx 0$ and increases near all boundaries for which $x_i(\alpha, \beta, \gamma, \delta) \rightarrow 0$. To avoid divergences at the simplex boundaries, the wave functions need to vanish there. More specifically, we will discuss only the cases with finite

$\psi V_{\text{cup}} \psi$, which requires vanishing $\psi(\text{near boundary}) \sim (A - A_{\text{boundary}})^p$ with power $p > 1/2$.

If the “nonfactorization potential” is small and neglected, then this Hamiltonian is a sum of transverse and longitudinal parts, and the WFs factorize into parts depending on $k_\perp$ and the longitudinal momentum fractions x_i . The transverse part is approximated by an oscillator

$$H_\perp = 5 \sum k_{i\perp}^2 - \frac{\sigma_T M^2}{a} \sum (\partial/\partial k_{i\perp})^2$$

so, e.g., its ground state wave function in 1D is

$$\psi_0 \sim \exp(-k_\perp^2 \beta_{\text{osc}}^2/2), \quad \beta_{\text{osc}} = \left(\frac{5a}{25\sigma_T M_q^2} \right)^{1/4} \quad (11)$$

with energy $E_0 = 5^{3/2} M_q \sqrt{\sigma_T/a} \approx 0.67 \text{ GeV}^2$. We use constituent quark mass $M_q = 0.35 \text{ GeV}$, the tension of confining string $\sigma_T = 0.42^2 \text{ GeV}^2$, as used in previous works. The value of $a \approx 6$ as follows after minimization in a , see [12]. Since we will need P-shell states as well, note that once-excited transverse oscillator WF is $\psi_1 \sim k_\perp \exp(-k_\perp^2 \beta^2/2)$ with energy $E_1 = 3E_0$.

In a separate paper [13] we discussed a well-known and somewhat related problem, that of N fermions on a circle. Since fermions cannot pass each other in 1D, their coordinates s_i are ordered $s_1 < s_2 < \dots < s_N < s_1 + 1$. Furthermore, one may fix the CM by a condition $\sum_i s_i = 0$, so its physical domain becomes a 4-dimensional simplex. However, it is important to note that it is similar (but *different*) from the symmetric simplex [14] A_4 for the pentaquark problem we are interested in. The fermions-on-circle problem is useful because it has known analytic solutions. Its WFs have the form of Slater determinants [15] containing four momenta $n_{21}, n_{31}, n_{41}, n_{51}$ which should all be nonzero and different, so that Slater determinant does not vanish. Naturally, for the ground state one should select the “occupied Fermi sphere” set, taking their values to be the lowest possible, namely $-2, -1, 1, 2$ (recall that momentum zero is already included). The Laplacian eigenvalue is then proportional to the sum of their squares $0^2 + 2 \cdot 1^2 + 2 \cdot 2^2 = 10$. Opening the determinant one finds sum of many waves, a very lengthy expression. Yet it can be simplified to the product of 10 sinusoidal terms for each $i \neq j$ pairs

$$\psi_{\text{sin}} \sim \prod_{i \neq j} \sin\left(\frac{s_i - s_j}{2}\right) \quad (12)$$

While the 4-d simplex of fermions-on-the-circle problem is *not* the same as we need for pentaquarks, it can be mapped to it. Unfortunately, this map changes the Laplacian into a different elliptic operator. Therefore, nice classic

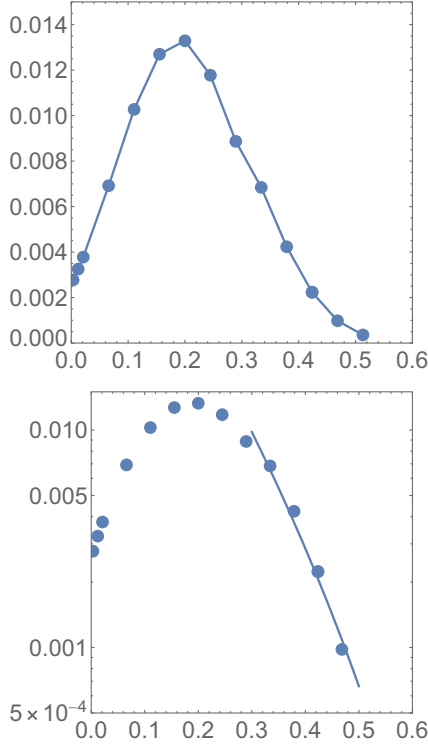


FIG. 3. The antiquark PDF in $L = 0$ pentaquarks defined in (15), (arbitrary units), versus the momentum fraction $x_{\bar{q}}$. The upper and lower plots show the same distributions, in linear and logarithmic plots. The lower plot is used to magnify the larger x region, with a fit to a form $\sim(1 - x_{\bar{q}})^8$.

fermions-on-circle problem does not solve the one we need. Yet its set of states can be used as an orthonormal basis, see our companion paper [13].

Since we will discuss the lowest S-shell and the next (negative parity) P-shell states only, we will use the variational approximation. For the ground state $\phi_0(\alpha, \beta, \gamma, \delta)$ we studied two different trial functions, one motivated by the fermion-on-circle problem (12) and another one with minimal enforcement of the boundary conditions. The first is the product of sinusoidal functions along ten axes, connecting the centers of the edges to the centers of the faces. Their arguments are such that they vanish at the above-mentioned ends.

$$\Psi_{\sin} = \prod_{i \neq j} \sin\left(\frac{2\pi}{5} + 2\pi(\text{coor} \cdot \mathbf{n}_{ij})\right), \quad (13)$$

where the coordinate vector $\text{coor} = (\alpha, \beta, \gamma, \delta)$ is multiplied by 10 unit vectors $\mathbf{n}_{ij}$ pointing to the centers of all 10 edges. Needless to say, it is symmetric for all quark permutations.

Another (and even simpler) trial function is the *product* of linear functions $x_i, i = 1..5$ [from (7)]

$$\Psi_{\text{faces}} = \prod_{i=1..5} x_i(\alpha, \beta, \gamma, \delta) \quad (14)$$

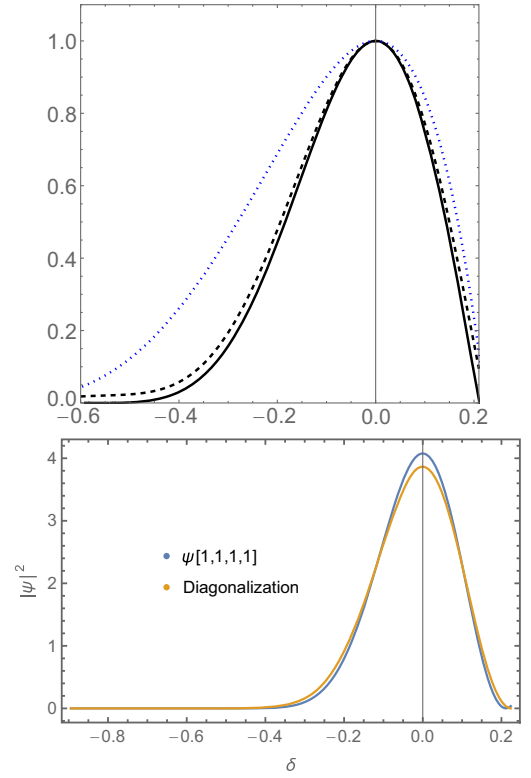


FIG. 4. Upper: the lowest Laplacian eigenfunction $\psi[1, 1, 1, 1]$ at $\alpha = \beta = \gamma = 0$ versus δ (solid line) compared to our simplest variational function (14) (dotted line), and its improvement by the Ritz variational procedure (dashed line). For comparison all eigenfunctions are normalized to their value at the origin. Lower: the same $\psi[1, 1, 1, 1]$ (blue) compared to the numerically exact wave function (orange) obtained in [13] and normalized.

It also enforces the Dirichlet boundary conditions. We found that not only it approximates the lowest eigenstate of the Laplacian, but with the inclusion of the “cup potential” it even becomes closer to it. This second trial function, appropriately modified, also gives an approximation to the negative parity P-wave WF. The details are in Appendix B.

Another technical point, discussed in Appendix D, is a study of the problem using *spheres*, to which the N -simplices tend at large N . While the average values of the Laplacians and Laplacian + V_{cup} may differ substantially, the corresponding wave functions are actually much closer. Our variational ground state wave function is a reasonable approximation to the (forward part) of the pentaquark wave function. This will be confirmed by a direct comparison to the exact diagonalization carried in [13].

The pentaquark wave function can provide a full description of the quantum states, with their associated observables. For example, the squared longitudinal part of the WF (depending on momentum fractions) is the PDF, the probability of one particle to have definite momentum fraction, with all others integrated out. Also, the antiquark PDF $P(x_{\bar{q}})$ is obtained by integration of the wave function squared over α, β, γ variables, as a function of the last

Jacobi coordinate δ .

$$\text{PDF}_{\vec{q}}(\delta) = \int_{\alpha,\beta,\gamma} |\Psi|^2 \quad (15)$$

The results are shown in Fig. 3. As expected for pentaquarks, the maximum is close to $x_{\text{mean}} = 1/N = 1/5$. A nontrivial prediction is that it gets very small for $x > 0.5$, and its hard edge behavior is $\sim(1-x)^8$.

IV. ORBITAL-COLOR-SPIN-FLAVOR WAVE FUNCTIONS ON LF FROM THE PERMUTATION GROUP S_4

In the CM frame the ground state is approximately spherical, justifying the “hyperdistance” approximation. In the LF spherical symmetry is absent, and the WF is approximated by a product of two parts, depending on the transverse and longitudinal momenta, respectively.

So far we discussed the latter for the ground state, as a function of the hyperdistance, which is symmetric under quark permutations. There is also a nontrivial part in the WFs that carry the color-spin-flavor indices. The only difference is that in the traditional (CM frame) spectroscopy there is *spherical* symmetry $O(3)$, while on the LF there is only *axial* symmetry $O(2)$. Therefore, the usual classification based on quantum numbers J, L, S is now unavailable. Instead, we should use only their longitudinal components J_z, L_z, S_z .

Considering states with nonzero orbital momentum $L \neq 0$, as we will do below, one finds further differences between the spectroscopy in the CM and the LF. In the former case the wave function with fixed total angular momentum and projection J, J_z (e.g., $J = 1/2, J_z = 1/2$) are built by a standard sum weighted by Clebsch-Gordan coefficients

$$\sum_{L_z} \binom{1/2, 1/2}{L, L_z, S, 1/2-L_z} Y_{L, L_z}(\theta, \phi) |S, S_z\rangle$$

corresponding to $\vec{L} + \vec{S} = \vec{J}$. In contrast, on the LF the L, S quantum numbers are absent. In particular, for the negative parity P-shell there are three values of $L_z = -1, 0, 1$, to be combined with six possible values of $S_z = -5/2, -3/2, \dots, 5/2$. The only global quantum number is their sum $J_z = L_z + S_z$. Mixing of all states with the same J_z happens, e.g., defined by the spin-dependent terms in the Hamiltonian. Components with $L_z = 1$ and $L_z = -1$ add one quantum of transverse oscillator, and the last $L_z = 0$ contains an extra linear coordinate $z = -i \frac{\partial}{\partial p_z}$.

A. $L=0$ pentaquarks on LF

In our previous paper [12] we derived fully antisymmetric WFs for pentaquarks, belonging to S- and P-shells. The procedure was based on representations of the permutation group S_4 . Now we need to apply similar procedures on the LF.

TABLE I. Table of the spin projection S_z versus isospin I of antisymmetric pentaquark states with $L = 0$. The numbers in brackets refer to the dimensions of the “good basis,” in which permutation matrices are calculated and diagonalized.

I/S_z	1/2	3/2	5/2
-5/2	1 (15)	1 (12)	0
-3/2	4 (75)	4 (60)	1 (15)
-1/2	7 (150)	7 (120)	2 (30)
1/2	7 (150)	7 (120)	2 (30)
3/2	4 (75)	4 (60)	1 (15)
5/2	1 (15)	1 (12)	0

In the S-shell ($L = 0$) the only changes are in the spin classification. For 5 quarks there are six choices, $S_z = -5/2, 3/2, \dots, 5/2$. In the extreme $S_z = 5/2$ case, the spin WF is a single symmetric monom $\uparrow\uparrow\uparrow\uparrow\uparrow$. Therefore Fermi statistics must be satisfied by (still nontrivial) *color* $\otimes$ *flavor* wave functions. Those were already found in [12] for $I = 1/2$ and $I = 3/2$ (there is no state with $I = 5/2$.)

With the new “axially symmetric” I, S_z classification the number of states and permutation generators are given by different matrices, yet the procedure is generally the same. The number of appropriate pentaquark antisymmetric states as a function of isospin and S_z is given in Table I.

It is instructive to explain how this table is related to the one we had derived in the CM frame in [12]. Let us count the total number of states with $I = 1/2$. The table in [12] would give a total number of states

$$2 * 3 + 4 * 3 + 6 * 1 = 24$$

where the first numbers are the number of components, $2S+1$ for fixed spin S . Now there is no spherical symmetry and the total spin is not used. In the current table the number of states with $I = 1/2$ (the first column) is simply

$$2(7 + 4 + 1) = 24$$

where the 2 in front stands for doubling due to negative S_z . These are the same states but counted differently. Different $2S+1$ orientations of spin- S states related by rotations are no longer related, as the only rotations allowed on the LF are the axial ones, preserving S_z .

Let us introduce the (relativistic correction) color-spin Hamiltonian which is based on products of color and spin matrices for each particle pair

$$H_{\lambda\lambda\sigma\sigma} = - \sum_{i>j} V_{\lambda\lambda\sigma\sigma}^{ij} (\vec{\lambda}_i \vec{\lambda}_j) (\vec{\sigma}_i \vec{\sigma}_j) \quad (16)$$

and calculate the Hamiltonian matrix including all states. Here we are substituting the average value of the potential, taken with the main symmetric WF, ignoring the back

reaction on the spins. All matrix elements are nonzero $O(1)$ numbers and look complicated, but after its diagonalization we found that its eigenvalues reproduce exactly the same values as we got using the “spherical” classification in [12]. In particular, the 7 states with $S_z = 1/2, I = 1/2$ have energies

$$E_n = 4.6667, 3.0000, 1.4415, -2.7749, \\ -0.33333, -3.3333, -3.3333 \quad (17)$$

all of which have been reported there. Four states with $S_z = 3/2, I = 1/2$ have energies

$$E_n = 3.000000, -0.333333, -3.333333, -3.333333 \quad (18)$$

Repetitions of the same values as for $S_z = 1/2$ is required by (now “hidden”) $O(3)$ symmetry, not used but still present in the Hamiltonian.

B. $L = 1$ pentaquarks on the LF

Moving up, to what used to be called P -shell states in the CM frame, we recall that negative parity requires some linear combination of coordinates, either transverse or longitudinal, times some functions of their squares. Therefore looking for the antisymmetric representation of the permutation group S_4 , one needs to work in higher dimensions of monoms, $N_{\text{monoms}} = 4 * 3^6 * 2^5 * 2^5$. For the WFs as in all other cases, are obtained by explicit use of the SparseArray command in *Mathematica*. Unfortunately, those are still way too large to be presented in print.

The multiplicities of the P -shell pentaquark states were detailed in our previous paper [12], see Table IV. On the LF, linear functions of coordinates get changed into linear functions of momenta. However, the permutation properties defined via Jacobi remains unchanged. Thus the orbital-color-spin-flavor structure of the states remains the same. The analog to the Table IV of [12] is the new Table II.

There are now two types of $L = 1$ states: (i) with $L_z = \pm 1$ corresponding to rotations in the transverse plane, and

TABLE II. Table of antisymmetric pentastates with spin projection S_z versus isospin I , with $L = 1$. The numbers in brackets are the dimensions of the “good basis,” in which the permutation generators are calculated and diagonalized.

I/S_z	1/2	3/2	5/2
$-5/2$	3 (60)	3 (48)	1 (12)
$-3/2$	14 (300)	13 (240)	4 (60)
$-1/2$	27 (600)	24 (480)	7 (120)
$1/2$	27 (600)	24 (480)	7 (120)
$3/2$	14 (300)	13 (240)	4 (60)
$5/2$	3 (60)	3 (48)	1 (12)

(ii) $L_z = 0$ case, in which the linear terms are made of the longitudinal momenta. Because the transverse part of the Hamiltonian is the 2D oscillator, type-1 states contain combinations of transverse momenta $\alpha_1 \pm i\alpha_2$ times the ground state Gaussian WFs in transverse momenta. The same applies to other Jacobi combinations of β, γ, δ , with four possibilities altogether. Clearly, there are certain 4-by-4 matrices of permutation generators. Those are the same as for longitudinal momenta (fractions) which we call simply $\alpha, \beta, \gamma, \delta$, omitting the longitudinal index. Evaluating the Kronecker products of permutation matrices for coordinates, color, spin and flavor, we obtain the global permutations. The antisymmetric eigenvectors are found by diagonalization.

Let us mention a set of energies for $I = 1/2, S_z = 1/2, L = 1$ states of the color-spin Hamiltonian

$$E_n = 9.05, 9.05, -7.33, 4.67, 4.67, -3.33, \\ -3.33, -3.33, -3.33, -3.33, -3.33, -3.33, \\ 3.00, 3.00, 3.00, -2.77, -2.77, -2.77, \\ 1.67, 1.67, 1.62, 1.62, 1.44, 1.44, \\ 1.44, -0.333, -0.333 \quad (19)$$

with repetitions for larger values of S_z , due to the various components at the same J .

A mixing with the proton, $I = 1/2$ and spin up $S_z = 1/2$ is only possible for pentaquark states with the same quantum numbers. In the CM frame those include the total angular momentum and its projection, and mixing with the nucleon is only possible if $J = 1/2$. The selection of such states is done by standard Clebsch-Gordan coefficients, combining, e.g., $S_z = 1/2, L_z = 0$ with $S_z = -1/2, L_z = 1$ states into unique combination.

However on the LF J is not defined and the states with the same $J_z = 1/2$, are treated as independent. The ones with $L_z = 0$ are the type-2 states, and there are 27 of them with $S_z = 1/2$ as listed in the table. They all have antisymmetric WFs under S_4 . The linear combinations of longitudinal momenta get combined with appropriate color, spin and isospin structures, as defined in our previous papers. The type 1 states with $L_z = +1, S_z = -1/2$ are also 27, with the transverse momenta occurring linearly in place of the longitudinal ones.

The type-1 and type-2 states formally do not have the same energy, as the LF Hamiltonian lacks spherical symmetry. There is no statement that the full angular momentum operator $\vec{J} = \vec{S} + \vec{L}$ has some fixed values at the resulting WFs. In some cases it is possible to show that different orientations do lead to the same mass (e.g., we did it for ρ meson in [16]). However, in the pentaquark case due to rather heavy calculations involved we do not study type-1 states in any detail.

The ground state transverse WF was given in (11), and that of the 1st excited state is just this one times the coordinate. The additional energy of these states, compared to the $L = 0$, comes from one quantum excitation of some transverse oscillators, with a value

$$M^2(L = 1, \text{type } 1) - M^2(L = 0) = 2\sqrt{\frac{5\sigma_T M^2}{a}} \quad (20)$$

where σ_T is the string tension, and a is the variational parameter of the einbein trick. (Its value at minimum is $a \approx 6$, see [13].

C. The shapes of the forward wave functions, for S- and P-shells

The longitudinal part of the LF Hamiltonian is not an oscillator, but a Laplacian (confining) and the irreducible potential V . For 5 quarks those are defined on the on the pentachoron (“starfish”) manifold of $\alpha, \beta, \gamma, \delta$ momentum fractions.

In Appendix A we recall how a similar problem for the 3 bodies (baryons) and 4 bodies (tetraquarks) can be solved using fermions on the circle. The actual shapes of the wave functions for 5 bodies is determined by a variational evaluation in Appendix B and by using a more elaborate (but complicated) WFs in Appendix C. Relegating all the details to those Appendices, we would like to present here some results in graphical form.

In Fig. 4 we show the shape of the ground state Laplacian eigenfunction on A_4 , and in Fig. 5 that of the first excited states. While the variational Ritz method proves successful in reproducing of the overall shapes of these wave functions (they were actually done prior to our discovery of semiexact eigensystem in the mathematical

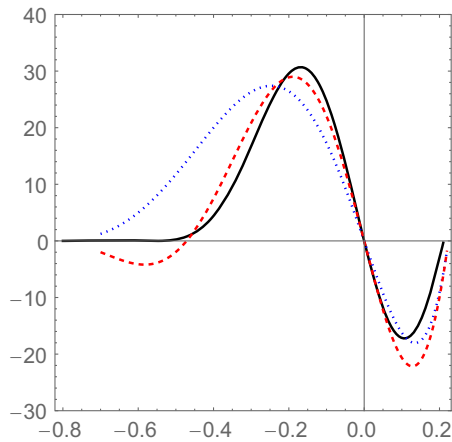


FIG. 5. Comparison between the shapes of the P-shell ($L = 1$) wave functions (arbitrary units) on the 4-simplex A_4 , at $\alpha = \beta = \gamma = 0$ as a function of δ . The black solid line is the exact Laplacian eigenstate with $\psi[2, 1, 1, 1]$, the blue dotted is the initial trial function (B2) for $C = 0$, and the red dashed line is the variational function with a single parameter and a value $C = 4.5$.

literature), one also needs to note that they do not accurately describe their tails.

V. BARYON-PENTAQUARK MIXING ON THE LIGHT FRONT

A. Strategy and definitions

In general, the problem of “dressing” the qqq sector of the baryon WFs with extra $\bar{q}q$ pairs, can be addressed in apparently different ways. For example, in [7] we used a “radiative” (DGLAP-like) description, with $\bar{q}q$ pair produced from one of the quarks. The mixing was ascribed by [17] to the ’t Hooft 4-fermion Lagrangian, producing $u \rightarrow u\bar{d}d$ reaction but *not* $u \rightarrow uu\bar{u}$. Another mechanism considered in literature (e.g., [18]) is production via an intermediate pion. The probability, as a function of the momentum fraction of the initial quark, is convoluted with the PDF of the mother nucleon, e.g., $u_v^{\text{proton}}(x)$. In this approach the interference of the newly created quark with those in the nucleon is ignored, in contradiction with Fermi statistics. This may perhaps be justified only if the “pion-sigma clouds” extend well beyond the original qqq wave function. Another issue with this approach is that the calculations are carried at the probability (density matrix, PDF) level. Our approach is based on the evaluation of the full 5-body wave function as described in our recent study [12], where one can find extensive introduction and references.

Here we briefly note that the puzzling features of the nucleon PDFs observed experimentally (such as flavor asymmetry of the antiquark sea, or the apparent presence of significant orbital motion) can find natural explanations if the extra $\bar{q}q$ pairs are described as admixture generated by σ and π mesons. The technical description is detailed in [12], as it follows earlier studies where the baryon wave function times pertinent mesonic operators T_σ, T_π is folded onto pentaquark states. The so called meson-baryon-penta overlap is the integral over all momentum fractions $\vec{x}_i, i = 1..5$. The coordinates allocation is given by

$$C_n = \int_x \Psi_n^{\text{penta}}(\vec{x}_i) \hat{T}(\vec{x}_4 - \vec{x}_5) B(\vec{x}_1, \vec{x}_2, \vec{x}_3) \quad (21)$$

with the antiquark coordinate identified as $\vec{x}_5$, and with all the partonic coordinates reexpressed in terms of Jacobi coordinates $\vec{\alpha} \dots \vec{\delta}$. Each factor, in turn, consists of the “main” (radial) wave function expressed in hyperdistances, Y_5 for the pentaquark and Y_3 for the baryon, times pertinent color-spin-flavor structures (vectors in respective “monom” spaces). Here $n = 1..27$ counts the suitable pentaquark states of the P-shell.

The new task is to translate these expressions to the LF formulation. The baryon and pentaquark wave functions are modified as we now detail. The nucleon color-spin-flavor structure remain as described in previous papers.

The radial wave function $B(Y)$ which depends on the hyperdistance $Y_B^2 = \vec{\alpha}^2 + \vec{\beta}^2$, splits into the *products* of two functions depending on the transverse and longitudinal momenta

$$B(\vec{x}_i) \rightarrow B_\perp(\vec{p}_i^\perp) B_{\Delta 3}(\alpha, \beta)$$

The latter function $B_{\Delta 3}(\alpha, \beta)$ is the ground state baryon wave function in the 2-simplex or the equilateral triangle, a function of two momentum fractions. We recall that for different baryons those are different, e.g., in [19] we extensively discussed the difference between Δ and nucleon states, due to a “good diquark” effect in the latter.

The procedure for pentaquark is similar. The wave function splits into transverse and longitudinal parts

$$\Psi(\vec{x}_i) \rightarrow \Psi_\perp(\vec{p}_i^\perp) \Psi_{\Delta 5}(\alpha, \beta, \gamma, \delta)$$

This splitting reflects the independence of the transverse and longitudinal variables in the LF Hamiltonian. The longitudinal wave functions defined in the 4-simplex $\Delta 5$, was discussed above.

As noted, the nucleon-meson admixture must involves P-shell pentaquarks, by parity. It is not a small issue, since there is also a part of the WF in the CM frame which contains *linearly* dependent coordinates. Since those have a nontrivial transformation matrix under quark permutations, it led to extensive calculations for orbital-color-spin-flavor WFs satisfying Fermi statistics [12]. The longitudinal momenta $p_z \gg p_\perp$ follow by boosting to the LF. The component with $L_z = \pm 1$, built from transverse momenta, is subleading, and only $L_z = 0$ survives, with a simplification to $J_z = S_z$. The longitudinal part is generalized to the orbital-color-spin-flavor WFs found earlier, with the obvious change of the coordinates to longitudinal momentum fractions (also called $\alpha, \beta, \gamma, \delta$).

The mesonic operators $\hat{T}$ includes the $\bar{q}q$ vertex which traditionally is written in coordinates as Gaussian

$$\exp[(\vec{x}_4 - \vec{x}_5)^2/2\rho^2].$$

Its momentum representation is also a Gaussian. We assume that the boost to the LF compresses hadrons longitudinally, leaving only the factor with transverse momenta.

The qualitative operators providing proper quantum numbers in $\hat{T}$ are $(\vec{S}\vec{L})$ in the T_σ and $(\vec{S}\vec{p})$ in the T_π , respectively. Spin is the total spin of q and $\bar{q}$, $\vec{S} = \vec{S}_4 + \vec{S}_5$. The boost leaves rotations only in the transverse plane, so we use

$$(\vec{S}\vec{L}) \rightarrow (S_z L_z)$$

In contrast, for the pion-related operator, the longitudinal momenta should be dominant. Obviously, the longitudinal

TABLE III. Some properties of the three 5-quark admixtures to the nucleon (24) defined as average values of the operators listed in the first column.

Operator	$p\sigma$	$N\pi$	$\Delta\pi$
$S_z(5)$	-0.177	-0.112	-0.0338
$\bar{d} = I_+(5)$	0.81	0.858	0.333
$\bar{u} = I_-(5)$	0.19	0.142	0.666

momenta of the 4th and 5th (antiquark, $5 \rightarrow \bar{q}$) constituents are to be reexpressed in terms of Jacobi variables γ, δ .

Quantitatively, projecting these structures onto the intrinsic δ coordinate amounts to identifying the transferred momentum with the momentum conjugate to that Jacobi transverse variable, $\mathbf{K} \rightarrow \mathbf{k}_\delta$. The effective transition operators are therefore taken in the form

$$T_\sigma = \frac{1}{2i} [(S_+^{(4)} + S_+^{(\bar{q})})k_{\delta,-} - (S_-^{(4)} + S_-^{(\bar{q})})k_{\delta,+}], \quad (22)$$

$$T_\pi = \left[(S_z^{(4)} + S_z^{(\bar{q})})k_{\delta,z} + \frac{1}{2} ((S_+^{(4)} + S_+^{(\bar{q})})k_{\delta,-} + (S_-^{(4)} + S_-^{(\bar{q})})k_{\delta,+}) \right] \tau^a, \quad (23)$$

where $k_{\delta,\pm} = k_{\delta,x} \pm ik_{\delta,y}$ are the transverse circular components in the δ mode. Equation (22) is the intrinsic realization of the light-front scalar spin-orbit structure $\mathbf{S} \cdot (\hat{z} \times \mathbf{k})$ and therefore excites purely transverse orbital components ($m_L = \pm 1$). In contrast, Eq. (23) is the intrinsic form of $\mathbf{S} \cdot \mathbf{k} \tau^a$ and contains both transverse ($m_L = \pm 1$) and longitudinal ($m_L = 0$) components.

In an early simplified model (with no projections to pentaquarks) with just the $N\pi, \Delta\pi$ states, Garvey [20] pointed out that two observables, the antiquark isospin asymmetry and the orbital momentum, are the same. In our analysis they are not. In Garvey’s model, Fermi statistics is *not* enforced, while in our approach it is paramount and strictly enforced, as all pentaquark states satisfy permutation symmetry.

Finally, the isospins in channels $N + \pi, \Delta + \pi$, should be analyzed carefully. While the total spin S , orbital momentum L and total J do not exist in the LF formulation, the isospin quantum numbers remain unchanged. Adding a sigma meson creates no issues, but adding a pion, with $I_\pi = 1$ to the nucleon N or delta Δ resonances still needs to be done in such a way that the total isospin quantum numbers matches that for the proton, namely $I = 1/2, I_z = 1/2$. This is done using standard Clebsch-Gordan coefficients

$$|N\pi\rangle = \frac{1}{\sqrt{3}} |p\pi^0\rangle - \sqrt{\frac{2}{3}} |n\pi^+\rangle$$

$$|\Delta\pi\rangle = \frac{1}{\sqrt{2}} |\Delta^{++}\pi^-\rangle - \sqrt{\frac{1}{3}} |\Delta^+\pi^0\rangle + \frac{1}{\sqrt{6}} |\Delta^-\pi^+\rangle \quad (24)$$

In general, there are three possible admixtures to the nucleon, with overlap coefficients called $C_{p\sigma}$, $C_{N\pi}$, $C_{\Delta\pi}$. While at the end their contributions will be added, it is instructive to study the three admixtures separately, see Table III.

B. The 5-quark sector of the proton

Using 27 pentaquark $L = 1$ states defined in Sec. III, we have evaluated the projections of the $p + \sigma$, $N + \pi$, $\Delta + \pi$ states onto them. The overlap coefficients C_n (21) are calculated as explained in the previous section, using proper combination of coordinates in $|P_n\rangle$, with the ansatz function (14). In this calculation we included only the functions normalized in momentum fractions $\alpha, \beta, \gamma, \delta \in \text{starfish}$. Their histogram is given in Fig. 6. It is rather flat, with a maximal modulus around 0.2. There is no dominance of particular states.

Ignoring the spin-dependent splitting in the denominator of the first order perturbation theory, we define various 5-quark admixtures to the nucleon as

$$\Delta\psi_A = \sum_{n=1}^{27} C_A^n \Psi_n \quad (25)$$

where the index refers to the three channels $A = p\sigma, N\pi, \Delta\pi$, separately normalized to 1.

Justification of usage of 1st order mixing can be argued as follows. Generally, it is possible that out of $3 \times 27 = 81$ coefficients some happened to be dominantly large and would require nonlinear diagonalization of the Hamiltonian. Yet, this is not the case, as is demonstrated, e.g., in Fig. 6. The largest values of C_A^n are shared between eight largest contributions. In fact none of admixtures exceed the level of two percent or so.

Structure of the admixed LFWFs can be studied both by averaging of certain operators, or by integrating over partonic variables. Each has 7200 nonzero components, in the standard 746496 monom basis. While rather sparse, it is still not small enough to be given here.

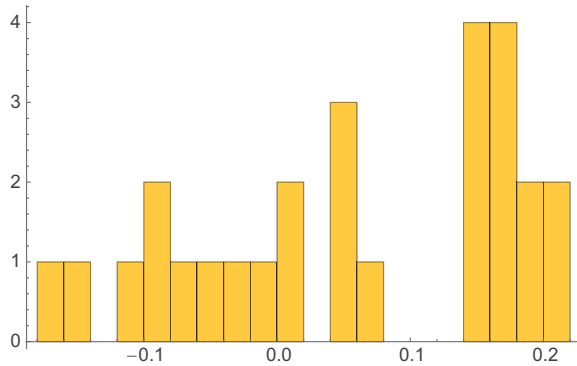


FIG. 6. The distribution of the projection coefficients C_n of the 27 pentaquark states to $N + \sigma$.

The individual 27 pentaquark states are not spherical, and we project to baryon + meson WF which designates by 1-2-3 particles in the baryon, and by 4-5 in the mesons. Yet one of the admixtures turned out to be “hyperspherical,” with a density (squared WF)

$$\Delta\psi_{p\sigma} \cdot \Delta\psi_{p\sigma} \sim (\alpha^2 + \beta^2 + \gamma^2 + \delta^2) \quad (26)$$

However, the appearance of this factor changes the shape of the wave function, so the analog to Fig. 7 now looks as shown in Fig. 8. Furthermore, the $N\pi, \Delta\pi$ projections generate densities $\Delta\psi \cdot \Delta\psi$ which are not hyperspherical, with more complicated shapes

$$\Delta\psi_{N\pi} \cdot \Delta\psi_{N\pi} \sim \alpha^2\gamma^2 + \beta^2\gamma^2 + \gamma^4 + 2.68\delta^4 \quad (27)$$

$$\Delta\psi_{\Delta\pi} \cdot \Delta\psi_{\Delta\pi} \sim \alpha^2\gamma^2 + \beta^2\gamma^2 + \gamma^4 + 3.70\delta^4 \quad (28)$$

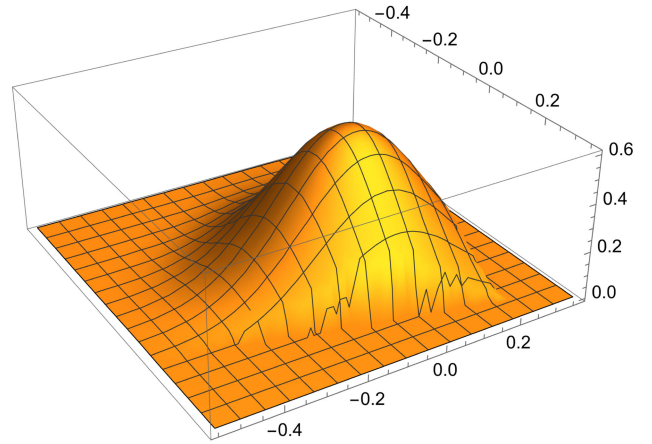


FIG. 7. The shape of the $\Psi 0^2$, at $\alpha = \beta = 0$ at the γ, δ plane. Note that this plane contains two corners a_4, a_5 , but not the opposite center of the face 123: thus it does not sit on an equilateral triangle.

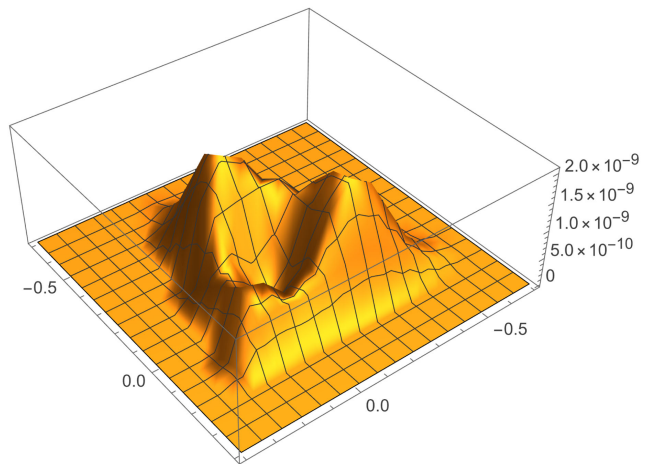


FIG. 8. The shape of $\Delta\psi_{N\pi}^2$ on the $\alpha = \beta = 0$ and γ, δ plane.

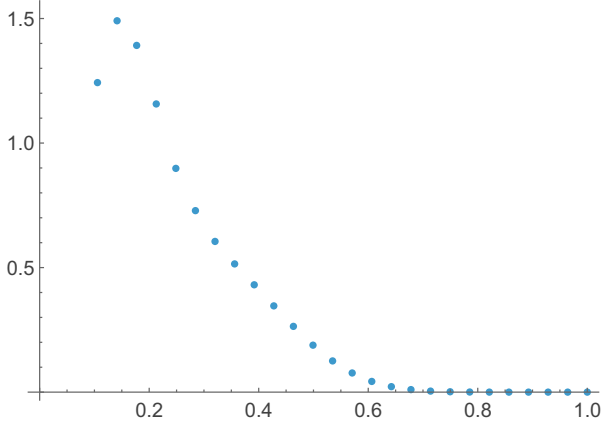


FIG. 9. Predicted PDF of the antiquarks, from $\Delta\psi_{p\sigma}$ (arbitrary units) versus Bjorken momentum fraction x . This plot is linear, emphasizing the small x region.

The antiquark PDF (integral of $\Delta\psi^2$ over α, β, γ) changes rapidly, see Fig. 9. Compared to the predictions for the ground ($L = 0$) shell pentaquarks in Fig. 3, it gets more narrow, and shifts to smaller x . Yet the large- x part (better seen in the lower plot) show the same $\sim(1-x)^8$ trend.

Some properties of three different 5-quark admixtures to the proton are listed in the Table III. Let us start with the mean spin of the antiquark (particle number 5)

$$\langle S(5) \rangle = \Delta\psi_A \cdot S(5) \cdot \Delta\psi_{p\sigma} \quad (29)$$

where $A = p\sigma, N\pi, \Delta\pi$. All show negative contributions, which should be compared to the total nucleon spin $S_z = +0.5$.

Other important matrix elements refer to the isospin projection of the antiquarks. Let us define the operators projecting to $\bar{d}, \bar{u}$ flavors

$$I_{\pm} = (\hat{1} \pm \tau_z)/2 \quad (30)$$

applied to particle 5 (antiquark). (Note that the isospin components for antiquarks are inverted.) In this case $p\sigma$ and $N\pi$ show preference for $\bar{d}$ and the $\Delta\pi$ the other way.

In summary: the antiquarks in the nucleon are described by baryon-pentaquark mixing, at the WF level. The three contributions to LF wave function $\Delta\psi_A$, were used to calculate various matrix elements related to antiquarks, such as the mean S_z and isospin, and related those with *moments* of the observed PDFs.

VI. ANTIQUARKS IN THE NUCLEON SEA: EMPIRICAL INFORMATION VERSUS THEORY

We now briefly recall some theoretical ideas on the PDFs and similar partonic observables in general. Those are not WFs but *density matrices*, the probabilities to have a given quark/gluon with momentum fraction x , assuming characteristic resolution scale Q , e.g., $x\bar{u}(x, Q)$. Compressing

a multidimensional wave function to a single-body density matrices is associated with certain loss of information, thus appearance of the *entanglement entropy* [21]. Similar phenomena are well known in other fields, e.g., in atomic and nuclear physics.

The benefit of the LF formulation developed in the previous sections, including the three $\Delta\psi_A, A = p\sigma, N\pi, \Delta\pi$ 5-quark admixtures to the basic 3-q baryons, is that one can calculate *all* partonic observables (PDFs, TMDs, GPDs etc.) within common setting.

Let us also briefly recall, that the dependence on the momentum transfer Q (resolution scale) was developed in QCD first via calculation of the *anomalous dimensions* of certain operators using perturbative QCD. That lead to prediction of the Q -dependence of the PDF moments. The moments values and scale dependence were calculated on the lattice, see [22] and references therein.

Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) [23–25] developed a “radiative evolution” of PDFs, in the form of integrodifferential equations. In order to compare experimental data taken at various resolutions, the results are usually “evolved” to some standard scales.

We cannot refer to huge literature on “global PDF fits.” Specific tutorials/references are given, e.g., at the web pages of the CTEQ collaboration. For a particular example of some global fit to PDFs and its application see, e.g., [26]. Here, we briefly illustrate some features using the *ManeParse Mathematica* package [27]. Those are based on fits to whole volume of data, for deep inelastic scattering and Drell-Yan processes, which are mostly sensitive to antiquark PDFs.

As we discussed in our previous papers, the theory we tried to develop aims at including the “instanton-based chiral physics,” which is dominant at the resolution of $Q^2 \sim 1 \text{ GeV}^2$, or of the order of the instanton size $\rho \sim 1.4 \text{ GeV}^{-1}$. To match to partonic observables, the latter should be evolved downward, to this “low scale.” We now proceed to directly interpolate the empirical PDFs to this low scale.

In Fig. 10 we first show examples of the perturbative evolution of the *gluon* PDF, from the highest Q on the r.h.s. to the resolution of $Q = 1 \text{ GeV}$. Note that it is in log-log scale. The chief observation is that multiple soft gluons are present at the experimental resolution, that totally disappear at the scale $Q = 1 \text{ GeV}$. This scale is dominated by the semiclassical glue, not the perturbative gluons.

The right plot shows the empirical antiquark PDFs. The solid lines are taken at the scale $Q^2 \sim 50 \text{ GeV}^2$, more or less the scale in Drell-Yan processes. In order to be compared with our theory (keeping only quarks and antiquarks), they need to be evolved down to the scale $Q^2 \sim 1 \text{ GeV}^2$ where gluons disappear. The evolved results are illustrated by the dashed curves.

A clear feature is the different $\bar{u}$ and $\bar{d}$ curves, or flavor asymmetry to be discussed below. The shape of the

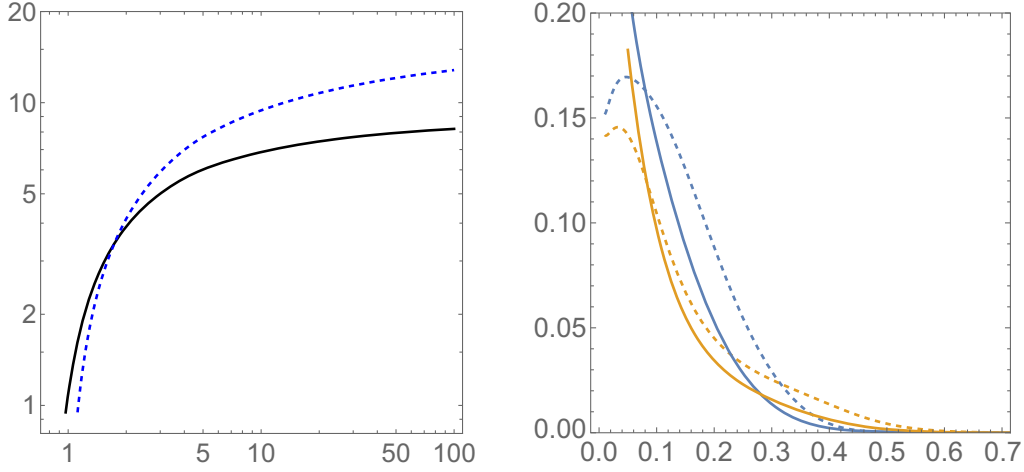


FIG. 10. Left: evolution of the gluon PDF $xg(x, Q)$ versus Q , at $x = 0.01$ (black solid) and $x = 0.005$ (blue dashed). It shows the disappearance of the soft gluons at $Q \rightarrow 1 \text{ GeV}$. Right: the antiquark $x\bar{d}(x, Q)$, $x\bar{u}(x, Q)$ distributions (blue and orange) in x , at the resolution $Q^2 = 50 \text{ GeV}^2$ (solid lines) evolving down to low resolution $Q^2 \sim 1 \text{ GeV}^2$ (dashed lines). Both plots were generated by ManeParse *Mathematica* package for generating PDFs, [27]. It includes DGLAP evolution and interpolation of multiple datasets done by the CTEQ collaboration.

distributions changes, as they move to the right (larger x), as shown in Fig. 10 (right).

Another feature is the very rapid decrease toward larger x . This feature is in qualitative agreement with our pentaquark admixed WFs. Yet the precise powers of this cutoff disagree: the empirical (fit) to the antiquark PDF in Fig. 11 (logarithmic scale), give the large- x tail $\sim (1-x)^8$, while our LFWFs give sharper one $\sim (1-x)^{12}$. The issue clearly needs more work. The experimental data for antiquark PDFs for $x > 1/2$ are not yet available, so PDFs used rely on extrapolations. Our calculations are also not reliable near the edge of the simplex, as the Hamiltonian used includes so far only kinetic and confinement terms, without Coulomb and spin-dependent (diquark) effects.

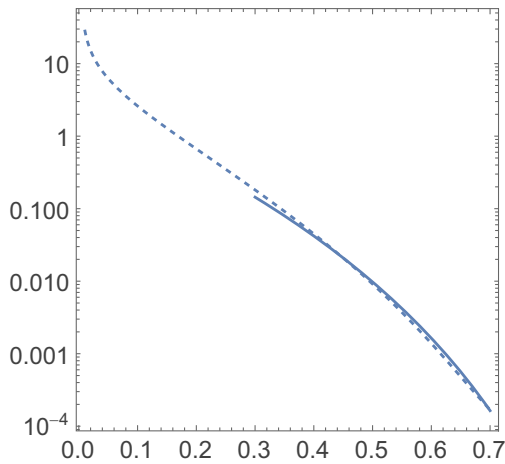


FIG. 11. The PDFs of all the antiquarks $u(x, 1) + d(x, 1)$ at the low scale $Q = 1 \text{ GeV}$ (no x prefactor here), from ManeParse [27], compared to the fit $\sim (1-x)^8$ shown by the solid line.

VII. SUMMARY AND DISCUSSION

This paper follows our previous work [12], in which we worked out the WFs and spectra of light pentaquarks, as well as their role in explaining the 5-quark sector of the nucleon.

$$|N\rangle = C_3|3q\rangle + \sum_i C_5^A|5q, A\rangle + \dots$$

The nontrivial aspects of the calculation was the requirement that the admixed pentaquarks must belong to their P-shell with $L = 1$, thus leading to a complex orbital-color-spin-flavor wave function. It was obtained by fixing the anti-symmetric representation of the permutation group S_4 , as required by Fermi statistics.

Our earlier work was done in the CM frame, and in this work we generalized it to the light front. Pentaquark wave function have longitudinal WFs valued in 4-simplex (starfish), in a 4D space of Jacobi momentum fractions. We obtained this wave functions both variationally and using an exact mathematical procedure, including a sum over many sinusoidal waves.

The 5-quark sector is the first one with antiquarks in the nucleon wave function. The chief goal was to extract their pertinent PDFs, and compare them to available experimental data. The antiquark PDF for the ground S-shell pentaquark were shown in Fig. 11, those for 5-quark admixture to the nucleon in Fig. 9. The experimental antiquark PDFs evolved to the low resolution point $Q^2 = 1 \text{ GeV}^2$ were given in Fig. 10, in qualitative agreement with our results.

ACKNOWLEDGMENTS

This work is supported by the Office of Science, U.S. Department of Energy under Contract No. DE-FG-88ER40388. This research is also supported in part within

the framework of the Quark-Gluon Tomography (QGT) Topical Collaboration, under Contract No. DE-SC0023646.

DATA AVAILABILITY

The data are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: BARYONS AND TETRAQUARKS ON THE LIGHT FRONT

Our investigation of multiquark hadrons on the LF, started with qqq baryons and their WFs [8]. In this Appendix we briefly recall some of the facts and notations for completeness.

At the start of [8] we addressed the issue of perturbative and nonperturbative interquark forces in baryons. While in the literature these forces are mostly discussed using *ansatz* Y (three strings joined by a color junction), both lattice-based studies as well as our own instanton-based results suggest that the so called *ansatz* A seems to be preferred, for which a “half-perimeter” appears. We used the einbein trick

$$\begin{aligned} H_{\text{confinement}} &= (\sigma/2)[r_{12} + r_{13} + r_{23}] \rightarrow \\ &= (\sigma/4) \left[\frac{3}{a} + a(r_{12}^2 + r_{13}^2 + r_{23}^2) \right] \rightarrow \\ &= (\sigma/4) \left[\frac{3}{a} + 3a(r_\rho^2 + r_\lambda^2) \right] \end{aligned} \quad (\text{A1})$$

and factored the CM out by using Jacobi coordinates, e.g., $\vec{r}_\rho = i\partial/\partial\vec{p}_\rho$ in recast in the momentum representation ($\hbar = 1$). For *ansatz* Y the same expression appears, with a different tension $\sigma_A \approx 0.87\sigma_Y$. The longitudinal part of the kinetic energy takes the part of Laplacian on the triangle

$$H_{\text{triangle}} = -\frac{\sigma_Y}{a} \left[\frac{\partial^2}{\partial \rho^2} + \frac{\partial^2}{\partial \lambda^2} \right] \quad (\text{A2})$$

and the transverse part of the Hamiltonian

$$H_\perp = 3(\vec{p}_\rho^2 + \vec{p}_\lambda^2) - 3m_Q^2 \left(\frac{\sigma_Y}{a} \right) \left(\frac{\partial^2}{\partial \vec{p}_\rho^2} + \frac{\partial^2}{\partial \vec{p}_\lambda^2} \right) \quad (\text{A3})$$

is that for harmonic oscillator. The einbein parameter a is found by minimization, as illustrated in Fig. 8 in [8]. Its value depends on the quark mass in question.

The only nonfactorizable part of the problem—which makes it challenging—is the “effective potential” depending on both the transverse and longitudinal variables

$$V_{\text{cup}} \sim \sum_{i=1}^3 (\langle \vec{p}_{i,\perp}^2 \rangle + m_i^2) \left[\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} - 9 \right] \quad (\text{A4})$$

The approximation is in changing the transverse momenta squared to their average value. We then subtract a constant so that at the center of the triangle the parton momenta $x-1 = x_2 = x_3 = 1/3$ are null.

In [8] the “cup potential” V was recast in matrix form $\langle n|V|n' \rangle$, with a finite cutoff in the basis set. However, the sensitivity to this cutoff was not investigated. Alternatively, the Schrodinger equation on a triangle with Dirichlet boundary conditions can be carried numerically, using in *Mathematica* the *NDEigensystem* command, with the results shown in Fig. 12.

For three bodies, $N = 3$, all eigenstates of the Laplacian free of the CM of motion were derived analytically in [8]. We now report further progress in this direction. These states can be written in the form of a Slater determinant

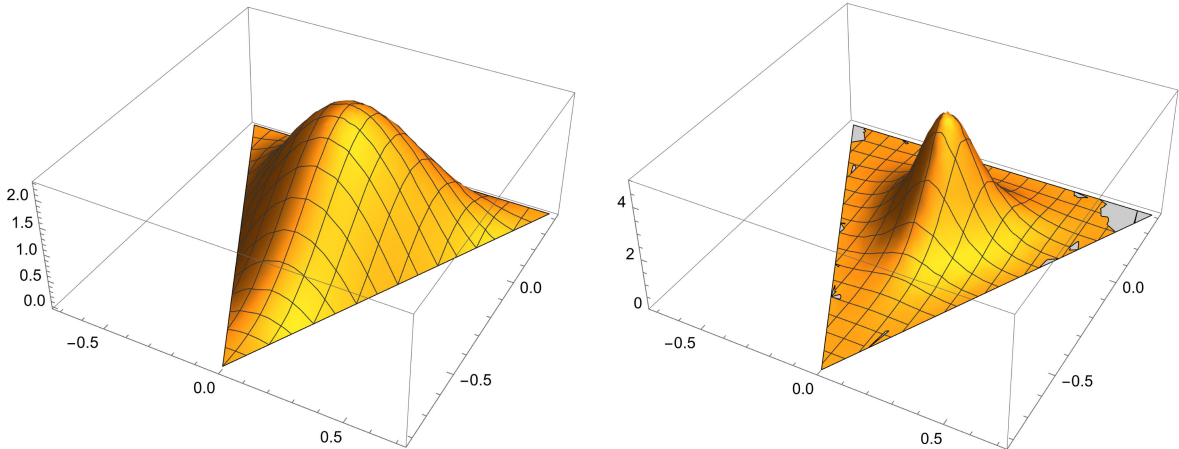


FIG. 12. Forward part of baryon wave function on the LF. The left plot is for only the Laplacian, and the right plot for the Laplacian plus the cup potential.

$$\varphi_n[\tilde{s}] = \frac{1}{(2\pi)^{3/2}\sqrt{3!}} \begin{vmatrix} 1 & 1 & 1 \\ e^{i\tilde{n}_{21}\tilde{s}_1} & e^{i\tilde{n}_{21}\tilde{s}_2} & e^{i\tilde{n}_{21}\tilde{s}_3} \\ e^{i\tilde{n}_{31}\tilde{s}_1} & e^{i\tilde{n}_{31}\tilde{s}_2} & e^{i\tilde{n}_{31}\tilde{s}_3} \end{vmatrix} \quad (\text{A5})$$

with the spectrum

$$\tilde{E}[\tilde{n}] = \frac{16\pi^2}{3} (n_{21}^2 + n_{31}^2 - n_{21}n_{31}) \quad (\text{A6})$$

The spectrum in our work

$$e_{m_L, n_L}^D = \frac{16\pi^2}{9L^2} (m_L^2 + n_L^2 - m_L n_L) \quad (\text{A7})$$

with $L = \sqrt{2}$ coincides with it. The normal coordinates can be identified with the standard Jacobi $\rho = \alpha$, $\lambda = \beta$ coordinates used in [8]

$$\begin{aligned} s_1 - \frac{S}{N} &= \frac{1}{\sqrt{6}} (\lambda + \sqrt{3}\rho) \\ s_2 - \frac{S}{N} &= \frac{1}{\sqrt{6}} (\lambda - \sqrt{3}\rho) \\ s_3 - \frac{S}{N} &= -\sqrt{\frac{2}{3}}\lambda \end{aligned} \quad (\text{A8})$$

If we set $\tilde{n}_{21} = 2\pi n_L$ and $\tilde{n}_{31} = 2\pi(2n_L)$ to compare with the nondegenerate Dirichlet states reported in [8], then the determinantal wave function reduces to

$$\begin{aligned} \varphi_{n_L}[\rho, \lambda] &= \frac{2^3(-1)^{n_L+\frac{1}{2}}}{(2\pi)^{\frac{3}{2}}\sqrt{3!}} \sin\left(\frac{2\pi n_L \rho}{L}\right) \cos\left(\frac{\pi n_L}{L}(\rho + \sqrt{3}\lambda)\right) \\ &\quad \times \cos\left(\frac{\pi n_L}{L}(\rho - \sqrt{3}\lambda)\right) \end{aligned} \quad (\text{A9})$$

For the ground state with $n_L = 1$, it can be recast in terms of 3 standing sine waves after a shift in $\rho = \tilde{\rho} + L/2$, which define the boundary or 3 edges of the 2-simplex (triangle)

$$\begin{aligned} \varphi_1[\rho, \lambda] &\rightarrow \frac{2^3}{(2\pi)^{\frac{3}{2}}\sqrt{3!}} \sin\left(\frac{2\pi\tilde{\rho}}{L}\right) \sin\left(\frac{\pi}{L}(\tilde{\rho} + \sqrt{3}\lambda)\right) \\ &\quad \times \sin\left(\frac{\pi}{L}(\tilde{\rho} - \sqrt{3}\lambda)\right) \end{aligned} \quad (\text{A10})$$

The ground state WF for the $N = 4$ Laplacian is again a product of six standing sine functions,

$$\begin{aligned} \varphi_1[\rho, \lambda, \xi] &= \frac{2^4}{\sqrt{(2\pi)^4 4!}} \sin\left(\frac{2\pi\rho}{L}\right) \sin\left(\frac{2\pi\lambda}{L}\right) \sin\left(\frac{\pi}{L}(\lambda + \rho + \sqrt{2}\xi)\right) \\ &\quad \times \sin\left(\frac{\pi}{L}(\lambda + \rho - \sqrt{2}\xi)\right) \sin\left(\frac{\pi}{L}(\lambda - \rho + \sqrt{2}\xi)\right) \sin\left(\frac{\pi}{L}(\lambda - \rho - \sqrt{2}\xi)\right) \end{aligned} \quad (\text{A11})$$

For $L = 1$, the waves vanish at the edges of the regular tetrahedron delimited by the four planes

$$\begin{aligned} \lambda + \rho + \sqrt{2}\xi &= 1 \\ \lambda - \rho + \sqrt{2}\xi &= 1 \\ \lambda + \rho - \sqrt{2}\xi &= 1 \\ -\lambda + \rho + \sqrt{2}\xi &= 1 \end{aligned} \quad (\text{A12})$$

Alternatively, for the regular tetrahedron embedded in the 3-cube with coordinates $x, y, z \in [0, 1]$ and defined by the four planar face conditions

$$\begin{aligned} 1 - x + y - z &\geq 0 \\ 1 - x - y + z &\geq 0 \\ 1 + x - y - z &\geq 0 \\ x + y + z - 1 &\geq 0 \end{aligned} \quad (\text{A13})$$

we may identify the coordinates s_i in the Fermion determinant, with the distances to the four planar faces of the tetrahedron of edge length $\sqrt{2}$,

$$\begin{aligned} s_1 &= \frac{1}{\sqrt{3}}(1 - x + y - z) \\ s_2 &= \frac{1}{\sqrt{3}}(1 - x - y + z) \\ s_3 &= \frac{1}{\sqrt{3}}(1 + x - y - z) \\ s_4 &= \frac{1}{\sqrt{3}}(x + y + z - 1) \end{aligned} \quad (\text{A14})$$

with constant

$$\sum_{i=1}^4 s_i = S = \frac{2}{\sqrt{3}}$$

by Fermat theorem. The cartesian variables x, y, z are related linearly to the Jacobi variables $\rho, \lambda, \xi = \alpha, \beta, \gamma$.

In terms of (A14), the free Laplacian is no longer diagonal in the new $x_i = x, y, z$ and S coordinates, reducing to the general elliptic operator

$$\sum_{i=1}^4 \partial_{s_i}^2 = \frac{1}{4} (7\partial_x^2 + 7\partial_y^2 + 7\partial_z^2 + 13\partial_S^2 8\partial_{xy}^2 + 8\partial_{xz}^2 + 8\partial_{yz}^2 + 14\partial_{xS}^2 + 14\partial_{yS}^2 + 14\partial_{zS}^2) \quad (\text{A15})$$

Note that the stationary WFs

$$\begin{aligned} \varphi_{n_1 n_2 n_3 n_4}(x, y, z) \\ = \sin\left(\frac{n_1 \pi s_1}{S}\right) \sin\left(\frac{n_2 \pi s_2}{S}\right) \sin\left(\frac{n_3 \pi s_3}{S}\right) \sin\left(\frac{n_4 \pi s_4}{S}\right) \end{aligned} \quad (\text{A16})$$

diagonalize the 4-dimensional free Laplacian (CM unsubtracted) with energies

$$e_{n_1 n_2 n_3 n_4} = \sum_{i=1}^4 \left(\frac{n_i \pi}{S}\right)^2 \quad (\text{A17})$$

and vanish identically on the 4 faces of the regular tetrahedron. The generalization of this approach to 5-bodies is much more involved, a subject of a separate analysis in [13]. We now will turn to a simpler variational approach.

APPENDIX B: VARIATIONAL CALCULATIONS OF THE LONGITUDINAL PART OF THE PENTAQUARK STATES

We start by recalling that for five bodies, $N = 5$, the kinematical domain is that of a regular *pentachoron*, a 4D simplex with 5 corners, 10 edges, 10 equilateral faces and 5 regular tetrahedrons, we also referred to as a starfish. There are basically two ways to solve Schrodinger equation in this geometry: (i) use a variational Ritz method, to be followed in this work, and (ii) use a full basis set for a different pentachoron (appearing in the problem of N fermions on the circle) which we presented in [13].

The solution of the elliptic PDEs by the Ritz method is well known. It consists of minimizing the corresponding energy functional over a set of parameters, usually coefficients of certain basis functions. For the ground state, we found a relatively simple set of Gaussian's trial functions

$$\Psi_{\text{Ritz}} = \Psi_{\text{faces}} * \left(1 + \sum_i^N C_i \exp(-Y^2/2R_i^2)\right) \quad (\text{B1})$$

where the first term is the linear ansatz (14), and the bracket gives a “correction function.” The average value of the Laplacian (with and without the cup potential) divided by a normalization

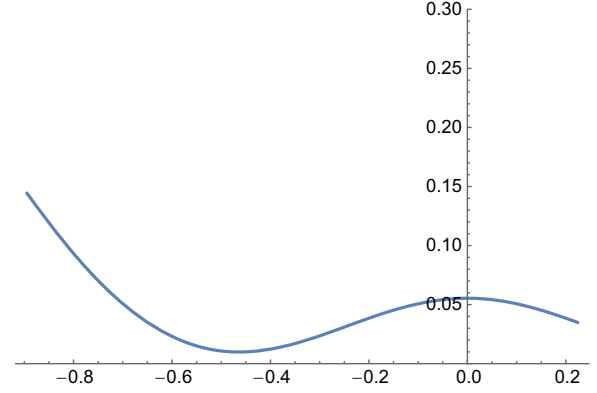


FIG. 13. The sum of Gaussians in the correction function (B1) after minimization of the Laplacian/norm ratio, $N = 3$. The plot is for $\alpha = \beta = \gamma = 0$ as a function of the last Jacobi momentum fraction δ . The left side is the “corner” and the right side the “face” of the simplex, where the ansatz itself vanishes.

$$\langle -\nabla^2 \rangle = \frac{\int_{\text{simplex}} \Psi_p (-\nabla^2) \Psi_p}{\int_{\text{simplex}} \Psi_p^2}$$

is minimized over C_i (using the *NMinimize* command). We used various number of correction terms and different sets of radii, eventually returning to just three Gaussians. For a Laplacian/norm we found the improvement, from the initial value of 182. to 165, improvement by about 10%, with a differential correction shown in Fig. 13. The corrections are relatively small, at the scale of few percents.

We found that adding the cup potential, $V_{\text{cup}} \sim (1/x_1 + 1/x_2 + 1/x_3 + 1/x_4 + 1/x_5 - 25)$ amounts to a *decrease* of the correction function, making the ground states WF close to the original linear ansatz (14). So, we have used it in the calculations to follow.

The P-shell pentaquark states have negative parity, due to the multiplication by the extra line coordinates. The simplest trial function that minimizes the Laplacian is of the form

$$\psi_1(C) \sim \Psi_{\text{faces}} \delta[1 - C(\alpha^2 + \beta^2 + \gamma^2 + \delta^2)] \quad (\text{B2})$$

The unmodified function $\psi_1(0)$ gives $\langle -\nabla^2 \rangle / \langle 1 \rangle \approx 312.0$. The modified one with $C = 4.5$ reduces it by about 1/10. More complicated trial functions we have tested lead to some improvements. The actual shape of the functions are compared in Fig. 5. (Note that the change of sign near zero is a robust feature, while another zero near $\delta \approx -0.5$ in the variational result is not as robust and is ansatz dependent. Clearly, the second zero should not be present in the true P1 state. (Yet we checked that this function is orthogonal to the S-shell $L = 0$ variational function, within small integration errors.) We conclude that the variational method is not to be trusted for $\delta < -0.5$.

APPENDIX C: EXACT LAPLACIAN EIGENSTATES

The “equilateral” (or regular) 4-simplex on which we need to solve the LF Schrodinger equation is known in mathematical literature as the A_4 simplex. It is related to the representations of the permutation group S_4 . The theory of tiles in arbitrary dimensions are related with diagonalization of the Laplacian, that we only learned of when most of the present paper was completed.

For the presentation, we follow the derivation in [28]. The so called *fundamental weights*, in $\dim = 5$ are four vectors ω_i in the 5D space

$$\begin{aligned} \omega = \{ & \{4/5, -1/5, -1/5, -1/5, -1/5\}, \\ & \{3/5, 3/5, -2/5, -2/5, -2/5\}, \\ & \{2/5, 2/5, 2/5, -3/5, -3/5\}, \\ & \{1/5, 1/5, 1/5, 1/5, -4/5\} \} \end{aligned} \quad (C1)$$

Together with the origin $\{0, 0, 0, 0, 0\}$, these vectors make five corners of a simplex, which is similar but different from the starfish simplex we need. In Jacobi coordinates its corners lie at five different points a_i

$$\begin{aligned} & \{1/\sqrt{2}, 1/\sqrt{6}, 1/(2\sqrt{3}), 1/(2\sqrt{5})\}, \\ & \{-(1/\sqrt{2}), 1/\sqrt{6}, 1/(2\sqrt{3}), 1/(2\sqrt{5})\}, \\ & \{0, -\sqrt{2/3}, 1/(2\sqrt{3}), 1/(2\sqrt{5})\}, \\ & \{0, 0, -(\sqrt{3}/2), 1/(2\sqrt{5})\}, \\ & \{0, 0, 0, -(2/\sqrt{5})\} \end{aligned} \quad (C2)$$

The affine transformation from one simplex to the other can be found numerically, using the *Mathematica* command *FindGeometricTransform*. It follows from rotation and shifts.

$$\begin{aligned} & \{ \{0.4 + 0.228825\alpha + 0.213762\beta + 0.237755\gamma + 0.473607\delta\}, \\ & \{0.2 - 0.59907\alpha - 0.0816497\beta + 0.0288675\gamma + 0.111803\delta\}, \\ & \{+0.228825\alpha - 0.559635\beta - 0.11547\gamma\}, \\ & \{-0.2 + 0.0707107\alpha + 0.305049\beta - 0.518007\gamma - 0.111803\delta\}, \\ & \{-0.4 + 0.0707107\alpha + 0.122474\beta + 0.366854\gamma - 0.473607\delta\} \} \end{aligned} \quad (C3)$$

In any dimension, the eigenstates of the Laplacian can be written as a sum over permutations over the whole $S_{\dim-1}$ group. In *Mathematica* it has a compact form

$$\text{Sum}[\text{Signature}[w] \text{Exp}[2I * \text{Pi} * (\text{Permute}[p, w].y)], \{w, Sn\}] \quad (C4)$$

or

$$\sum_{w \in S_d} \text{sign}[w] e^{2\pi i (wp).y} \quad (C5)$$

The “momenta” $p = \sum_1^4 m_i \omega_i$ are defined with m ranging over a 4-dimensional set of positive integers, Sn being all the $(d-1)!$ permutations of the permutation group. Here y are the coordinates, either in the original simplex, or those projected to the starfish as explained above. The permutations of the components of p according to w (not ω) are necessary to account for every edge of the simplex.

The ground state correspond to $m = \{1, 1, 1, 1\}$ or $p = \{-2, -1, 0, 1, 2\}$, a set of momenta in the “Fermi sphere” with five elements. The exact form of its real part is given at the end of the section due to its complexity. The imaginary part of the sum is identically zero, so we only quoted the real part.

These set of functions are eigensstates of the Laplacian. The lowest eigenvalue of $\psi[1, 1, 1, 1]$ is

$$-\nabla^2 \psi[1, 1, 1, 1] / \psi[1, 1, 1, 1] = 186.972 \quad (C6)$$

which is in the ballpark of the values found using the variational minimization. Recall that the numerically exact lowest eigenstate yields a lowest eigenvalue of 164 [13]. The shape of $\psi[1, 1, 1, 1]$ is compared to the variational and numerically exact solution in Fig. 4. The $\psi[1, 1, 1, 1]$ eigenfunction is concentrated near the origin, and is smaller near the corner (left-hand side of the plot).

However, we note that the two set of functions exhibit differences. The set $\psi[m_1, m_2, m_3, m_4]$ enforces the Laplacian at *any point* inside the 4-simplex, but only enforces the boundary conditions on a subset of the boundary conditions (e.g., corners and edges of the 4-simplex). In contrast, all variational functions are designed to vanish on the whole boundary, but with varying Laplacian in bulk. (Recall that the minimization in the previous Appendix, was defined as a ratio of *integrals* over the simplex, as suggested by the Ritz method.)

To quantify the deviations of $\psi[1, 1, 1, 1]$ from the strict Dirichlet conditions, let us compare its values at the boundary to that at the center. Here, and for clarity, the arguments are the set of five Bjorken momentum fractions, with the normalization set by the value of the WF at the center of the 4-simplex or $\psi_0 = \psi\{1/5, 1/5, 1/5, 1/5, 1/5\}$

$$\begin{aligned}\psi\{1, 0, 0, 0, 0\}/\psi_0 &= -6.75737 * 10^{-6}(\text{corner}) \\ \psi\{1/2, 1/2, 0, 0, 0\}/\psi_0 &= 0.000133(\text{edge}) \\ \psi\{1/3, 1/3, 1/3, 0, 0\}/\psi_0 &= 0.007927(3 \text{ face}) \\ \psi\{1/4, 1/4, 1/4, 1/4, 0\}/\psi_0 &= 0.094441(4 \text{ face})\end{aligned}\quad (C7)$$

The first value at the corners is consistent with the expected numerical accuracy, while the others are not. We also note that the 5-fold symmetry is holding strictly, e.g., the values at all corners are the same.

The nonzero value of the function at the boundary leads to a logarithmic divergence of the mean value of the potential $\int_{\text{simplex}} \psi^2 V$, which in the infinite momentum frame is singular at the boundary $V \sim m^2/X$. However, once the potential is regulated $X \rightarrow \sqrt{X^2 + 1/\gamma^2}$ via a finite gamma factor, the boundary term turns out to be numerically small, as it follows from the *squared* small values tabulated above.

This procedure was extended to the P-shell of the pentaquarks. The shape of the function $\psi[\{2, 1, 1, 1\}]$ (black solid line) is compared to the variational ones in Fig. 5. Note that even a simple one-parameter variational function (red dashed) takes the shape close to the exact one, except at the left side of the plot (where $x_5 \rightarrow 1$). This observation is quite typical. While the Ritz variational method can lead to reasonable wave functions overall, their predictions of the tails are not reliable since they play little role in the minimization of the integrals.

The general formula for the eigenvalues on a unit d -simplex is

$$\begin{aligned}E(m_1, m_2, \dots, m_{d-1}) &\propto \pi^2 |p|^2 \\ &= \pi^2 \sum_{i=1}^{d-1} \sum_{j=1}^{d-1} \left(\min(i, j) - \frac{ij}{d} \right) m_i m_j,\end{aligned}\quad (C8)$$

which for integer values, become

$$\begin{aligned}E(m_1, m_2, m_3, m_4) &\sim (2m_1^2 + 3m_2^2 + 3m_3^2 + 3m_3m_4 + 2m_4^2 \\ &\quad + 4m_2m_3 + 2m_2m_4 + 3m_1m_2 + 2m_1m_3 + m_1m_4)\end{aligned}\quad (C9)$$

with a common scaling factor from the coordinate transform from x to y .

Therefore the ratio of the P-wave to the S-wave eigenvalues is predicted to be

$$\frac{P_{\text{wave}}}{S_{\text{wave}}} = (74/5)/10 = 1.48 \quad (C10)$$

This ratio is lower than the ratio of the values obtained from (14) prior to the variational corrections

$$\frac{P_{\text{wave}}}{S_{\text{wave}}} = 312./182. = 1.71 \quad (C11)$$

The latter tend to lower this ratio toward the exact eigenvalues, reaching about 1.6.

The Laplacian eigenfunctions are smaller near the corners (the left side of the plots versus δ) than the variational ones. If we evaluate the PDFs (the integral of the corresponding squared WF over the three momentum fractions α, β, γ), a logarithmic plot shows a larger hard edge behavior $\sim (1-x)^p$, $p \approx 12$. A similar behavior was observed in [13].

The averaging of the potential V_{cup} leads to smaller values for the Laplacian functions, in comparison to the variational ones, to the extent that it can even be neglected within a 10% accuracy. One may wonder why V_{cup} is more important for baryons (on a triangle or 2-simplex) compared to pentaquarks (on a pentachoron or 4-simplex). It may be a general trend as N increases.

Finally and for definiteness, let us give the explicit form of the lowest Laplacian eigenstate $\psi(1, 1, 1, 1)$.

$$\begin{aligned}Re \psi[1, 1, 1, 1] &= \cos(2.5651\beta + 13.3389\gamma - 1.5708\delta) - \cos(-10.4036\alpha + 4.85939\beta + 7.37354\gamma - 0.868315\delta + 1.25664) \\ &\quad - \cos(-3.2149\alpha - 12.7221\beta + 3.74594\gamma - 0.868315\delta + 1.25664) \\ &\quad + \cos(-8.41672\alpha - 9.71879\beta + 4.65284\gamma - 0.165833\delta + 2.51327) \\ &\quad + \cos(-9.41018\alpha - 0.573574\beta + 9.90275\gamma - 0.165833\delta + 2.51327) \\ &\quad + \cos(9.41018\alpha + 7.99807\beta + 5.65549\gamma + 1.5708\delta) \\ &\quad - \cos(-5.20182\alpha + 0.438171\beta - 12.432\gamma + 2.27328\delta + 1.25664) \\ &\quad - \cos(-6.19528\alpha + 9.58339\beta - 7.18206\gamma + 2.27328\delta + 1.25664)\end{aligned}$$

$$\begin{aligned}
& -\cos(0.993459\alpha - 7.99807\beta - 10.8097\gamma + 2.27328\delta + 1.25664) \\
& -\cos(4.20836\alpha + 11.0013\beta + 6.56239\gamma + 2.27328\delta + 1.25664) \\
& -\cos(10.4036\alpha + 2.5651\beta + 8.1847\gamma + 2.27328\delta + 1.25664) \\
& -\cos(11.3971\alpha - 6.58012\beta + 2.93479\gamma + 2.27328\delta + 1.25664) \\
& -\cos(9.41018\alpha + 5.43297\beta - 7.68338\gamma + 3.14159\delta) \\
& +\cos(-9.41018\alpha - 1.99153\beta - 8.99585\gamma + 3.67824\delta + 3.76991) \\
& +\cos(-10.4036\alpha + 7.15369\beta - 3.74594\gamma + 3.67824\delta + 3.76991) \\
& +\cos(-3.2149\alpha - 10.4278\beta - 7.37354\gamma + 3.67824\delta + 3.76991) \\
& +\cos(8.57164\beta + 9.9985\gamma + 3.67824\delta + 3.76991) \\
& +\cos(7.18874\alpha - 9.00981\beta + 6.3709\gamma + 3.67824\delta + 3.76991) \\
& +\cos(6.19528\alpha + 0.135402\beta + 11.6208\gamma + 3.67824\delta + 3.76991) \\
& -\cos(5.20182\alpha - 0.708976\beta - 12.0264\gamma + 3.84407\delta + 1.25664) \\
& -\cos(9.41018\alpha + 9.14522\beta + 0.0957439\gamma + 3.84407\delta + 1.25664) \\
& -\cos(-8.41672\alpha - 7.42449\beta - 6.46664\gamma + 4.38072\delta + 5.02655) \\
& -\cos(0.993459\alpha + 3.13867\beta + 12.5277\gamma + 4.38072\delta + 5.02655) \\
& +\cos(-0.993459\alpha + 11.4395\beta - 5.86958\gamma + 4.54656\delta + 2.51327) \\
& +\cos(6.19528\alpha - 6.14194\beta - 9.49718\gamma + 4.54656\delta + 2.51327) \\
& +\cos(4.20836\alpha + 12.1485\beta + 1.00264\gamma + 4.54656\delta + 2.51327) \\
& +\cos(11.3971\alpha - 5.43297\beta - 2.62496\gamma + 4.54656\delta + 2.51327) \\
& +\cos(-9.41018\alpha + 5.43297\beta + 6.56239\gamma + 5.0832\delta + 6.28319) \\
& +\cos(-2.22144\alpha - 12.1485\beta + 2.93479\gamma + 5.0832\delta + 6.28319) \\
& +\cos(-5.20182\alpha + 5.29757\beta - 10.2126\gamma + 5.24904\delta + 3.76991) \\
& +\cos(11.3971\alpha - 1.72072\beta + 5.15417\gamma + 5.24904\delta + 3.76991) \\
& -\cos(-7.42326\alpha - 9.14522\beta + 3.84169\gamma + 5.78569\delta + 7.53982) \\
& -\cos(-8.41672\alpha + 9.0916\gamma + 5.78569\delta + 7.53982) \\
& -\cos(-9.41018\alpha + 2.86787\beta - 6.77648\gamma + 6.654\delta + 6.28319) \\
& -\cos(7.18874\alpha - 4.15042\beta + 8.59028\gamma + 6.654\delta + 6.28319) \\
& +\cos(5.20182\alpha + 4.15042\beta - 9.80701\gamma + 6.81983\delta + 3.76991) \\
& +\cos(10.4036\alpha + 4.85939\beta - 2.93479\gamma + 6.81983\delta + 3.76991) \\
& -\cos(-9.41018\alpha + 6.58012\beta + 1.00264\gamma + 7.35648\delta + 7.53982) \\
& -\cos(-2.22144\alpha - 11.0013\beta - 2.62496\gamma + 7.35648\delta + 7.53982) \\
& -\cos(-4.20836\alpha + 7.28909\beta + 7.87487\gamma + 7.35648\delta + 7.53982) \\
& -\cos(2.98038\alpha - 10.2924\beta + 4.24727\gamma + 7.35648\delta + 7.53982) \\
& -\cos(7.15369\beta - 8.90011\gamma + 7.52231\delta + 5.02655) \\
& -\cos(11.3971\alpha - 0.573574\beta - 0.405578\gamma + 7.52231\delta + 5.02655) \\
& +\cos(-7.42326\alpha - 7.99807\beta - 1.71806\gamma + 8.05896\delta + 8.79646) \\
& +\cos(-3.2149\alpha + 1.85612\beta + 10.4041\gamma + 8.05896\delta + 8.79646)
\end{aligned}$$

$$\begin{aligned}
& -\cos(10.8659\beta - 1.12099\gamma + 8.2248\delta + 6.28319) \\
& -\cos(7.18874\alpha - 6.71552\beta - 4.74859\gamma + 8.2248\delta + 6.28319) \\
& +\cos(-7.42326\alpha - 4.28582\beta + 6.06107\gamma + 8.76144\delta + 10.0531) \\
& +\cos(-4.20836\alpha + 8.43624\beta + 2.31512\gamma + 9.62976\delta + 8.79646) \\
& +\cos(2.98038\alpha - 9.14522\beta - 1.31248\gamma + 9.62976\delta + 8.79646) \\
& +\cos(-8.41672\alpha + 2.29429\beta - 2.02789\gamma + 10.3322\delta + 10.0531) \\
& +\cos(2.98038\alpha - 5.43297\beta + 6.46664\gamma + 10.3322\delta + 10.0531) \\
& -\cos(6.19528\alpha + 3.57684\beta - 5.05842\gamma + 10.4981\delta + 7.53982) \\
& -\cos(-7.42326\alpha - 3.13867\beta + 0.501322\gamma + 11.0347\delta + 11.3097) \\
& -\cos(-2.22144\alpha - 2.4297\beta + 7.37354\gamma + 11.0347\delta + 11.3097) \\
& +\cos(0.993459\alpha + 6.58012\beta - 4.15152\gamma + 11.2006\delta + 8.79646) \\
& +\cos(7.18874\alpha - 1.85612\beta - 2.52921\gamma + 11.2006\delta + 8.79646) \\
& -\cos(-3.2149\alpha + 4.15042\beta - 0.715412\gamma + 12.6055\delta + 11.3097) \\
& -\cos(2.98038\alpha - 4.28582\beta + 0.9069\gamma + 12.6055\delta + 11.3097) \\
& +\cos(-2.22144\alpha - 1.28255\beta + 1.8138\gamma + 13.308\delta + 12.5664)
\end{aligned} \tag{C12}$$

APPENDIX D: LARGE N AND THE SPHERICAL APPROXIMATION

If the number of corners N is large, the corresponding simplices are bracketed by spheres. The outer or “large” sphere with the radius at the corners or $R_{\text{large}} = |a_i|$, and thinner or “small” sphere with the radius normal to the edges. This spherical approximation maybe useful for higher Fock components, but for now $N = 5$ is enough to provide the first glimpse for the antiquarks.

For $N = 5$ the number of edges, 10, can be considered large. The numerical values of the radii are

$$\begin{aligned}
R_{\text{large}} &= \frac{2}{\sqrt{5}} \approx 0.894, \\
R_{\text{small}} &= \sqrt{\frac{3}{10}} \approx 0.548
\end{aligned} \tag{D1}$$

which are not that close, but they should become closer at larger N

The radial Schrodinger equation for 4D spheres, is of the form

$$-u''(r) + \left(\frac{3}{4r^2} + V(r)\right)u(r) = \lambda u(r) \tag{D2}$$

where $u = \psi * r^{3/2}$. Without potential (pure Laplacian) the ground state eigenvalues are 18.353 for our large sphere, and 48.94 for the small sphere. With the potential $3/(R - r)$ (imitating our V_{cup}), those values increase substantially, to 27.10 and 63.51 respectively. The values for our variational ansatz $\Psi_{0.8}$ are larger still, about 70 and 150, respectively.

Yet, as Fig. 14 shows, the wave functions are hardly modified by the potential. Perhaps this is also the case for the actual 4-simplex.

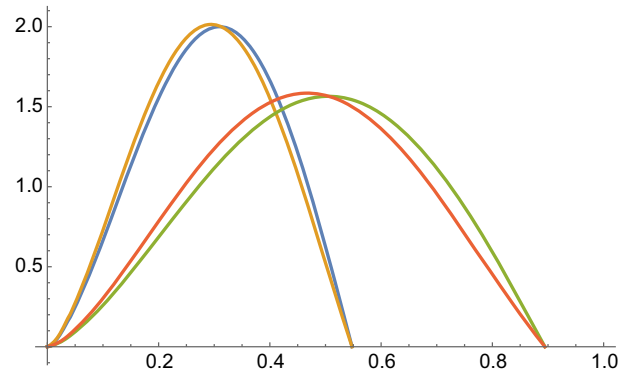


FIG. 14. The ground state wave functions $u(r)$, for the large and small spheres, with (red and orange) and without the cup potential (blue and green).

- [1] S. D. Drell and T.-M. Yan, *Phys. Rev. Lett.* **24**, 181 (1970).
- [2] M. Diehl, *Phys. Rep.* **388**, 41 (2003).
- [3] S. J. Brodsky, G. F. de Téramond, and H. G. Dosch, *Few-Body Syst.* **55**, 407 (2014).
- [4] J. P. Vary, H. Honkanen, J. Li, P. Maris, S. J. Brodsky, A. Harindranath, G. F. de Téramond, P. Sternberg, E. G. Ng, and C. Yang, *Phys. Rev. C* **81**, 035205 (2010).
- [5] S. Jia and J. P. Vary, *Phys. Rev. C* **99**, 035206 (2019).
- [6] E. Shuryak and I. Zahed, *Phys. Rev. D* **107**, 034023 (2023).
- [7] E. Shuryak and I. Zahed, *Phys. Rev. D* **107**, 034027 (2023).
- [8] E. Shuryak and I. Zahed, *Phys. Rev. D* **107**, 034026 (2023).
- [9] N. Miesch, E. Shuryak, and I. Zahed, *Phys. Rev. D* **108**, 094033 (2023).
- [10] N. Miesch, E. Shuryak, and I. Zahed, *Phys. Rev. D* **112**, 094001 (2025).
- [11] N. Miesch, E. Shuryak, and I. Zahed, *Phys. Rev. D* **109**, 014022 (2024).
- [12] N. Miesch, E. Shuryak, and I. Zahed, *Phys. Rev. D* **112**, 074017 (2025).
- [13] F. He, E. Shuryak, W. Wu, and I. Zahed, [arXiv:2510.02076](#).
- [14] In Wolfram *Mathematica* there is a command `Simplex[ $a_1, \dots, a_5$ ]` creating simplex between given number of points. It can be used in integrals as the domain.
- [15] M. Girardeau, *J. Math. Phys. (N.Y.)* **1**, 516 (1960).
- [16] W.-Y. Liu, E. Shuryak, and I. Zahed, *Phys. Rev. D* **109**, 074029 (2024).
- [17] A. E. Dorokhov and N. I. Kochelev, *Phys. Lett. B* **304**, 167 (1993).
- [18] G. A. Miller, [arXiv:2208.12873](#).
- [19] E. Shuryak, *Phys. Rev. D* **100**, 114018 (2019).
- [20] G. T. Garvey, *Phys. Rev. C* **81**, 055212 (2010).
- [21] D. E. Kharzeev and E. M. Levin, *Phys. Rev. D* **95**, 114008 (2017).
- [22] W. Detmold and C. J. D. Lin, *Phys. Rev. D* **73**, 014501 (2006).
- [23] V. N. Gribov and L. N. Lipatov, *Phys. Lett.* **37B**, 78 (1971).
- [24] Y. L. Dokshitzer, *Sov. Phys. JETP* **46**, 641 (1977).
- [25] G. Altarelli and G. Parisi, *Nucl. Phys.* **B126**, 298 (1977).
- [26] P. M. Nadolsky, H.-L. Lai, Q.-H. Cao, J. Huston, J. Pumplin, D. Stump, W.-K. Tung, and C. P. Yuan, *Phys. Rev. D* **78**, 013004 (2008).
- [27] D. B. Clark, E. Godat, and F. I. Olness, *Comput. Phys. Commun.* **216**, 126 (2017).
- [28] Y. Doumerc and J. Moriarty, *Probab. Theory Relat. Fields* **145**, 351 (2008).